New methods for left exact localisations of topoi

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Introduction

A new dimension of study of topoi has recently been open with the notion of ∞-topos.

∞-topoi have some specific features not present in 1-topoi.

My purpose today is to illustrate some of them and to convince you that ∞-topoi are exciting new objects.

All of this is a joint work with Georg Biedermann, Eric Finster and André Joyal.

The problem

My pretext to talk about ∞-topoi is going to be the following problem.

One of the most basic tool of topos theory is left exact localizations.

For ordinary topoi—or 1-topoi—it is a classical theorem that left exact localizations are generated by Grothendieck/Lawvere-Tierney topologies.

I will explain why and give a remedy.

This will led us to see some new creatures of the ∞-world.

Disclaimer

I am going to talk about $(\infty, 1)$-categories in a model independent approach.

This is fine provided only universal constructions are used.

An isomorphism in an $(\infty, 1)$-category is a map $f : A \to B$ such that there exists

- a left inverse $gf \simeq 1_A$, and
- a right inverse $fh \simeq 1_B$.

PLAN

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– I –

∞-Categories

Why $(\infty, 1)$-categories

First, a word on $(\infty, 1)$-CT.

The langage of $(\infty, 1)$-CT is the same as CT (objects, arrows, diagrams, functors, colimits, adjunction, Kan extensions...)

Surprisingly perhaps, higher arrows do not play a fundamental role.

All structural results are the same (completions, SAFT, ...)

In fact, it is interesting to look at $(\infty, 1)$-CT and CT are two different semantics for the same syntax.

What is new is not gonna be found in new notions
but in new behavior of classical notions.

New behavior – ∞-colimits

In ∞-CT, colimits are still defined by the same universal property but are computed a different way.

Essentially the idea is to replace

the logical notation

$$a = b$$

by the topo-logical notation

$$a \;\text{———}\; b.$$

When identifying two points, do not collapse them together but draw a line between them.

This a way to force them to be isomorphic and not equal.

New behavior – ∞-colimits

Consider a pushout

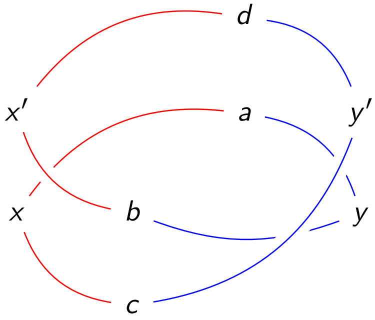
$$\begin{array}{ccc} \begin{Bmatrix} a & b \\ c & d \end{Bmatrix} & \xrightarrow{p_2} & \begin{Bmatrix} y \\ y' \end{Bmatrix} \\ \downarrow{\scriptstyle p_1} & & \\ \{x, x'\} & & \end{array}$$

where p_1 and p_2 are vertical and horizontal projections.

The classical pushout is a singleton.

New behavior – ∞-colimits

Here is the diagram of relations (category of elements)



The ∞-colimit of the diagram is the homotopy type of this circle.

It is intended to remember all the ambiguities of identification.

Doing so will provide a more regular object.

New behavior – ∞-colimits

The colimits of a diagram of sets is no longer a set.

This is because we have embedded *Set* in the ∞-category S of homotopy types (or ∞-groupoids).

$$Set \to S$$

This embedding do not preserve colimits. Only its left adjoint does

$$\pi_0 : S \to Set$$

The classical colimit is obtained as the π_0 of the ∞-colimit.

New behavior – Effectivity of colimits

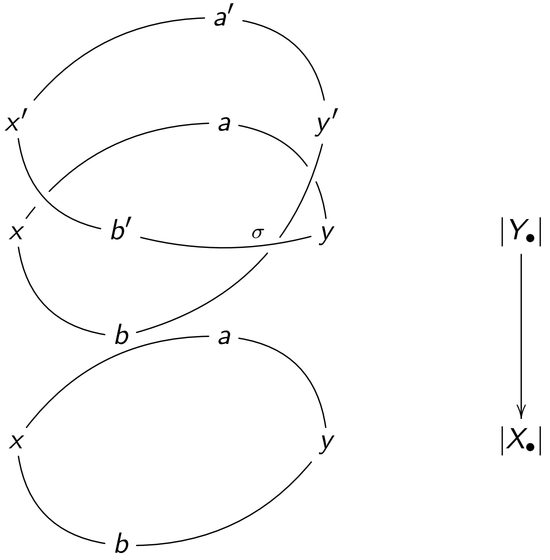
Colimits in ∞-categories has a property not held by classical colimits: effectivity.

Consider the following cartesian morphism of diagrams

$$\begin{array}{ccccccc} \{x, x'\} & \xleftarrow{(id,id)} & \{a, a'\} \coprod \{b, b'\} & \xrightarrow{(id,\sigma)} & \{y, y'\} & & Y_\bullet \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \{x\} & \longleftarrow & \{a, b\} & \longrightarrow & \{y\} & & X_\bullet \end{array}$$

The induce map between the homotopy colimits is a two-fold cover of a circle.

New behavior – Effectivity of colimits



New behavior – Effectivity of colimits

The map between the classical colimit is an isomorphim between two points.

Something has been lost : the fact that the fibers had two elements.

The cartesian nature of the map of diagram say that the fiber of the maps $Y_i \to X_i$ are the same.

Effectivity is the property that the fibers between two colimits are the same as the fibers of the map of the diagram.

This is powerful computational tool.

E.g., when colimits are effective, every group object G admits a classifying object BG.

New behavior – Effectivity of colimits

Effectivity of colimit is the characteristic property of ∞-topoi.

An ∞-category E is an ∞-topos if

1. it is presentable (in particular has small colimits and finite limits) and
2. colimits are universal,
3. colimits are effective.

New behavior – Stability

Let C be a 1-category with finite limits and finite colimits.

If we assume that sums commute with products, then C is a additive category.

If we assume that finite limits commutes with finite colimits, then C collapse to a point.

But there are plenty of $(\infty, 1)$-categories where finite limits commutes with finite colimits!

They are call stable $(\infty, 1)$-categories since the archetype is the ∞-category Sp of spectra (in the sense of algebraic topology).

Anither example is the $(\infty, 1)$-category $C(k)$ of chain complexes.

New behavior – Stability

Stable homotopy theory is very different from unstable homotopy theory.

The category of spectra Sp is very much not a ∞-topos (effectivity fails for sum).

Nonetheless, it is not so far from the world of ∞-topos.

New behavior – Stability

Let B be an ∞-groupoid.

A spectra parametrized by B is a functor $E : B \to Sp$.

B is called the base of the object, it is useful to think of E as a bundle (or local system) of spectra over B.

There is a category PSp of parametrized spectra over arbitrary bases. It is equipped with a fibration over the category of S of ∞-groupoids.

$$base : PSp \to S$$

The fiber over B is the category Sp^B of spectra parametrized by B.

The fiber over 1 is Sp.

New behavior – Stability

The following result has come as a shock for all experts in homotopy theory.

Theorem (Goodwillie theory)

The category PSp of parametrized spectra is an ∞-topos.

Parametrized spectra crossbreed the stable and unstable homotopy theories of spaces into a generalized unstable homotopy theory (i.e. an ∞-topos).

New behavior – Stability

PSp is arguably the main protagonist of ∞-topos theory.

The proof that is it is an ∞-topos is simply the fact, extracted from Goodwillie theory, that PSp is a lex localization of the topos classifying pointed objects

$$S[X^\bullet] = [Fin^\bullet, S] \xrightarrow{\text{lex loc.}} PSp.$$

($Fin^\bullet$ = pointed finite ∞-groupoids)

What kind of pointed object does PSp classifies ?

We shall give an answer later.

– II –

1-Topos & ∞-Topos

Definition

Here is the shortest introduction to ∞-topoi.

Let Set be the category of sets. A topos is a left exact localisation of a presheaf category $[C, Set]$, for C a small category.

Let S be the ∞-category of ∞-groupoids (= homotopy types of spaces). An ∞-topos is a left exact localisation of a presheaf category $[C, S]$, for C a small ∞-category.

An algebraic morphisms of topoi $E \to E'$ is a cocontinuous (cc) and left exact (lex) functor.

$$E \xrightarrow{\text{cc lex}} E'.$$

The category of topoi and geometric morphisms is the opposite of the category of topoi and algebraic morphisms. I am not going to use this category here.

Intuition

A topos can be thought as a generalized category of sets.

For example as a category of sets parametrized continuously by a space (= sheaf).

In particular, there is always an algebraic morphism $Set \to E$ (constant sheaves).

An ∞-topos can be thought as a generalized ∞-category of homotopy types (i.e. a generalized unstable homotopy theory).

For example as a ∞-category of homotopy types parametrized continuously by a space (= stacks).

In particular, there is always an algebraic morphism $S \to E$ (constant stacks).

∞-topoi are more regular that 1-topoi

So far the theory of 1-topoi and ∞-topoi look pretty similar.

Essentially, we have just replaced *Set* by *S*, which is a way to change the computation of colimits (ordinary colimit v. homotopy colimits).

The big difference between them concern the behavior of the slice functor

$$\begin{aligned} E_{/-} : E^{op} &\longrightarrow \widehat{Cat} \\ X &\longmapsto E_{/X} \end{aligned}$$

∞-topoi are more regular that 1-topoi

If E is a ∞-topos, $E_{/-}$ is a continuous functor:

$$E_{/\operatorname{colim} X_i} = \lim E_{/X_i}$$

Up to size issues, $E_{/-}$ is then representable by the object classifier (or universe) U.

If E is a 1-topos, U preserve only covers of the canonical topology (it is a stack). Only the following subfunctor $Sub \subset E_{/-}$ is continuous

$$\begin{aligned} Sub : E^{op} &\longrightarrow \widehat{Cat} \\ X &\longmapsto Sub(X) \end{aligned}$$

Sub is then representable by the subobject classifier Ω.

∞-topoi are more regular that 1-topoi

The condition $E_{/\operatorname{colim} X_i} = \lim E_{/X_i}$ is equivalent to universality and effectivity of colimits.

$$E_{/\operatorname{colim} X_i} \underset{cst_I}{\overset{\operatorname{colim}_I}{\leftrightarrows}} \lim E_{/X_i} = (E^I_{cart})_{/X_\bullet}.$$

1. Colimits are universal if, for all $X : I \to E$, cst_I is fully faithful (colim_I is a localization).
2. Colimits are effective of, for all $X : I \to E$, colim_I is fully faithful.

1-topos & ∞-topos

1-topos	∞-topos
category *Set* of sets	∞-category S of ∞-groupoids
$Pr(C) = [C^{op}, Set]$	$P(C) = [C^{op}, S]$
All 1-topoi are lex loc. of $Pr(C)$	All ∞-topoi are lex loc. of $P(C)$
subobject classifier Ω	object classifier/universe U
Grothendieck topology on C	?
Lawvere-Tierney top. on $Pr(C)$	lex modalities

– III –

Equations within a topos

Two sides

The theory of topos has two sides:

- a geometric side : a topos is a space X
- a algebraic/logical side : a topos is a category E of generalized sets (or generalized ∞-groupoids).

The relation between both sides is given by the idea that E is the category of continuous functions on X with values in the space A\ of sets (or the space of ∞-groupoids).

$$E = C^0(X, A\backslash).$$

Today, I'm gonna focus on the second side.

The algebraic side

From the logical side, a topos is a category where to get semantics for logical theories.

The algebraic point of view on this, is to say that a topos is a category where to get solutions to some equations of the type

a given map $A \to B$ is an isomorphism.

Examples of equations

1. $U \rightarrowtail 1$ an isomorphism
   (= the proposition U is true)

2. $X \to 1$ is surjective $\Leftrightarrow$ $im(X) \rightarrowtail 1$ is an isomorphism
   (= X is non-empty)

3. $X \to X^2$ is an isomorphism
   (= X is a proposition)

Examples of equations

4. The square

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

is cartesian : $A \to B \times_Y X$ is an isomorphism.

5. The square

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle g} \\ B & \longrightarrow & Y \end{array}$$

has a unique diagonal lift : the map

$$\langle f, g \rangle = [B, X] \to [A, X] \times_{[A,Y]} [B, Y]$$

is an isomorphism.

Funny equations

Today, I'm gonna be interested in equations with not enough solutions in *Set* or any topos.

Let $1 \to X$ be a pointed object, then we have maps

1. $X \vee X \to X \times X$
2. $X \to \Omega\Sigma X$
3. $\Sigma\Omega X \to X$

which we can force to be isomorphisms.

In *Set*, and in any topos, the only solution is $X = 1$.

This says that the classifying topos of such an equation is trivial

$$Set[X]//(X \to \Omega\Sigma X) = Set.$$

Funny equations

How about if we replace *Set* with spaces $\mathcal{S}$?

Unfortunately, the situation is the same, the only solution is $X = 1$.

Are there non-trivial solutions in some other ∞-topoi ?

Yes.

Funny equations

The equation $X \vee X \simeq X \times X$ is true in any additive category.

In particular within chain complexes, or spectra.

The equations $X \to \Omega\Sigma X$ and $\Sigma\Omega X \to X$ are also true in chains complexes where Σ and Ω correspond to the shift of chain complexes.

In fact they are true in any stable category, in particular in the category Sp of spectra.

Funny equations

Recall the topos PSp of parametrized spectra.

We have an inclusion

$$Sp \subset PSp.$$

This functor commutes with all limits and contractible colimits. In particular, it preserves all relations

1. $X \vee X \to X \times X$ iso
2. $X \to \Omega\Sigma X$ iso
3. $\Sigma\Omega X \to X$ iso

Any spectra provide a solution to these equations in Sp and hence in PSp.

So the classifying ∞-topoi of these equations are not trivial!

Funny equations

Recall that we started with a pointed object $1 \to X$.

The ∞-topos classifying objects is $S[X] = [Fin, S]$ where Fin is the category of finite ∞-groupoids.

The ∞-topos classifying objects is $S[X^\bullet] = S[X]_{/X} = [Fin^\bullet, S]$ where $Fin^\bullet$ is the category of finite pointed ∞-groupoids.

We proved that there exists a non-trivial lex localisation of $S[X^\bullet]$ generated by any of the equations

1. $X \vee X \to X \times X$ iso
2. $X \to \Omega\Sigma X$ iso
3. $\Sigma\Omega X \to X$ iso

But how to describe an ∞-topoi such as $S[X^\bullet]/\!/(X \simeq \Omega\Sigma X)$?
To what full subcategory of $S[X^\bullet]$ does it corresponds ? What are the "sheaves" for the condition $X \simeq \Omega\Sigma X$?

– IV –

Left exact localizations of ∞-Topos

The problem

Given a topos E and $f : A \to B$ in E,

we have the cc lex localization of E generated by inverting f

$$E \longrightarrow L^{lex}_{cc}(E, f)$$

where the localisation functor is cocontinuous (cc) and left exact (lex).

Because of the presentability assumptions, this functor has a fully faithful right adjoint and the problem is to find a description of its image

$$L^{lex}_{cc}(E, f) = \{X \in E \text{ such that what?}\}.$$

Before to review the answer to this question for 1-topoi, we need to fix some notations.

Pullback hom

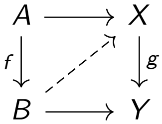
Given two maps $f : A \to B$ and $g : X \to Y$ in a category C,

the pullback hom of f and g is defined as the map

$$\langle f, g \rangle = [B, X] \to [A, X] \times_{[A,Y]} [B, Y].$$

The object $[A, X] \times_{[A,Y]} [B, Y]$ is also the set (or space) of commutative squares with f and g as vertical edges.

And the map $\langle f, g \rangle$ produces the square associated to a diagonal filler



The map $\langle f, g \rangle$ is an isomorphism iff all squares have a unique diagonal filler.

Orthogonality

We define two notions of orthogonality.

1. The external orthogonality

$$f \perp g \qquad \text{if } \langle f, g\rangle \text{ is an iso.}$$

2. The fiberwise orthogonality

$$f \perp\!\!\!\perp g \qquad \text{if, for any base change } f' \to f,\ \langle f', g\rangle \text{ is an iso.}$$

Factorisation systems & modalities

Within a topos, we can always use the small object argument to transform orthogonality conditions into factorisations.

Let S be a set of maps in C

1. The pair $({}^{\perp}(S^{\perp}), S^{\perp})$ is a unique factorisation system
2. The pair $({}^{\perp\!\!\perp}(S^{\perp\!\!\perp}), S^{\perp\!\!\perp})$ is a unique factorisation system stable by base change.

This second type of factorisation system is called a modality.

Examples of modalities

Let consider the topos $\mathcal{S}$.

For $n \geq -1$, let S^n be the n-sphere ($S^{-1} = 0$) and $s^n : S^n \to 1$ be the canonical map.

For a map $f : A \to B$, we have

$$\left\langle s^0, f \right\rangle = \Delta f \qquad \text{and} \qquad \langle s^n, f \rangle = \Delta^{n+1} f.$$

The modality generated by s^0 is $(surj, mono)$.

A map f is a mono iff $\left\langle s^0, f \right\rangle = \Delta f$ is an iso.
A map f is a surjection iff $\left\langle s^{-1}, f \right\rangle = \Delta^0 f = f$ is a surjection.

Examples of modalities

The modality generated by s^1 is $(connected, discrete)$.

A map f is discrete iff $\langle s^1, f \rangle = \Delta^2 f$ is an iso.

A map f is connected iff f is surjective and $\langle s^0, f \rangle = \Delta f$ is surjective.

In general, the modality generated by s^{n+1} is $(n - connected, n - truncated)$.

A map f is n-truncated iff $\langle s^{n+1}, f \rangle = \Delta^{n+2} f$ is an iso (= fiber have no homotopy $> n$).

A map f is n-connected iff $\langle s^k, f \rangle = \Delta^{k+1} f$ are surjective for $k \leq n$ (= fibers have no homotopy $\leq n$)

Examples of modalities

The previous modalities make sense in any ∞-topos E.

A map f in E is *n-truncated* if $\Delta^{n+2} f$ is an iso.

A map f in E is *n-connected* iff $\Delta^k f$ are surjective for $k \leq n+1$.

There are inclusions

$$\begin{array}{llllll} \dots (n+1)\text{-conn.} \subset n\text{-conn.} & \subset \dots \subset 0\text{-conn.} & \subset (-1)\text{-conn.} & = \text{surj.} \\ \dots (n+1)\text{-tr.} \supset n\text{-tr.} & \supset \dots \supset 0\text{-tr.} & \supset (-1)\text{-tr.} & = \text{mono.} \end{array}$$

The factorisation associated to these modalities can be put together into the *Postnikov tower* of a map $f : A \to B$

$$A \to \dots \xrightarrow{n-tr} P_n f \xrightarrow{(n-1)-tr} \dots \xrightarrow{1-tr} P_1 f \xrightarrow{disc} P_{-1} f \xrightarrow{mono} B.$$

Examples of modalities

The class of ∞-connected maps is defined by

$$\infty\text{-connected} = \bigcap n\text{-connected}.$$

A map f is ∞-connected iff all $\Delta^n f$ are surjective.

The only ∞-connected maps in $\mathcal{S}$ are the isomorphisms.

But in $Sp \subset PSp$ any map between spectra is ∞-connected.

The class of ∞-truncated maps is defined by

$$(\infty\text{-connected})^{\perp\!\!\!\perp} = (\infty\text{-connected})^{\perp}$$

There is a (lex) modality (∞-connected, ∞-truncated).

Other examples of modalities

- If $L : E \to E'$ is a lex localization of topoi, then $(L - equiv, L - local)$ is a lex modality.
  All lex modalities are of this kind.

- If a stable category C has a (lex) t-structure, then it extends to a (lex) modality on the topos PC of parametrized objects.

- In internal logic, a modality (L, R) is a reflexive sub-universe

$$U \rightleftarrows R$$

The solution for 1-topoi

Given a 1-topos E and $f : A \to B$ in E, what is the condition in

$$L_{cc}^{lex}(E, f) = \left\{X \in E \text{ such that what?}\right\}$$

A remark first: for lex localisations, inverting a map f is equivalent to invert two monomorphisms

- the image $im(f) : C \rightarrowtail B$ of f (forces f to be surjective)
- and the diagonal $\Delta f : A \rightarrowtail A \times_B A$ of f (forces f to be a mono)

The solution for 1-topoi

Theorem (classical)

For E a 1-topos

$$L_{cc}^{lex}(E, f) = \left\{ X \in E \mid \left(im(f) \coprod \Delta f \right) \perp\!\!\!\perp X \right\}$$

Proof.

For a monomorphism m, the condition $(m \perp\!\!\!\perp -)$ describe the LT-topology generated by m. □

For a mono m, we have simply

$$L_{cc}^{lex}(E, m) = \{ X \in E \mid m \perp\!\!\!\perp X \}$$

The solution for 1-topoi

In a 1-category all maps f are discrete (0-truncated).

This is why the diagonals Δf are always monomorphisms.

And this is why lex localizations are controled by monomorphisms (ie by G/LT topologies).

This is no longer the case in ∞-topoi.

The solution for 1-topoi

It is a fact that the functor

$$base : PSp \to S$$

is a left exact localization of topoi

inverting no monomorphisms.

The class of inverted maps is actually ∞-conn.

There is no way this localization can be studied/controled by a G/LT topology.

We need a new approach.

Lurie's factorization

Lurie distinguishes two types of lex localizations of topoi

- the topological ones that can be generated by monomorphisms
- the cotopological ones that inverts no monomorphisms

Any lex localization $E \to L^{lex}_{cc}(E, W)$ (with W the class of all inverted maps) can be factored into

$$\begin{array}{ccc} E & \xrightarrow{\text{loc.}} & L^{lex}_{cc}(E, W) \\ & {\scriptstyle \text{cotop. loc.}} \searrow \quad \nearrow {\scriptstyle \text{top. loc.}} & \\ & L^{lex}_{cc}(E, W \cap Mono) & \end{array}$$

The theorem

For a map f in a topos E, we introduce the notation

$$f^{\Delta} = \coprod_{n \geq 0} \Delta^n f.$$

f^{Δ} is surjective iff f is ∞-connected.

Theorem (ABFJ)

$$L_{cc}^{lex}(E, f) = \left\{X \in E \mid f^{\Delta} \perp\!\!\!\perp X\right\}$$

$$L_{cc}^{lex}(E, f)^{top} = \left\{X \in E \mid im(f^{\Delta}) \perp\!\!\!\perp X\right\}$$

For a mono m, we have $(m^{\Delta} \perp\!\!\!\perp -) \Leftrightarrow (m \perp\!\!\!\perp -)$ and we recover

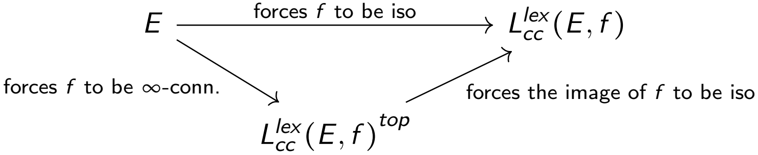
$$L_{cc}^{lex}(E, m) = \{X \in E \mid m \perp\!\!\!\perp X\}.$$

but now E is an ∞-topos.

Corollary

A localization is topological iff it forces some map f to become ∞-connected.

Lurie's factorization then the following



Presentations of topoi

	Site	*Presentation*
Generators	cat. of representables C	cat. of generators G
"Free" object	$Pr(C)$	$S[G] = [G^{lex}, S]$
Relations	topology τ	relation $r : F \to G$
Quotient	$Pr(C)/\!/(\tau) = Sh(C,\tau)$ $= \{X \in Pr(C) \mid m \perp\!\!\!\perp X\}$	$S[G]/\!/(r)$ $= \{X \in S[G] \mid r^{\Delta} \perp\!\!\!\perp X\}$

Presentations of topoi

The difference between the two notions can be understood as follows.

Relations in a site are of the type

$$\operatorname{colim} \textit{representables} = \textit{representable}.$$

Relations in a presentation are of the type

$$\operatorname{colim} \lim \textit{generators} = \operatorname{colim} \lim \textit{generators}.$$

Hence presentations makes it easier to write conditions involving limits, such as $X \simeq \Omega\Sigma X$. In a site, such conditions must be integrated by hand to the construction of C.

Examples of presentations

- free topos on no generator (initial topos) S
- free topos on one generator (object classifier)

$$S[X] = [Fin, S]$$

- free topos libre on a category C (classifying C-diagrams) :

$$S[C] = Pr(C^{lex}, S)$$

- topos classifying pointed objects:

$$S[X^\bullet] = S[X]_{/X} = [Fin^\bullet, S]$$

Examples of presentations

- if **2** is the Sierpiński space, we have

$$Sh(\mathbf{2}) = S[X]//(X \to X \times X)$$

- open quotient

$$E//(U \rightarrowtail 1)$$

- complemented closed quotient: for an object A in E

$$E//(\varnothing \to A) = E//(\varnothing \to im(A))$$

- another way to pointed objects

$$S[X^{\bullet}] = S[Z \to X]//(Z \to 1)$$

- object equal to its free group

$$S[X^{\bullet}]//(X \to \Omega\Sigma X)$$

Examples of presentations

- topos classifying sub-objects :

$$S[X]/\!/(\Delta X)$$

$(\Delta X = X \to X \times X)$

- topos classifying discrete objects (0-truncated) :

$$S[X]/\!/\big(\Delta^2 X\big)$$

$(\Delta^2 X = X \to X^{S^1})$

- topos classifying n-truncated objects :

$$S[X]/\!/\big(\Delta^{n+2} X\big)$$

$(\Delta^{n+1} X = X \to X^{S^{n+1}})$

Examples of presentations

- topos classifying non-empty objects :

$$S[X]/\!/(im(X \to 1)) = [Fin^{\circ}, S]$$

- topos classifying connected objects :

$$S[X]/\!/\Big(im(\Delta X) \coprod im(X \to 1)\Big)$$

- topos classifying pointed connected objects :

$$S[X^{\bullet}]/\!/(im(\Delta X^{\bullet}))$$

This is also the topos classifying groups.

- topos classifying pointed n-connected objects :

$$S[X^{\bullet}]/\!/\Big(\forall 0 \leq k \leq n+1, im(\Delta^{k} X^{\bullet})\Big)$$

This is also the topos classifying E_{n+1}-groups.

– VI –

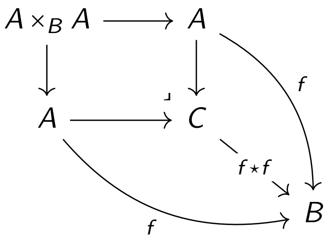
Application to Goodwillie Calculus

Applications

The canonical localization $L_0 : S[X^\bullet] \to S$ sending $X^\bullet$ to 1, is generated by the map $x : 1 \to X^\bullet$

$$L_{cc}^{lex}(S[X^\bullet], x) = \left\{ F \mid x^\Delta \perp\!\!\!\perp F \right\} = S.$$

The join power of a map $f : A \to B$ is the map $C \to B$ defined as the cocartesian gap map



$(1 \to B) \star (1 \to B) = \Sigma\Omega B \to B$

Applications

Theorem (ABFJ)

The Goodwillie localization

$$L_1 : S[X^\bullet] \to PSp$$

is generated by the map $(x^\Delta)^{\star 2}$

$$PSp = \left\{ F \mid (x^\Delta)^{\star 2} \perp\!\!\!\perp F \right\}$$

Concretely, this means that PSp classifies pointed objects $X^\bullet$ satisfying, for all m, n in $\mathbb{N}$,

$$\Omega^m X^\bullet \vee \Omega^n X^\bullet \simeq \Omega^m X^\bullet \times \Omega^n X^\bullet$$

i.e. objects such that the category generated by the $\Omega^n X$ is additive.

Applications

Theorem (ABFJ)

The topological part of the Goodwillie localization

$$L_1 : S[X^\bullet] \to PSp$$

is the topos

$$S[X^\bullet_{>\infty}]$$

classifying ∞*-connected pointed objects.*

This means that PSp classifies in particular ∞-connected pointed objects.

So there are no non-trivial models of PSp in Set, a 1-topos or in S, where 1 is the only ∞-connected object.

Applications

Theorem (ABFJ)

The Goodwillie localization

$$L_n : S[X^\bullet] \to \{\textit{n-excisive functors}\}$$

is generated by the map $(x^\Delta)^{\star(n+1)}$

$$\{\textit{n-excisive functors}\} = \left\{ F \mid (x^\Delta)^{\star(n+1)} \perp\!\!\!\perp F \right\}.$$

Applications

The Goodwillie localizations $L_n = L_0^{\star(n+1)}$ are completely determined by the localization $L_0 : S[X^\bullet] \to S$.

Theorem (ABFJ)

There is a tower $L^{\star(n+1)}$ of localizations associated to any $L : E \to E'$.

This tower is trivial if the localization $L : E \to E'$ is topological.

Applications

In our approach, no cubical diagram are needed anymore to describe the n-excisive objects.

Theorem (ABFJ)

The Weiss tower of localizations of

$$[\textit{Orthogonal category}, S]$$

in his orthogonal calculus is another application of our setting.

– V –

Presentations of topoi

Presentations of topoi

Here is an alternative to the notion of site, best suited for ∞-topoi.

A presentation of a topos is the data of

- a category G of generators, from which we get the free topos $S[G] = [G^{lex}, S]$
- a relation which is simply a morphism $r : F \to G$ dans $S[G]$.

The topos associated to the presentation (G, r) is defined to be

$$S[G]/\!/(r) = L^{lex}_{cc}(S[G], r) = \{X \in S[G] \mid r^{\Delta} \perp\!\!\!\perp X\}.$$

The free topos $S[G]$ classifies G-diagrams. The topos $S[G]/\!/(r)$ classifies G-diagrams satisfying the equation r.

Thanks !