The homotopy quotient

When the notations mean more than what we make them say

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The purpose of this talk is to explain how of the replacement of

the logical notation

$$a = b$$

by the topo-logical notation

$$a \longrightarrow b$$

has brought mathematics beyond set theory, into a new paradigm.

PLAN

I — Colimits

Limits and colimits

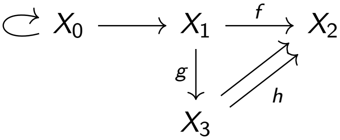
Contemporary mathematics is based on the notion of set.

The two elementary operations on sets are

colimits (pasting)	limits (crossing)
disjoint union (sum) $A \coprod B$	cartesian product $A \times B$
union of subsets $A \cup B$	intersection of subsets $A \cap B$
pushout $A \coprod_X B$	fiber product $A \times_X B$
quotient $A_{/f(b) \simeq g(b)}$	equalizer $\{a : A \mid f(a) = g(a)\}$
"OR synthesis"	"AND synthesis"

Diagrams

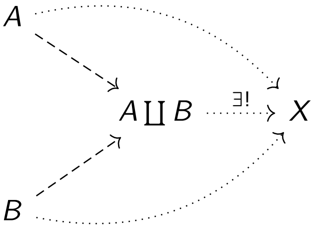
A diagram of set is the data of some sets X_i and morphisms $f : X_i \to X_j$



such that for all $X_i \xrightarrow{f} X_j \xrightarrow{g} X_k$ the composition $X_i \xrightarrow{gf} X_k$ is part of the diagram,

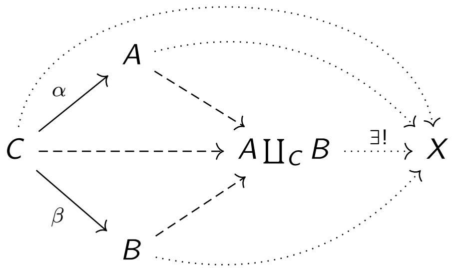
and such that certain "commutation relations" holds (like $f = hg$)
The drawing of a diagram leaves often implicit these relations.

Sum



Pushout

Diagrammatic definition

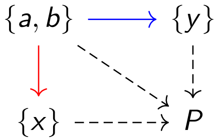


Computation

$$A \coprod_C B = \left(A \coprod C \coprod B\right) /_{c=\alpha(c),\ c=\beta(c)}$$
$$= \left(A \coprod B\right) /_{\alpha(c)=\beta(c)}$$

Pushout – example 1

We consider the following pushout



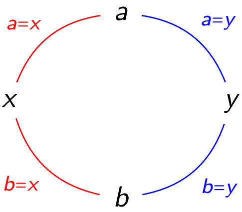
The computation of of the pushout gives

$$P = \{x\} \coprod_{\{a,b\}} \{y\} = \{x, a, b, y\}/_{(a=x,\ b=x,\ a=y,\ b=y)} = \{\star\}$$

Pushout – example 1

A graphical representation help to better see what is going on.

It is useful to represent the previous four identifications relations in a diagram (this is were the change of symbol is happening):



The fact that the pushout is a singleton correspond to the fact that this graph is connected: all elements are equivalent.

Pushout – example 2

Another example

$$\begin{array}{ccc} \begin{Bmatrix} a & b \\ c & d \end{Bmatrix} & \xrightarrow{p_2} & \begin{Bmatrix} y \\ y' \end{Bmatrix} \\ {\scriptstyle p_1}\big\downarrow & & \\ \{x, x'\} & & \end{array}$$

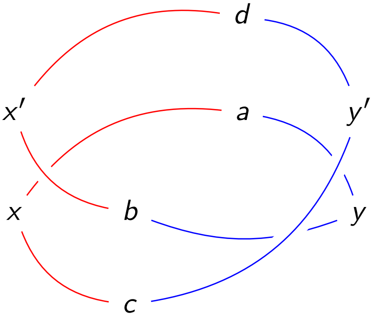
where p_1 and p_2 are vertical and horizontal projections.

The pushout is still a singleton, but this is no longer easy to see: a computation needs to be made.

There again a graphical representation help to see what is going on.

Pushout – example 2

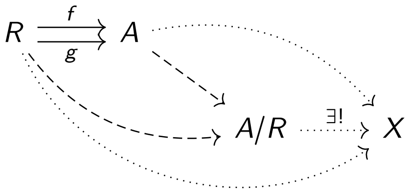
Here is the diagram of relations



This graph is connected, this shows that the pushout is a singleton (all elements are equivalent).

Quotient (coequalizer)

Diagrammatic definition



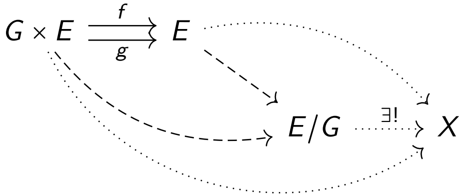
Computation

$$A/R = (A \coprod R)/_{r=f(r),\ r=g(r)}$$
$$= A/_{f(r)=g(r)}$$

Quotient (coequalizer)

Coequalizer appear when quotients are involved.

For example, let a group G act on a set E, then the quotient is defined by the coequalizer



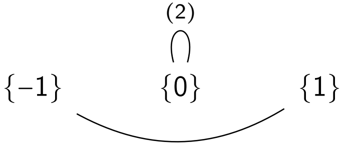
$$
\begin{aligned}
E/G &= \left((G \times E) \coprod E\right) /_{(g,e)=e,\ (g,e)=ge} \\
&= E/_{e=ge}
\end{aligned}
$$

Quotient – example 1

Let us look at the action of $G = \mathbb{Z}_2$ on $E = \{-1, 0, 1\}$ by change of sign.

Again it is convenient to look at a graphical representation of what is going on.

The arrows represent the action of $1 \in \mathbb{Z}_2$ (the action of 0 being trivial)

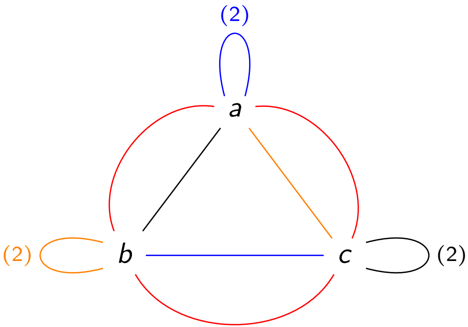


(The (2) over the loop is the order -1 in $\mathbb{Z}_2$.)

This shows the two orbits of the action, and that the quotient E/G has two elements.

Quotient – example 2

The (simplified) graphical representation of the action of $\mathfrak{S}_3$ on the set $\{a, b, c\}$ is



(The three substitutions are in black, orange and blue, the cyclic permutation is in red. The (2) are the order of the loop in the group $\mathfrak{S}_3$.)

We can see the single orbit but also that each element has a nontrivial stabilizer (represented by the loop).

Remarks

Why dwell on these trivial constructions?

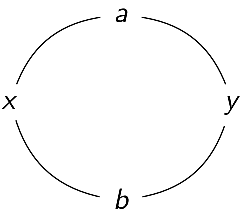
Because, in both cases, even though the pushout is trivial, the proof of its triviality is not.

We had to use the graph of relations and to look at its shape to conclude it was connected.

The replacement of $x = a$ by x —— a has led us to a topological object and to the use of topological features.

Remarks

Also, these graphs reveal an interesting ambiguity of identification.



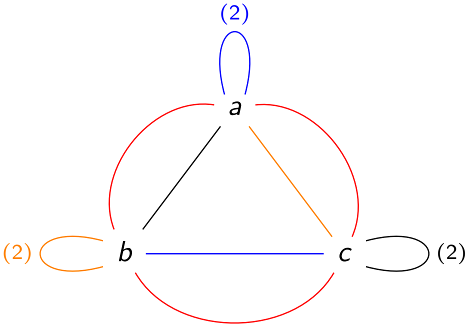
There is *two ways* to prove that $x = y$:

$$\begin{array}{rclcclcc} \text{the upper way} & : & x = a & \text{and} & a = y & \text{then} & x = y, \\ \text{or the lower way} & : & x = b & \text{and} & b = y & \text{then} & x = y. \end{array}$$

In other words x and y are not canonically equivalent.

Remarks

Same thing in the group action:



each element can be connected to another one in exactly two ways. This is related to the fact that the stabilizer of the orbit is non-trivial (as shown by the loops).

Remarks

The classical notion of colimit does not remember the shapes (and the ambiguities of identification),

only the connected components (and the existence of identifications).

This is the purpose of the so-called homotopy colimit to do so.

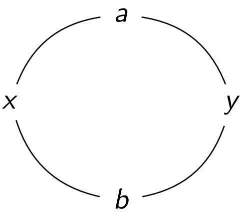
II — Homotopy colimits

Homotopy pushout – example 1

Let us come back to the pushout

$$\begin{array}{ccc} \{a, b\} & \longrightarrow & \{y\} \\ \downarrow & & \\ \{x\} & & \end{array}$$

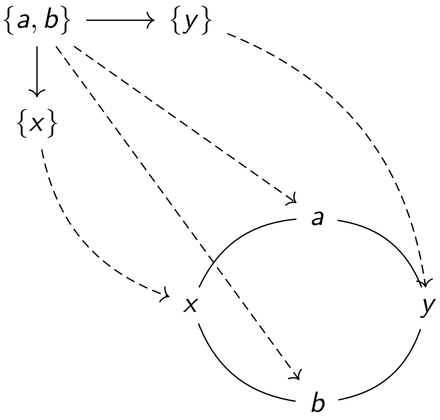
The homotopy colimit of this diagram is the circle



viewed up to homotopy.

Homotopy pushout – example 1

The canonical cone is



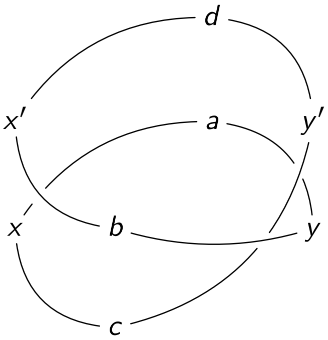
It does not commute strictly, but only up to homotopy. This is the point.

Homotopy pushout – example 2

Similarly the homotopy colimit of

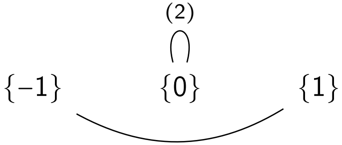
$$\begin{array}{ccc} \begin{Bmatrix} a & b \\ c & d \end{Bmatrix} & \xrightarrow{p_2} & \begin{Bmatrix} y \\ y' \end{Bmatrix} \\ {\scriptstyle p_1}\big\downarrow & & \\ \{x, x'\} & & \end{array}$$

is the circle

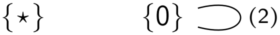


Homotopy quotient – example 1

The homotopy colimit (homotopy quotient) of the action of $\mathbb{Z}_2$ on $\{-1, 0, 1\}$ is



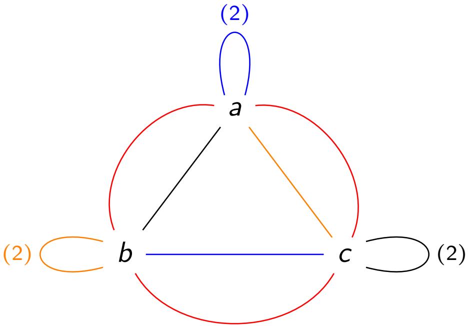
which is homotopy equivalent to



This is the union a point together with the classifying space $B\mathbb{Z}_2$ of the group $\mathbb{Z}_2$ (a space having $\mathbb{Z}_2$ as π_1 and no other homotopy invariant). This space is homotopic to $\mathbb{P}^\infty_{\mathbb{R}}$.

Homotopy quotient – example 2

The homotopy colimit (homotopy quotient) of the action of $\mathfrak{S}_3$ on the set $\{a, b, c\}$ is



which is homotopy equivalent to

This is again the classifying space $B\mathbb{Z}_2$.

Homotopy colimit – general construction

The homotopy colimit of a diagram X_i is constructed as follows

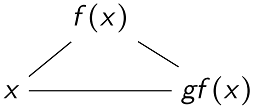
- every element $x : X_i$ define a vertex

$$x$$

- every pair $(x : X_i, X_i \xrightarrow{f} X_j)$ defines an edge

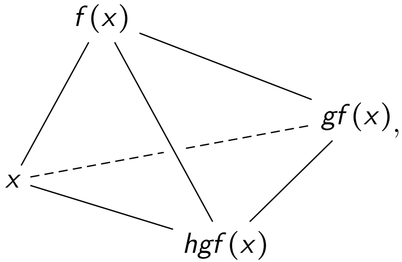
$$x — f(x)$$

- every triplet $(x : X_i, X_i \xrightarrow{f} X_j \xrightarrow{g} X_k)$ defines a triangle,



Homotopy colimit – general construction

- every quadruplet $(x : X_i, X_i \xrightarrow{f} X_j \xrightarrow{g} X_k \xrightarrow{h} X_l)$ defines a tetrahedron,



- etc.

Homotopy colimit – general construction

This build, little by little, a space of possibly infinite dimension (a simplicial set) endowed with a triangulation,

This space will be considered only up to homotopy (up to continuous deformation without any cutting or pasting).

This means in particular that the data of the triangulation is forgotten.

All that is remembered is a kind of shape.

The homotopy type of any topological space can be constructed as the homotopy colimit of a diagram of sets.

Remark

The idea behind the homotopy colimit is not to delete the identifications done when computing the classical colimit, not to project them onto mere equalities, but to write them as isomorphisms.

This is the purpose of the replacement of $x = a$ by $x — a$.

Getting rid of the logical symbol "=" for the topological symbol " — " acknowledges that identifying two things is an action, a structure, or a process, rather than a mere statement.

III — ∞-groupoids

∞-groupoids

The homotopy colimit of a diagram of sets is no longer a set.

It is a notion called an ∞-groupoid.

Intuitively this is the following structure

- the data of some objects (represented as points)
- equipped with isomorphisms between them (represented as paths)
- equipped with isomorphisms between isomorphisms (represented as homotopies between paths)
- equipped with isomorphisms between isomorphisms between isomorphisms (represented as homotopies between homotopies)
- etc.

The simplest definition of ∞-groupoid is as topological spaces up to homotopy. The words ∞-groupoids and homotopy types are synonyms.

∞-groupoids

Sets are particular ∞-groupoids, they have objects but no isomorphisms nor higher isomorphism.

Reciprocally, any ∞-groupoid can be truncated into a set by taking the set of connected components.

$$\mathrm{Set} \underset{}{\overset{\pi_0}{\leftarrow}} \hookrightarrow \infty\text{-}\mathrm{Gpd}$$

Replacement for sets

Within a set the only possible statement is $a = b$.

Within an ∞-groupoid, the situation is richer: for two objects a and b, there is an ∞-groupoid $iso(a, b)$ of isomorphisms between a and b.

In particular, a and b can have multiple ways to be equal.

∞-groupoids must be thought as an enhancement of sets, suited for encoding the ambiguity of identifications between some objects.

∞-groupoids

For any two objects x and y in a ∞-groupoids X, we define

$$\pi_1(X; x, y) = \pi_0(iso_X(x, y))$$

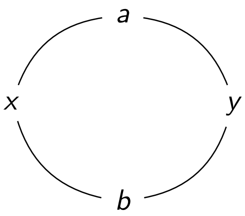
This is the set of paths x — y up to homotopy.

This set capture the ambiguity to identify x and y.

x and y are canonically isomorphic iff $\pi_1(X; x, y)$ has a single element.

∞-groupoids

In the example



the number of turns around the circle gives

$$\pi_1(X; x, y) \simeq \mathbb{Z}.$$

∞-groupoids

For any two paths $f, g : x \longrightarrow y$, we define

$$\pi_2(X; x, y; f, g) = \pi_0\left(iso_{iso_X(x,y)}(f, g)\right)$$

This is the set of homotopies between f and g up to homotopies of homotopies.

This set capture the ambiguity to identify f and g.

Similarly, at each level we can define some sets π_n recording the classes of homotopy of identification of level n.

These are the so-called homotopy invariants of the homotopy type.

The judiciary metaphor

Let us call an isomorphism $x \xrightarrow{f} y$ a witness of the identity of a and b.

Let us say that two witnesses f and g agree if there exist a witness $f \xrightarrow{\alpha} g$.

Then $\pi_1(X; , x, y)$ is the set of discordances between witnesses of the identity of x and y.

The judiciary metaphor

The specificity of the homotopy paradigm as opposed to the set paradigm, is to record all identifications that have been done, to minute them.

One can think to an ∞-groupoid as a judiciary file recording all witnessing.

The homotopy type is trivial when all witness agree (in which case it is contractible, i.e. equivalent to a set with one element).

But, in general, not all witnesses agree.

The non-triviality of the homotopy type is the obstruction to find the truth.

Homotopy colimit – What is it about?

It has been a big event to discover that mathematical constructions have more regular properties when this ambiguity structure is considered.

For example this is behind all the so-called derivation processes in homological algebra (derived functors, derived categories, syzygies in commutative algebra, Euler characteristic...)

Homotopy colimit – What is it about?

Replacing sets by ∞-groupoids ask to enhance and replace old notions

sets	∞-groupoids
vector space	chain complex
top. space, manifold	topos, stacks

Replacement for sets

Another motivation to replace sets by ∞-groupoid is because the notions of set and equality are too naive for certain purposes.

For example, the proper notion of identification for sets is bijection.

But sets do not provide a structure able of classifying sets and their bijections.

They can only classify things up to equality.

Sets and their bijections form a groupoid, not a set.

Replacement for sets

Sets and their bijections form a groupoid, not a set.

Groupoids and their equivalence form a 2-groupoid.

Only ∞-groupoids are able to self-classify.

This is why they are so important.

They are the simplest notion, containing that of set, and able to self-classify.

IV — Conclusion

Conclusion

In relation with semiotics, my purpose was to emphasize that the notation "=" contains more structure than the way it is usually thought.

Something was hidden, but not so much in the symbol "=" than in its interpretation.

When replacing the logical symbol $x = a$ by the "better" topological symbol x — a , the interpretation changes and reveal the structure of ambiguity when identifying two things.

This "superstructure" of identification is of a topological nature.

The price to pay to access this structure is to enhance sets into ∞-groupoids. Bye bye sets...

This is a big revolution, a shift of paradigm.

But this change stays consistent with the axiomatic method, so the practice of mathematics is not affected.

Thanks !

Appendix

Homotopy colimit – general construction

One way to understand the HC is to look at it as the underlying shape of the diagram.

This is a bit subtle since composition are usually left implicit when picturing a category, but with this in mind, the shape of the diagram is obtained by

- forgetting the name of objects
- forgetting the direction of arrows
- materializing all composition of 2 arrows with triangle (thus forgetting which side is the composite of the two others),
- materializing all composition of n arrows with an n-simplex, etc.
- forgetting the resulting triangulation

After all these forgettings, all is left is a pure shape.

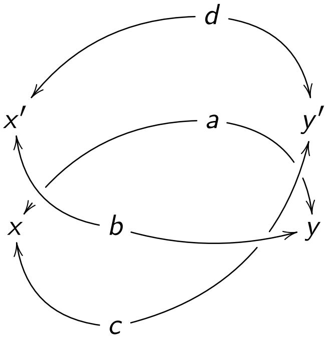
Homotopy colimit – general construction

From a diagram

$$\begin{array}{ccc}
\left\{\begin{matrix} a & b \\ c & d \end{matrix}\right\} & \xrightarrow{p_2} & \left\{\begin{matrix} y \\ y' \end{matrix}\right\} \\
{\scriptstyle p_1}\big\downarrow & & \\
\{x, x'\} & &
\end{array}$$

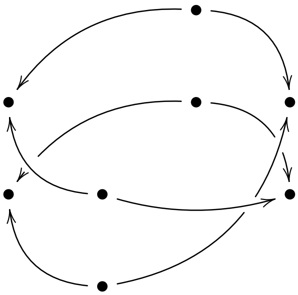
Diagram of elements

we start by drawing all the arrows (so-called diagram of elements)



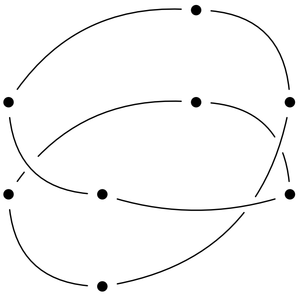
Homotopy colimit – general construction

then we forget the names of the objects



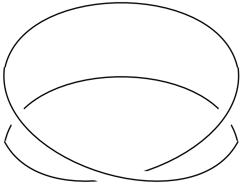
Homotopy colimit – general construction

then the direction of arrows



Homotopy colimit – general construction

then the embedding in space



Homotopy colimit – general construction

then the embedding in space