Exponentiable ∞-topoi

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Abstract

The main result of this paper is a characterization of exponentiable ∞-topoi $\mathbf{X}$ as those with a continuous ∞-category of sheaves $\mathcal{S}\mathrm{h}(\mathbf{X})$. Our proof follows and simplifies the original one of [JJ82] by going around the use of Grothendieck topologies and wavy arrows. We use this result as a pretext to develop some aspects of ∞-topos theory. This allows us to provide several interpretations of the continuity condition: in terms of distributivity of limits and colimits; in terms of Leray's original definition of sheaves; and in terms of geometric theories. Finally, we apply our main theorem to show that when $\mathbf{X}$ is exponentiable, its ∞-category of stable sheaves $\mathcal{S}\mathrm{h}(\mathbf{X}, \mathcal{S}\mathrm{p})$ is a dualizable object in the ∞-category of presentable stable ∞-categories.

Contents

1 Introduction

1.1 Exponentiability of ∞-topoi

The purpose of this work is to characterize the exponentiable objects in the ∞-category ∞-topoi and to give a number of interpretations of the extra condition of continuity they satisfy. We shall say that an $\mathcal{I}$nd-cocomplete ∞-category $\mathcal{C}$ is *continuous* if the canonical map $\varepsilon : \mathcal{I}\mathrm{nd}(\mathcal{C}) \to \mathcal{C}$ from its $\mathcal{I}$nd-completion has a left adjoint $\beta : \mathcal{C} \to \mathcal{I}\mathrm{nd}(\mathcal{C})$. The characterization of exponentiable ∞-topoi is then the same as the original result of [JJ82] for 1-topoi.

Theorem 3.2.1. *An ∞-topoi* $\mathbf{X}$ *is exponentiable if and only if its ∞-category of sheaves* $\mathcal{S}\mathrm{h}(\mathbf{X})$ *is continuous.*

The proof of the theorem follows the steps of the original one from [JJ82], but the use of Grothendieck topologies had to be circumvented since they no longer control all left exact localizations of ∞-topoi. We have in fact managed to go around the whole apparatus of wavy arrows of [JJ82] and to provide a simplified proof using a certain clever argument (see Lemma 2.7.10).

Since the first version of the proof, given in the PhD thesis of the second author [Lej16], the result has been independently proven by in Lurie's SAG book [Lur17a]. The new proof provided here, centered on Lemma 2.7.10, simplifies both the one of [Lej16] and some of Lurie's arguments.

The condition of continuity is arguably not very intuitive and we provide three interpretations to make sense of it. The formulation of these interpretations has led us to develop some aspects of the theory of ∞-topos not directly involved in the proof of Theorem 3.2.1.

Interpretation in terms of distributivity If $\mathcal{C}$ is a presentable continuous ∞-category the canonical functor $\mathcal{I}\mathrm{nd}(\mathcal{C}) \to \mathcal{C}$ is in particular continuous, i.e. commute with limits. For posets this condition is equivalent to the existence of a left adjoint, but for ∞-categories, it is weaker. However, this is where the name of the condition is coming from and we have chosen to respect the existing terminology. This condition of "weak" continuity has a nice interpretation in terms of distributivity of *arbitrary limits* over *filtered colimits* for which we refer the reader to [ALR03]. For example, if I is a set and C_i a family of filtered ∞-categories, for any diagram $X_\bullet : \prod_i C_i \to \mathcal{C}$ we have an isomorphism

$$\prod_i \operatorname*{colim}_{c\in C_i} X_c = \operatorname*{colim}_{c_\bullet \in \prod_i C_i} \prod_i X_{c_i}\,.$$

In the case where $\mathcal{C} = \mathcal{S}\mathrm{h}(\mathbf{X})$ is the category of sheaves on an ∞-topoi, this condition has to be compared with a certain characterization of ∞-topoi.

Theorem 2.1.4. *A presentable ∞-category* $\mathcal{E}$ *is an ∞-topos if and only if the canonical functor* $\mathcal{P}(\mathcal{E}) \to \mathcal{E}$ *is left exact.*

The condition that $\mathcal{P}(\mathcal{E}) \to \mathcal{E}$ is left exact says that *finite limits* distribute over *arbitrary colimits* [ALR03, GL12]. For example, for any finite diagram $X_\bullet : I \to \mathcal{C}$ in a presheaf topos, we have an isomorphism (see Proposition 2.1.2):

$$\lim_i \operatorname*{colim}_{c\to X_i} c = \operatorname*{colim}_{c_\bullet \to X_\bullet} \lim_i c_i\,.$$

Such formulas generalize the better known condition of universality of colimits (which is the case of finite products). In the end, these considerations show that an exponentiable ∞-topos can be understood as an ∞-topos where an extra distributivity relation holds in $\mathcal{S}\mathrm{h}(\mathbf{X})$.

Interpretation in terms of Leray sheaves When a presentable ∞-category is continuous, we prove in Proposition 2.5.9 that it can be obtained as a coreflective localisation of a presentable ∞-category of ind-objects:

$$\mathcal{I}\mathrm{nd}(D) \underset{\varepsilon}{\overset{\beta}{\leftrightarrows}} \mathcal{S}\mathrm{h}(\mathbf{X}),$$

where both functors ε and β are cocontinuous. We deduce a description of $\mathcal{S}\mathrm{h}(\mathbf{X})$ as the fixed points of the idempotent comonad $W = \beta\varepsilon$. Moreover, since W is cocontinuous, it is associated to a bimodule $w : D^{op} \times D \to \mathcal{S}$ called the bimodule of *wavy arrows* in [JJ82]. We then have the following description of sheaves on $\mathbf{X}$.

Proposition 4.1.6. *Let $\mathbf{X}$ be an exponentiable ∞-topos. Then the ∞-category $\mathcal{S}\mathrm{h}(\mathbf{X})$ is equivalent to the ∞-category of functors $F : D^{op} \to \mathcal{S}$ which are*

(i) left exact, and

(ii) fixed points of the coend with w: $F(c) \simeq \int^{d\in D^{op}} w(c,d) \otimes F(d)$, for all $a \in D$.

We call such functors *Leray sheaves* since they are very close to Leray's original definition [KS90]. Indeed, on a locally compact Hausdorff space X, Leray defines a sheaf as a contravariant functor from the poset of closed subsets $F : K \mapsto F(K)$ satisfying the condition:

$$F(K) \simeq \operatorname*{colim}_{K \ll K'} F(K'),$$

where $K \ll K'$ means that there exists an open subset U such that $K \subset U \subset K'$. Lurie called $\mathcal{K}$-sheaves this notion and proves in [**?**, Theorem 7.3.4.9] the equivalence with usual sheaves for a locally compact Hausdorff space X. If we define a *wavy arrow* $c \rightsquigarrow d$ as an object of $w(c,d)$, the coend condition can be written as a colimit

$$\int^{d\in D^{op}} w(c,d) \otimes F(d) = \operatorname*{colim}_{c \rightsquigarrow d} F(d)$$

which provide an analog of Leray's condition where D plays the role of an ∞-category of "compact spaces" (more details are given in Remark 4.1.10).

In the theory of locales, a locale X is exponentiable (in locales) if and only if it is locally quasi-compact, and there exists a wavy arrow $U \rightsquigarrow V$ (denoted $U \ll V$ in [Joh82]) between two open subspaces if and only if there exists a quasi-compact subspace K of X such that $U \subset K \subset V$. We believe it should be true that an ∞-topos $\mathbf{X}$ is exponentiable if and only if it has "locally enough compact objects". However, we do not know what could be a definition of these objects that would make such a statement precise. Anyway, another way to understand exponentiable ∞-topoi is as those ∞-topoi such that a description of sheaves à la Leray is possible.

The Leray description of sheaves generalizes to sheaves with values in other ∞-logoi than $\mathcal{S}$ but not in arbitrary ∞-categories $\mathcal{C}$, e.g. the ∞-category $\mathcal{S}\mathrm{p}$ of spectra. We have not been able to construct such a description without further hypothesis on the topos $\mathbf{X}$ and the ∞-category $\mathcal{C}$.

Theorem 4.2.9. *Let $\mathbf{X}$ be an exponentiable topos such that $\beta : \mathcal{S}\mathrm{h}(\mathbf{X}) \to \mathcal{I}\mathrm{nd}(\mathcal{S}\mathrm{h}(\mathbf{X}))$ is left exact and $\mathcal{C}$ be a bicomplete ∞-category where filtered colimits commute with finite limits. Then, the ∞-category $\left[\mathcal{S}\mathrm{h}(\mathbf{X})^{op}, \mathcal{C}\right]$ of sheaves on $\mathbf{X}$ with values in $\mathcal{C}$ admits a description à la Leray.*

In Sections 4.2.1 and 4.2.2 we prove also two variations of this result under weaker hypothesis on $\mathbf{X}$ but stronger hypothesis $\mathcal{C}$.

Interpretation in terms of geometric theories Finally, another understanding of Leray sheaves and the exponentiability result can be given in relation to logical theories. Since geometric logic is not yet developed in the setting of ∞-topoi, we are going to be a bit vague. Essentially, a *geometric theory* is a theory whose axioms involve only finite limits and arbitrary colimits. The models of such a theory do form the points of an

∞-topos and any ∞-topos can be thought as being the classifying ∞-topos of a geometric theory. Classically, sheaves on an ∞-topos are described as diagrams (presheaves) satisfying conditions involving infinite limits, so they are not a priori models of a geometric theory. But, in the case of exponentiable ∞-topoi, the previous description of sheaves à la Leray is precisely a geometric one! Thus, we can also understand exponentiable ∞-topoi has those whose sheaves do form a geometric theory. We make this more precise in Section 4.3.

1.2 Topos theory

As we said before, we have used the proof of Theorem 3.2.1 as a pretext to develop some aspects of the theory of ∞-topoi.

A characterisation of topoi We already mentionned Theorem 2.1.4. The full statement of the theorem is in fact stronger than what we said above. One of the most important property of ∞-topoi is that the left Kan extension of a left exact functor $C \to \mathcal{E}$ (with value in an ∞-topoi $\mathcal{E}$) is still a left exact functor $\mathcal{P}(C) \to \mathcal{E}$ [**?**, Prop. 6.1.5.2]. In Section 2.1, we prove in fact that this property characterizes ∞-topoi.

Theorem 2.1.4. *Let $\mathcal{E}$ be a presentable ∞-category. The following conditions on $\mathcal{E}$ are equivalent:*

(a) $\mathcal{E}$ is left exact localization of an ∞-category $\mathcal{P}(\mathcal{C})$ (i.e. $\mathcal{E}$ is an ∞-topos);

(b) the canonical functor $\mathcal{P}(\mathcal{E}) \to \mathcal{E}$ is left exact;

(c) for any ∞-category $\mathcal{C}$ with finite limits, we have an equivalence $[\mathcal{C}, \mathcal{E}]^{\mathrm{lex}} \simeq [\mathcal{P}(\mathcal{C}), \mathcal{E}]^{\mathrm{lex}}_{\mathrm{cc}}$.

Topoi and logoi In Section 2.2 and after, we push forward the analogy of topos theory with commutative algebra sketched in [**?**, Rem. 6.1.1.3]. We start by introducing the vocabulary, coined by André Joyal, of *topos* and *logos*. A logos is a left exact localisation of presheaves on a small ∞-category, a morphism of logos is a cocontinuous and left exact functor. Topoi and a geometric morphisms are the objects and morphisms in the opposite ∞-category. This construction mimics that of locales from frames and affine schemes from commutative rings. As in these two examples, the double vocabulary is useful to know exactly on which side—algebraic or geometric—one is working. The choice of the two names is a play on the word topo-logy.

Product of topoi Central to the proof of Theorem 3.2.1, we prove in Proposition 2.4.8 that the product of ∞-topoi, i.e. the sums of ∞-logoi, can be computed as the tensor product of the ∞-logoi as cocomplete ∞-categories. The result is similar to what happen in commutative algebra where the product of schemes corresponds to the tensor product of the corresponding rings.

Classes of topoi In Section 2.3, we introduce names for certain useful classes of topoi inspired by the analogy with commutative algebra. We start in Section 2.3 with the introduction of the *free logos* $\mathcal{S}[C]$ on a small ∞-category C. It has the universal property to classify C-diagrams (see Proposition 2.3.2). In analogy with free rings, the corresponding topoi are called *affine* and denoted $\mathbf{A}^C$. The topos corresponding to free logos on one generator is denoted by $\mathbf{A}$. This is the topos classifying objects. It plays a role analogous to the affine line in algebraic geometry. For example, we have $\mathcal{S}\mathrm{h}(\mathbf{X}) = \mathrm{Hom}(\mathbf{X}, \mathbf{A})$ (where $\mathrm{Hom}(\mathbf{X}, \mathbf{A})$ is the $(\infty, 1)$-category of maps in the $(\infty, 2)$-category $\mathcal{T}\mathrm{opos}$). Any logos is a colimit of free logoi and this is a key step in the proof of Theorem 3.2.1 to reduced the proof of the exponentiability of a topos $\mathbf{X}$ to the existence of the sole exponential $\mathbf{A}^{\mathbf{X}}$.

From there, we introduce the *quasi-free* logoi as being presheaves ∞-categories. The corresponding topoi are called *quasi-affine*. Another key step of the proof of Theorem 3.2.1 is a characterization of injective and quasi-injective topoi as retract of affine and quasi-affine ones done in Section 2.6. Finally, in Section 2.7, we introduce the class of *lean* topoi, whose corresponding logoi can be reconstructed from their ∞-category of points. Injective topoi, including quasi-affine and affine topoi provide examples of lean topoi and this plays a role in the proof Lemma 2.7.10 which is the heart of the proof of Theorem 3.2.1. The characterization of lean topoi is left as an open problem (see Remark 2.7.7).

Other In Section 4.3, we introduce a minimal piece of higher geometric logic, just enough to be able to talk about the theory of sheaves on an ∞-topos. Finally, since some of our computations and proofs are done using coends, we have added Appendix A where we define coends and prove all the required formulas.

1.3 Duality

In our last section, we study a consequence of our characterization of exponentiable objects. The ∞-category $\mathcal{T}\mathrm{opos}$ of ∞-topoi is cartesian but not closed, since not all objects are exponentiable. We saw that the cartesian product of topoi correspond to the tensor product of their ∞-categories of sheaves. This produces a symmetric monoidal functor $\mathcal{S}\mathrm{h}(-) : \mathcal{T}\mathrm{opos}^{op} \to \mathcal{CAT}_{\mathrm{cc}}$. In $\mathcal{CAT}_{\mathrm{cc}}$, all objects are "exponentiable", since it is monoidal closed ∞-category, however, not all objects are dualizable. The functor $\mathcal{S}\mathrm{h}(-) : \mathcal{T}\mathrm{opos}^{op} \to \mathcal{CAT}_{\mathrm{cc}}$ does not send exponentiable topoi to dualizable objects, but the main result of Section 5.1 proves that it becomes true after stabilisation.

In order to prove this, we first characterize dualizable cocomplete stable and unstable ∞-categories.

Theorem 5.1.3. *A cocomplete ∞-category is dualizable $\mathcal{CAT}_{\mathrm{cc}}$ if and only if it is a retract of an ∞-category $[C, \mathcal{S}]$ for some small ∞-category C.*

Theorem 5.2.2. *A cocomplete ∞-category is dualizable $\mathcal{S}\mathrm{t}\mathcal{CAT}_{\mathrm{cc}}$ if and only if it is a retract of an ∞-category $[C, \mathcal{S}\mathrm{p}]$ for some small ∞-category C if and only if it is continuous.*

Our last result is then a consequence of the fact that stabilisation preserve continuity (Lemma 5.2.6).

Corollary 5.2.7. *The stabilisation functor*

$$\begin{aligned} \mathcal{S}\mathrm{p} \otimes - : \mathcal{T}\mathrm{opos}^{op} &\to \mathcal{S}\mathrm{t}\mathcal{CAT}_{\mathrm{cc}} \\ \mathbf{X} &\longmapsto \mathcal{S}\mathrm{h}(\mathbf{X}) \otimes \mathcal{S}\mathrm{p} \end{aligned}$$

is symmetric monoidal and sends exponentiable objects to dualizable objects.

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Conventions

We fix $\alpha < \beta < \gamma$, three inaccessible cardinals bigger than the countable cardinal ω. We shall simply say that *small* means α-small, *normal* means β-small and *large* means γ-small. When no adjective is mentioned for a diagram, a limit or a colimit, the reader should assume it is a *small* one. But, when no adjective is mentioned for an ∞-category, the reader should assume it is a *normal* one.

The (normal) ∞-category of small spaces will be denoted $\mathcal{S}$. The large ∞-category of (normal) spaces is $\widehat{\mathcal{S}}$, The (normal) ∞-category of small ∞-categories is $\mathcal{C}\mathrm{at}$; the large one of (normal) ∞-categories will be denoted $\mathcal{CAT}$. Small ∞-categories are always denoted by roman capital $(C, D, \dots)$ and normal ∞-categories are always denoted by calligraphic capitals $(\mathcal{C}, \mathcal{D}, \dots)$.

We shall say that a map $f : A \to B$ in an ∞-category is an *isomorphism* if it admits both a left and a right inverse, i.e. if there exists $g, h : B \to A$ and two 2-cells $fg \simeq id_B$ and $hf \simeq id_A$. When an isomorphism between two objects is canonical, we use the notation $A = B$.

Lurie has chosen to call *continuous* the functors commuting with colimits. We think this choice creates confusion and we have prefered the classical terminology of a *cocontinuous* functor, keeping continuous for functors commuting with limits. Because of his terminology, Lurie had to rename *compacty assembled* the property of continuity of an ∞-category. Although this name is nicely suggestive, we have prefered to keep the original name in order to better underline its meaning in terms of distributivity (see the introduction).

2 ∞-Topoi

In this section we introduce a few notions useful for the proof of the exponentiability theorem, but we have taken this as an opportunity to introduce new material concerning ∞-topoi. In particular, we shall provide in Theorem 2.1.4 a new characterisation of ∞-topoi.

2.1 Characterization of ∞-topoi

2.1.1 Computation of limits in presheaves ∞-categories

In this section, we establish the useful formula of Proposition 2.1.2 to compute limits in a presheaf ∞-category. This will be useful in the proof of Theorem 2.1.4

Recall from [?, Prop. 4.1.1.8] that a functor $u : D \to E$ between small ∞-categories is called *cofinal* if, for any diagram $X_\bullet : E \to \mathcal{C}$ with value in any ∞-category $\mathcal{C}$, u induces an canonical isomorphism

$$\operatorname*{colim}_{e\in E} X_e = \operatorname*{colim}_{d\in D} X_{u(d)} \,.$$

Lemma 2.1.1. *A right adjoint functor $R : C \to D$ is always cofinal.*

Proof. From [?, Prop. 4.1.3.1], a functor $R : C \to D$ is cofinal if and only if, for any object d in D, the comma ∞-category $C_{d/}$, defined by the fibre product

$$\begin{tikzcd} C_{d/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near start] & C \arrow[d, "R"] \\ D_{d/} \arrow[r] & D \,, \end{tikzcd}$$

is contractible. But in the case where R had a left adjoint, $C_{d/}$ has a initial object. ☐

For ∞-category $\mathcal{C}$, we denote by $\mathcal{P}(\mathcal{C})$ its free cocompletion. When C is a small ∞-category, $\mathcal{P}(C)$ can be described as the ∞-category of presheaves $[C^{op}, \mathcal{S}]$. Recall that any object F in $\mathcal{P}(C)$ is the colimit of the diagram of its elements $C_{/F} \to C \to \mathcal{P}(C)$.

Let I be a finite ∞-category and C a small ∞-category with limits of I-diagrams. For any diagram $X_\bullet : I \to \mathcal{P}(C)$, the adjunction

$$C^I \underset{\lim_I}{\overset{\mathrm{cst}_I}{\leftrightarrows}} C$$

induces an adjunction

$$C^I{}_{/X_\bullet} \underset{\lim_I}{\overset{\mathrm{cst}_I}{\leftrightarrows}} C_{/\lim_i X_i} \,.$$

Applying Lemma 2.1.1 we get that, for the canonical diagram $C_{/\lim_i X_i} \to \mathcal{P}(C)$, the following formula

$$\operatorname*{colim}_{c\to X_i \in C_{/\lim_i X_i}} c \simeq \operatorname*{colim}_{c_\bullet \to X_\bullet \in C^I{}_{/X_\bullet}} \lim_i c_i \,.$$

Proposition 2.1.2 (Distributivity à la Day)**.** *Let I be a finite ∞-category and C a small ∞-category with limits of I-diagrams. Then for any diagram $X_\bullet : I \to \mathcal{P}(C) = [C^{op}, \mathcal{S}]$, the following distributivity formula holds*

$$\lim_i \operatorname*{colim}_{c \to X_i \in C_{/X_i}} c = \operatorname*{colim}_{c_\bullet \to X_\bullet \in C^I{}_{/X_\bullet}} \lim_i c_i \,.$$

Proof. From the previous considerations we have

$$\lim_i X_i = \operatorname*{colim}_{c \to X_i \in C_{/\lim_i X_i}} c = \operatorname*{colim}_{c_\bullet \to X_\bullet \in C^I{}_{/X_\bullet}} \lim_i c_i \,.$$

The conclusion follows from

$$\lim_i X_i = \lim_i \operatorname*{colim}_{c \to X_i \in C_{/X_i}} c \,.$$

□

Remark 2.1.3. This formula is to be compared to the usual distributivity formula in a commutative ring

$$\prod_{i \in I} \sum_{j \in J_i} c_{ij} = \sum_{\phi \in \prod_i J_i} \prod_i c_{i\phi(i)} \,.$$

2.1.2 Kan extensions of left exact functors

Let $\mathcal{C}$ be an ∞-category with finite limits and $\mathcal{P}(\mathcal{C})$ its free cocompletion. For $\mathcal{E}$ a cocomplete ∞-category, the left Kan extension along the Yoneda embedding $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ induces an equivalence $[\mathcal{C}, \mathcal{E}] = [\mathcal{P}(\mathcal{C}), \mathcal{E}]_{\mathrm{cc}}$. If $\mathcal{C}$ and $\mathcal{E}$ have finite limits, this equivalence does not restrict in general to an equivalence between left exact functors. It is one of the fundamental properties of ∞-topoi is that the left Kan extension of a left exact functor are still a left exact functor. When C is small, this is proven in [**?**, Prop. 6.1.5.2]. We shall give a new proof of this by proving the stronger result that this property characterizes ∞-topoi. The following theorem is an analog of [GL12, Prop. 2.6].

Theorem 2.1.4. *Let $\mathcal{E}$ be an ∞-category with small colimits and finite limits. The following conditions on $\mathcal{E}$ are equivalent:*

(a) For every ∞-category $\mathcal{C}$ with finite limits, the left Kan extension induces an equivalence

$$[\mathcal{C}, \mathcal{E}]^{\mathrm{lex}} \simeq [\mathcal{P}(\mathcal{C}), \mathcal{E}]^{\mathrm{lex}}_{\mathrm{cc}} \,.$$

(b) The canonical functor $\mathcal{P}(\mathcal{E}) \to \mathcal{E}$ is left exact.

(c) $\mathcal{E}$ is a reflective left exact localization of an ∞-category $\mathcal{P}(\mathcal{D})$ where $\mathcal{D}$ has finite limits.

Moreover, if $\mathcal{E}$ is presentable, the previous conditions are also equivalent to

(d) $\mathcal{E}$ is a reflective left exact localization of an ∞-category $\mathcal{P}(C)$, for a small ∞-category C.

Proof. (a) ⇒ (b) because $\mathcal{P}(\mathcal{E}) \to \mathcal{E}$ is the Kan extension of the identity of $\mathcal{E}$, which is left exact. And (b) ⇒ (c) is true by definition of $\mathcal{P}(\mathcal{E}) \to \mathcal{E}$.

We are left to prove (c) ⇒ (a). Let $f : \mathcal{C} \to \mathcal{E}$ be a left exact functor, and let $\mathcal{D}$ be a sub-∞-category of $\mathcal{E}$ containing the image of $\mathcal{C}$ and such that $\mathcal{E}$ is a left exact localization of $\mathcal{P}(\mathcal{D})$. The composite $g : \mathcal{C} \to \mathcal{E} \to \mathcal{P}(\mathcal{D})$ is left exact, and we get back the original functor f by composing g with the localization $\mathcal{P}(\mathcal{D}) \to \mathcal{E}$. Moreover, since the functor $\mathcal{P}(\mathcal{D}) \to \mathcal{E}$ is cocontinuous, the left Kan extension f' of f along $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ is the composition of the left Kan extension g' of g with $\mathcal{P}(\mathcal{D}) \to \mathcal{E}$.

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "g"] \arrow[rr, bend left, "f"] \arrow[d] & \mathcal{P}(\mathcal{D}) \arrow[r] & \mathcal{E} \\
\mathcal{P}(\mathcal{C}) \arrow[ur, "g'"] \arrow[urr, "f'"'] & &
\end{tikzcd}
$$

Now, since $\mathcal{P}(\mathcal{D}) \to \mathcal{E}$ is left exact by hypothesis, the map f' is left exact as soon as g' is. The conclusion follows from Lemma 2.1.5 (applied one universe up). This finishes to prove (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).

We assume now that $\mathcal{E}$ is presentable. Clearly (d) $\Rightarrow$ (c). We will finish the proof of the equivalence by showing that (a) $\Rightarrow$ (d). Let C be a generating small full sub-category of $\mathcal{E}$ stable by finite limits. By (a), the left Kan extension $L : \mathcal{P}(C) \to \mathcal{E}$ of the inclusion $C \subset \mathcal{E}$ is left exact. Moreover, C being generating, L has a fully faithful right adjoint. This proves (d). □

Lemma 2.1.5. *Let I be a finite ∞-category, C and D be two small ∞-categories with limits of I-diagrams, and $h : C \to D$ be a functor preserving limits of I-diagrams. Then, the functor $\mathcal{P}(h) : \mathcal{P}(C) \to \mathcal{P}(D)$ preserves limits of I-diagrams.*

Proof. We are going to use the calculus of coends recalled in Appendix A, particularly Remark A.1.5. For any small diagram $X : I \to \mathcal{P}(C)$, we have the following isomorphisms:

$$
\begin{aligned}
\lim_i \mathcal{P}(h) X_i &= \operatorname*{colim}_{d_\bullet \to \mathcal{P}(h)(X_\bullet)} \lim_i d_i && \text{by Proposition 2.1.2 in } \mathcal{P}(D) \\
&= \int^{d_\bullet} [d_\bullet, \mathcal{P}(h)(X_\bullet)] \times \lim_i d_i && \\
&= \int^{d_\bullet} \left[d_\bullet, \operatorname*{colim}_{c_\bullet \to X_\bullet} h(c_\bullet) \right] \times \lim_i d_i && \text{by construction of } \mathcal{P}(h) \\
&= \int^{d_\bullet} \operatorname*{colim}_{c_\bullet \to X_\bullet} [d_\bullet, h(c_\bullet)] \times \lim_i d_i && \text{by computation of colimits in } \mathcal{P}(D)^I \\
&= \int^{d_\bullet, c_\bullet} [c_\bullet, X_\bullet] \times [d_\bullet, h(c_\bullet)] \times \lim_i d_i && \\
&= \int^{c_\bullet} [c_\bullet, X_\bullet] \times \lim_i h(c_i) && \text{by Fubini \& Yoneda formulas for coends} \\
&= \operatorname*{colim}_{c_\bullet \to X_\bullet} \lim_i h(c_i) && \\
&= \mathcal{P}(h) (\lim_i X_i) && \text{by Proposition 2.1.2 in } \mathcal{P}(C) \text{ \& properties of } h.
\end{aligned}
$$

□

2.2 Topoi & logoi

This section recall the definition of an ∞-topos and introduce the terminology of ∞-logos after [AJ19]. The notion of ∞-topoi was first defined in [Sim99]. It was then developped in [TV05] and [Rez05], and more recently in [Lur09] which will be our main reference for the theory.

Definition 2.2.1 (Logos)**.** We shall say that an ∞-category $\mathcal{E}$ is an ∞-*logos* if it is presentable and satisfies the equivalent properties of Theorem 2.1.4. A morphism of ∞-logoi is defined as a cocontinuous and left exact functor. For $\mathcal{E}$ and $\mathcal{F}$ two ∞-logoi, the ∞-category of cocontinuous and left exact functors shall be denoted $[\mathcal{E}, \mathcal{F}]^{\mathrm{lex}}_{\mathrm{cc}}$. We denote by $\mathcal{L}\mathrm{ogos}$ the very large ∞-category of ∞-logoi.

Definition 2.2.2 (Topos)**.** The very large ∞-category of ∞-*topoi* is defined by:

$$
\mathcal{T}\mathrm{opos} = \mathcal{L}\mathrm{ogos}^{op} .
$$

Its objects are called ∞-*topoi*. The correspondence sends an ∞-topos $\mathbf{X}$ to its ∞-logos $\mathcal{S}\mathrm{h}(\mathbf{X})$ and a morphism $f : \mathbf{X} \to \mathbf{Y}$ to the "inverse image" $f^* : \mathcal{S}\mathrm{h}(\mathbf{Y}) \to \mathcal{S}\mathrm{h}(\mathbf{X})$.

Remark 2.2.3. Classically, the word ∞-*topos* is used abusively to refer both to a kind of space and an ∞-category of sheaves on that space, which is unfortunate. Introducing different names and notations for these two ∞-categories helps to understand the roles they play. The double vocabulary of ∞-topos and ∞-logos—which is a play of the word topo-logy—has the intended purpose to separate more clearly the geometric and algebraic sides of ∞-topos theory, in analogy with the theory of affine schemes and commutative rings or that of locales and frames.

Remark 2.2.4. Morphisms in $\mathcal{T}$opos are usually called *geometric morphisms* but we shall simply call them morphisms of ∞-topoi. Morphisms in $\mathcal{L}$ogos are sometimes called *algebraic morphisms* but we shall say morphisms of ∞-logoi.

Since we are only considering $(\infty,1)$-categories, there is no confusion on the meaning of $\mathcal{L}\mathrm{ogos}^{op}$. However, it is sometimes useful to consider the $(\infty,1)$-category of morphisms between two ∞-logoi or ∞-topoi. In this case, when $\mathcal{L}$ogos is viewed as an $(\infty,2)$-category, we shall defined the $(\infty,2)$-category of ∞-topoi as $\mathcal{T}\mathrm{opos} = \mathcal{L}\mathrm{ogos}^{1op}$ (inverting only 1-arrows), that is, for two ∞-topoi $\mathbf{X}$ and $\mathbf{Y}$, we define the $(\infty,1)$-category of morphisms of ∞-topoi by $[\mathbf{X},\mathbf{Y}] = [\mathcal{S}\mathrm{h}(\mathbf{Y}),\mathcal{S}\mathrm{h}(\mathbf{X})]^{\mathrm{lex}}_{\mathrm{cc}}$.

2.3 Affine and quasi-affine ∞-topoi

The ∞-category of commutative rings is generated under colimits by free rings $\mathbb{Z}[x_1,\dots,x_n]$, hence the ∞-category affine schemes is generated under limits by the affine spaces $\mathbf{A}^n$. We introduce the analog of affine spaces for ∞-topoi and prove the analoguous property of generation.

Definition 2.3.1 (Free logos)**.** Let D be a small ∞-category. Let D^{lex} be the free category generated by D by finite limits i.e $(D^{\mathrm{lex}})^{op}$ is the smallest subcategory in $\mathcal{P}(D^{op})$ containing D^{op} and closed under finite colimits. We shall call $\mathcal{S}[D] = \mathcal{P}(D^{\mathrm{lex}})$ the *free ∞-logos* generated by D.

The following proposition justifies the name of free ∞-logos.

Proposition 2.3.2 (Universal property of free ∞-logoi)**.** *Let D be a small ∞-category and $\mathcal{E}$ be an ∞-logos. Let $i : D \to \mathcal{S}[D]$ be the inclusion functor. Then the restriction functor i^* induces an equivalence between the ∞-category of cocontinuous left exact functors $\mathcal{S}[D] \to \mathcal{C}$ and the ∞-category of functors $D \to \mathcal{E}$.*

Proof. Using the universal property of D^{lex} and the fact that left Kan extensions of left exact functors with values in an ∞-topos are still left exact, we have natural equivalences

$$[D,\mathcal{E}] \simeq \left[D^{\mathrm{lex}},\mathcal{E}\right]^{\mathrm{lex}} \simeq \left[\mathcal{P}\big(D^{\mathrm{lex}}\big),\mathcal{E}\right]^{\mathrm{lex}}_{\mathrm{cc}}$$

where the last equivalence follows from Theorem 2.1.4. □

Recall that a morphism of ∞-topoi $f : \mathbf{X} \to \mathbf{Y}$ is an *immersion*, or that $\mathbf{X}$ is a *sub-∞-topos* of $\mathbf{Y}$, if the corresponding morphism of ∞-logoi $f^* : \mathcal{S}\mathrm{h}(\mathbf{Y}) \to \mathcal{S}\mathrm{h}(\mathbf{X})$ is a localization.

Proposition 2.3.3. *An ∞-category $\mathcal{E}$ is an ∞-logos if and only if it is a left exact and accessible localisation of a free ∞-logos:*

$$\mathcal{S}[D] \xrightarrow{\ L\ } \mathcal{E}\,.$$

In other words, every ∞-topos is a sub-∞-topos of a affine ∞-topos.

Proof. By definition an ∞-logos $\mathcal{E}$ is a left exact and accessible reflective localisation of a presheaf ∞-category $L : \mathcal{P}(D) \to \mathcal{E}$ with D a small ∞-category. The proposition we want to prove is just a slight variation. Indeed for any small ∞-category D, the Yoneda embedding $D \hookrightarrow \mathcal{P}(D)$ extends to a left exact and cocontinuous functor $T : \mathcal{S}[D] \to \mathcal{P}(D)$. Its right adjoint is the left extension of the inclusion $D \hookrightarrow \mathcal{P}\big(D^{\mathrm{lex}}\big) = \mathcal{S}[D]$, it is accessible and fully faithful and $LT : \mathcal{S}[D] \to \mathcal{E}$ is the desired reflective localisation. □

Definition 2.3.4 (Affine topos)**.** An *affine ∞-topos* is an ∞-topos $\mathbf{X}$ such that $\mathcal{S}\mathrm{h}(\mathbf{X})$ is a free ∞-logos. We denote by $\mathcal{A}\mathrm{ff}$ be the full subcategory of $\mathcal{T}\mathrm{opos}$ spanned by affine ∞-topoi and by $\mathcal{F}\mathrm{ree}$ the subcategory of $\mathcal{L}\mathrm{ogos}$ spanned by free ∞-logoi.

For a small ∞-category D, we denote by $\mathbf{A}^D$ the affine ∞-topos corresponding to the ∞-logos $\mathcal{S}[D]$. When $D = 1$ is the terminal ∞-category, we denote $\mathbf{A}^1$ simply by $\mathbf{A}$ and call it the *line ∞-topos*. The corresponding ∞-logos is $\mathcal{S}[X]$, free on one generator. When $D = 0$ is the initial ∞-category, we denote denote $\mathbf{A}^0$ simply by $\mathbf{1}$ and call it the *terminal ∞-topos*. The corresponding ∞-logos is $\mathcal{S}$, free on no generator.

Free ∞-logos are in particular presheaves ∞-categories, but not all presheaf ∞-categories are free ∞-logoi. Because of the importance presheaves in ∞-topos theory, it is useful to introduce the following vocabulary.

Definition 2.3.5 (Quasi-affine topoi)**.** An ∞-logos shall be called *quasi-free* if its corresponding ∞-logos is a presheaf ∞-category. Dually, the corresponding ∞-topos shall be called *quasi-affine*. The ∞-categories of quasi-free ∞-logoi and quasi-affine ∞-topoi shall be denoted $\mathcal{QF}\mathrm{ree}$ and $\mathcal{QA}\mathrm{ff}$.

Proposition 2.3.6. *The ∞-category $\mathcal{T}\mathrm{opos}$ is generated under pullbacks by affine ∞-topoi.*

Proof. We are going to prove the dual statement that the ∞-category $\mathcal{L}\mathrm{ogos}$ is generated under pushouts by the free ∞-logoi.

For any ∞-logos $\mathcal{E}$, there exists a free ∞-logos $\mathcal{S}[D]$ and a left exact and accessible reflective localisation functor $L : \mathcal{S}[D] \to \mathcal{S}\mathrm{h}(\mathbf{X})$. Let S be the set of morphisms f in $\mathcal{S}[D]$ such that $L(f)$ is an equivalence in $\mathcal{E}$, then S is strongly saturated. Because both $\mathcal{S}[D]$ and $\mathcal{E}$ are accessible ∞-categories, [**?**, Proposition 5.5.4.2] gives a small subset $S_0 \subset S$ such that S_0 generates S as a strongly saturated class.

We can now identify $\mathcal{E}$ as the localization $L(\mathcal{S}[D], S_0)$. We then obtain the following pushout in the ∞-category $\mathcal{L}\mathrm{ogos}$:

$$\begin{tikzcd} \mathcal{S}\left[\coprod_{S_0} \Delta^1\right] \arrow[r] \arrow[d] & \mathcal{S}[D] \arrow[d] \\ \mathcal{S}\left[\coprod_{S_0} \Delta^0\right] \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathcal{E}\,. \end{tikzcd}$$

This ends the proof that any ∞-logos is a poushout of free ∞-logoi: morphisms $f^* : \mathcal{S}\mathrm{h}(\mathbf{X}) \to \mathcal{S}\mathrm{h}(\mathbf{Y})$ are canonically equivalent to morphisms $g^* : \mathcal{S}[D] \to \mathcal{S}\mathrm{h}(\mathbf{Y})$ such that $g^*(s)$ is invertible for any s in S_0. ☐

2.4 Products of ∞-topoi

The purpose of this section is to prove that the product of two ∞-topoi is computed by the tensor product of cocomplete ∞-categories of their corresponding ∞-logoi.

We first recall some facts on tensor products of ∞-categories from [**?**, Ch. 5.5], and [Lur17b, Ch. 1.4 & 4.8]. Let $\mathcal{CAT}_{\mathrm{cc}}$ be the ∞-category of cocomplete ∞-categories and cocontinuous functors. For two objects $\mathcal{C}$ and $\mathcal{D}$ of $\mathcal{CAT}_{\mathrm{cc}}$, we denote by $[\mathcal{C}, \mathcal{D}]_{\mathrm{cc}}$ the ∞-category of cocontinuous functors from $\mathcal{C}$ to $\mathcal{D}$. The tensor product of two cocomplete ∞-categories $\mathcal{C}$ and $\mathcal{D}$ is the object $\mathcal{C} \otimes \mathcal{D}$ such that, for any cocomplete ∞-category $\mathcal{E}$, cocontinuous functors $\mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ are equivalent to functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ cocontinous in each variable, i.e. such that

$$[\mathcal{C} \otimes \mathcal{D}, \mathcal{E}]_{\mathrm{cc}} = [\mathcal{C} \times \mathcal{D}, \mathcal{E}]_{\mathrm{cc,\,cc}}$$

Theorem 2.4.1 ([Lur17b, Corollary 4.8.1.4])**.** *The tensor product of cocomplete ∞-categories exists and provide on ∞-category $\mathcal{CAT}_{\mathrm{cc}}$ the structure of a closed symmetric monoidal structure $\otimes$ with unit the ∞-category $\mathcal{S}$ and internal hom $[-,-]_{\mathrm{cc}}$.*

Theorem 2.4.2 ([Lur17b, Remark 4.8.1.18])**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be two presentable ∞-categories, then $\mathcal{C} \otimes \mathcal{D}$ is presentable. Moreover $[\mathcal{C}, \mathcal{D}]_{\mathrm{cc}}$ is also presentable, so that $\mathcal{P}\mathrm{res}$, the ∞-category of presentable ∞-categories, inherits a closed symmetric monoidal structure from $\mathcal{CAT}_{\mathrm{cc}}$.*

Remark 2.4.3. The ∞-category $\mathcal{C} \otimes \mathcal{D}$ can be constructed as follows. First consider the free cocompletion $\mathcal{P}(\mathcal{C} \times \mathcal{D})$ for α-small colimits, it is a β-small ∞-category. Then, $\mathcal{C}\otimes\mathcal{D}$ is defined as the localization of $\mathcal{P}(\mathcal{C} \times \mathcal{D})$ in $\mathcal{CAT}_{cc}$ generated by the β-small set of maps $\operatorname{colim}(c_i \times d) \to (\operatorname{colim} c_i) \times d$ and $\operatorname{colim}(c \times d_j) \to c \times (\operatorname{colim} d_j)$ for all colimit cones $c_i \to c$ in $\mathcal{C}$ and $d_j \to d$ in $\mathcal{D}$. This construction, proves that any object of $\mathcal{C} \otimes \mathcal{D}$ is a small colimit of pure tensors $c \otimes d$.

Remark 2.4.4. A straightforward computation proves that, for two small ∞-categories C and D, we have $\mathcal{P}(C) \otimes \mathcal{P}(D) = \mathcal{P}(C \times D)$. Another useful formula is $\mathcal{C} \otimes \mathcal{D} \simeq [\mathcal{C}^{op}, \mathcal{D}]^{c}$ when both $\mathcal{C}$ and $\mathcal{D}$ are presentable [Lur17b, Proposition 4.8.1.17].

For an ∞-category $\mathcal{C}$ and a β-small set S of arrows in $\mathcal{C}$, we denote by $\mathcal{C} \to L(\mathcal{C}, S)$ the localization of $\mathcal{C}$ by S.

Proposition 2.4.5 ([Lur17b, Proof of prop. 4.8.1.15])**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be presentable ∞-categories. Let $\mathcal{C} \to L(\mathcal{C}, S)$ and $\mathcal{D} \to L(\mathcal{D}, T)$ be accessible and reflective localisations. Let $f : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ be the canonical map and denote by $S \boxtimes T$ the set of arrows in $\mathcal{C} \otimes \mathcal{D}$ of the form $f(s \times b)$ with (s, b) in $S \times \mathcal{D}$ or $f(a \times t)$ with (a, t) in $\mathcal{C} \times T$. Then the localisation of $\mathcal{C} \otimes \mathcal{D}$ along $S \boxtimes T$ exists, is reflective and accessible. In addition:*

$$L(\mathcal{C} \otimes \mathcal{D}, S \boxtimes T) \simeq L(\mathcal{C}, S) \otimes L(\mathcal{D}, T)\,.$$

The universal property of the tensor product gives the following corollary.

Corollary 2.4.6. *The following square is a pushout in $\mathcal{CAT}_{cc}$,*

$$\begin{tikzcd}
\mathcal{C} \otimes \mathcal{D} \arrow[r] \arrow[d] & L(\mathcal{C}, S) \otimes \mathcal{D} \arrow[d] \\
\mathcal{C} \otimes L(\mathcal{D}, T) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & L(\mathcal{C}, S) \otimes L(\mathcal{D}, T)\,.
\end{tikzcd}$$

For $\mathcal{E}$ an ∞-logos, we shall say that a full sub-∞-category $\mathcal{E}' \subset \mathcal{E}$ is *accessible and left exact reflective* if the inclusion $\mathcal{E}' \to \mathcal{E}$ is accessible and admits a left exact left adjoint functor $\mathcal{E} \to \mathcal{E}'$.

Lemma 2.4.7 ([?, Lemma 6.3.3.4])**.** *Let $\mathcal{E}$ be an ∞-logos and let $\mathcal{E}_0 \subset \mathcal{E}$ and $\mathcal{E}_1 \subset \mathcal{E}$ be two accessible left exact reflective sub-∞-categories. Then the intersection $\mathcal{E}_0 \cap \mathcal{E}_1 \subset \mathcal{E}$ is again an accessible left exact reflective sub-∞-category.*

We now describe the coproducts inside $\mathcal{L}$ogos. The following theorem is stated in [Lur17b, Example 4.8.1.19] but left to the reader. A particular case is also proven [?, Theorem 7.3.3.9] where one of the two ∞-topoi is a topological space.

Proposition 2.4.8. *If $\mathcal{E}$ and $\mathcal{F}$ are two ∞-logoi, then $\mathcal{E} \otimes \mathcal{F}$ is a coproduct of $\mathcal{E}$ and $\mathcal{F}$ in $\mathcal{L}$ogos.*

Proof. Let C and D be two small ∞-categories, we first prove that

$$\mathcal{S}[C] \otimes \mathcal{S}[D] \simeq \mathcal{S}[C \amalg D]\,.$$

We have

$$\mathcal{S}[C] \otimes \mathcal{S}[D] = \mathcal{P}\big(C^{\text{lex}}\big) \otimes \mathcal{P}\big(D^{\text{lex}}\big) \simeq \mathcal{P}\big(C^{\text{lex}} \times D^{\text{lex}}\big)\,.$$

The finite completion functor $C \mapsto C^{\text{lex}}$ goes from $\mathcal{C}$at to $\mathcal{C}\text{at}^{\text{lex}}$, the ∞-category of finitely complete small ∞-categories with left exact functors. This functor is left adjoint to the forgetful functor. Hence it sends coproducts to coproducts. But in $\mathcal{C}\text{at}^{\text{lex}}$ products and coproducts coincide, and because the forgetful functor preserves limits, we have:

$$(C \amalg D)^{\text{lex}} \simeq C^{\text{lex}} \times D^{\text{lex}}\,.$$

Finally, we have

$$\mathcal{P}\big(C^{\text{lex}} \times D^{\text{lex}}\big) \simeq \mathcal{P}\Big((C \amalg D)^{\text{lex}}\Big) = \mathcal{S}[C \amalg D]\,.$$

By the universal property of free ∞-logoi, we deduce that $\mathcal{S}[C \amalg D]$ a coproduct of $\mathcal{S}[C]$ and $\mathcal{S}[D]$.

Let $\mathcal{E}$ and $\mathcal{F}$ be two ∞-logoi, we will now show that $\mathcal{E} \otimes \mathcal{F}$ is an ∞-logos. There exists two small ∞-categories C and D together with two accessible left exact reflective localisation functors $G : \mathcal{S}[C] \to \mathcal{E}$ and $H : \mathcal{S}[D] \to \mathcal{F}$. Then both

$$G^{(D^{\mathrm{lex}})^{op}} : \mathcal{S}[C]^{(D^{\mathrm{lex}})^{op}} \to \mathcal{E}^{(D^{\mathrm{lex}})^{op}} \quad \text{and} \quad H^{(C^{\mathrm{lex}})^{op}} : \mathcal{S}[D]^{(C^{\mathrm{lex}})^{op}} \to \mathcal{F}^{(C^{\mathrm{lex}})^{op}}$$

are left exact and accessible reflective localisation functors. By Corollary 2.4.6, we deduce that $\mathcal{E} \otimes \mathcal{F}$ is equivalent to the intersection $(\mathcal{E} \otimes \mathcal{S}[D]) \cap (\mathcal{S}[C] \otimes \mathcal{F})$ and is thus, by Lemma 2.4.7, an accessible and left exact localisation of $\mathcal{S}[C] \otimes \mathcal{S}[D]$. As we have just shown above, $\mathcal{S}[C] \otimes \mathcal{S}[D]$ is equivalent to a free ∞-logos, so that $\mathcal{E} \otimes \mathcal{F}$ is indeed an ∞-logos.

Let $p^* : \mathcal{S} \to \mathcal{E}$ be a morphism of ∞-logoi (unique up to contractible choice) and let $q^* : \mathcal{S} \to \mathcal{F}$ be another. We claim that the maps $p^* \otimes \mathrm{Id}_{\mathcal{F}} : \mathcal{F} \to \mathcal{E} \otimes \mathcal{F}$ and $\mathrm{Id}_{\mathcal{E}} \otimes q^* : \mathcal{E} \to \mathcal{E} \otimes \mathcal{F}$ exhibit $\mathcal{E} \otimes \mathcal{F}$ as a pushout of $\mathcal{E}$ and $\mathcal{F}$ in $\mathcal{L}$ogos. Notice that both maps are left exact and cocontinuous: the first is the localisation along left exact functors of the left exact cocontinuous map $\mathcal{S}[D] \to \mathcal{S}[C] \otimes \mathcal{S}[D] \simeq \mathcal{S}[C \amalg D]$ induced by the canonical map $D \hookrightarrow C \amalg D$. For a symmetric reason, the second map is also a morphism of ∞-logoi.

For any ∞-logos $\mathcal{G}$, those two maps induce a commutative square

$$\begin{tikzcd}
{[\mathcal{E} \otimes \mathcal{F}, \mathcal{G}]^{\mathrm{lex}}_{\mathrm{cc}}} \arrow[r] \arrow[d, hook] & {[\mathcal{E}, \mathcal{G}]^{\mathrm{lex}}_{\mathrm{cc}} \times [\mathcal{F}, \mathcal{G}]^{\mathrm{lex}}_{\mathrm{cc}}} \arrow[d, hook] \\
{[\mathcal{S}[C \amalg D], \mathcal{G}]^{\mathrm{lex}}_{\mathrm{cc}}} \arrow[r] & {[\mathcal{S}[C], \mathcal{G}]^{\mathrm{lex}}_{\mathrm{cc}} \times [\mathcal{S}[D], \mathcal{G}]^{\mathrm{lex}}_{\mathrm{cc}}}
\end{tikzcd}.$$

In the above diagram, the vertical arrows are inclusions and the bottom one is an equivalence as $\mathcal{S}[C \amalg D]$ is the coproduct $\mathcal{S}[C] \amalg \mathcal{S}[D]$.

In consequence, we only need to show that if (φ, ψ) is in $[\mathcal{S}[C], \mathcal{G}]^{\mathrm{lex}}_{\mathrm{cc}} \times [\mathcal{S}[D], \mathcal{G}]^{\mathrm{lex}}_{\mathrm{cc}}$ factorises through $\mathcal{E}$ and $\mathcal{F}$ then the associated map $\varphi \amalg \psi$ factorises through $\mathcal{E} \otimes \mathcal{F}$. Let S be a set of arrows of $\mathcal{S}[C]$ such that $\mathcal{E} \simeq L(\mathcal{S}[C], S)$ and let be T such that $\mathcal{F} \simeq L(\mathcal{S}[D], T)$. If φ and ψ factorise, it means that φ sends arrows in S to equivalences and ψ sends arrows in T to equivalences. Let $S \boxtimes T$ be the set of arrows of the form $s \otimes x$ for s in S, x in $\mathcal{S}[D]$ or $y \otimes t$ with t in T, y in $\mathcal{S}[C]$, in $\mathcal{S}[C] \otimes \mathcal{S}[D]$. By the proof that $\mathcal{S}[C] \otimes \mathcal{S}[D] \simeq \mathcal{S}[C \amalg D]$ above, we have that the map from $\mathcal{S}[C \amalg D]$ to $\mathcal{G}$ associated to (φ, ψ) is equivalent to the map $\varphi \otimes \psi : \mathcal{S}[C] \otimes \mathcal{S}[D] \to \mathcal{G}$. But $\varphi \otimes \psi$ sends arrows in $S \boxtimes T$ to equivalences so it factorises through $\mathcal{E} \otimes \mathcal{F} \simeq L(\mathcal{S}[C] \otimes \mathcal{S}[D], (S \boxtimes T))$. □

Recall that a morphism of ∞-topoi $\mathbf{Y} \to \mathbf{Z}$ is an *immersion* if the corresponding morphism of ∞-logoi $\mathcal{S}\mathrm{h}(\mathbf{Z}) \to \mathcal{S}\mathrm{h}(\mathbf{Y})$ is a left exact localization.

Proposition 2.4.9. *If $\mathbf{Y} \to \mathbf{Z}$ is a immersion of ∞-topoi, then $\mathbf{X} \times \mathbf{Y} \to \mathbf{X} \times \mathbf{Z}$ is also an immersion.*

Proof. Translated in terms of ∞-logoi, this means that, if $\mathcal{F} \to \mathcal{G}$ is a left exact localization of ∞-logoi, then, for any ∞-logos $\mathcal{E}$, the functor $\mathcal{E} \otimes \mathcal{F} \to \mathcal{E} \otimes \mathcal{G}$ is a left exact localization. This is a direct consequence of Proposition 2.4.5 and Lemma 2.4.7. □

2.5 Continuous ∞-categories

This section generalizes to ∞-categories, the notion of continuous ∞-category of [JJ82]. We shall prove the same structural results in the setting of ∞-categories.

For a ∞-category $\mathcal{C}$, we denote by $\mathcal{I}\mathrm{nd}(\mathcal{C})$ the free completion of $\mathcal{C}$ for ω-filtered small colimits (we shall say simply *filtered colimits* afterwards). The existence of this construction is an application of [**?**, Prop. 5.3.6.2]. If $\mathcal{CAT}_{\mathrm{filt}}$ is the ∞-category of ∞-categories with filtered colimits and functor preserving them, the construction $\mathcal{I}\mathrm{nd}$ is left adjoint to the forgetful functor $\mathcal{CAT}_{\mathrm{filt}} \to \mathcal{CAT}$. The ∞-category $\mathcal{I}\mathrm{nd}(\mathcal{C})$ comes equipped with a fully faithful functor $\alpha : \mathcal{C} \to \mathcal{I}\mathrm{nd}(\mathcal{C})$. An ∞-category $\mathcal{C}$ has filtered colimits if and only if $\alpha : \mathcal{C} \to \mathcal{I}\mathrm{nd}(\mathcal{C})$ has a left adjoint $\varepsilon : \mathcal{I}\mathrm{nd}(\mathcal{C}) \to \mathcal{C}$.

Definition 2.5.1. Let $\mathcal{C}$ be an ∞-category with filtered colimits. We will say that $\mathcal{C}$ is *continuous* if the functor $\varepsilon : \mathcal{I}\mathrm{nd}(\mathcal{C}) \to \mathcal{C}$ has a left adjoint $\beta : \mathcal{C} \to \mathcal{I}\mathrm{nd}(\mathcal{C})$. Altogether, a continuous ∞-category is equipped with three adjoint functors

$$\begin{tikzcd} \mathcal{I}\mathrm{nd}(\mathcal{C}) \arrow[r, "\varepsilon" description] & \mathcal{C} \arrow[l, bend right, hook', "\beta"'] \arrow[l, bend left, hook, "\alpha"] \end{tikzcd}$$

where both α and β are fully faithful. We denote by $\mathcal{CONT}_{\mathrm{filt}}$ the full sub-∞-category of $\mathcal{CAT}_{\mathrm{filt}}$ spanned by continuous ∞-categories.

Remark 2.5.2. Recall from Theorem 2.1.4 that an ∞-logos is such that the canonical functor $\mathcal{P}(\mathcal{E}) \to \mathcal{E}$ is left exact. This condition is a way to say that finite limits distribute over colimits [GL12, **?**]. Similarly, in a continuous ∞-category, the continuity of the functor $\mathcal{I}\mathrm{nd}(\mathcal{C}) \to \mathcal{C}$ says that all limits distribute over filtered colimits. When $\mathcal{C}$ is an ∞-logos, we get for free that finite limits do not only distribute on filtered colimits but commute with them. So the extra continuity assumption on $\mathcal{C}$ can be understood by saying that infinite products distribute on filtered colimits. That is, given for each element i of a set I, a diagram $F_i : C_i \to \mathcal{E}$ from a filtered ∞-category, we have an isomorphism

$$\prod_i \operatorname*{colim}_{c \in C_i} F_i(c) = \operatorname*{colim}_{c \in \prod_i C_i} \prod_i F_i(c_i).$$

In the end, a continuous ∞-logos is an ∞-category with distributivity of finite limits over all colimits and of all limits over filtered colimits.

Proposition 2.5.3. *Any retract by ω-accessible functors of a continuous ∞-category is continuous.*

Proof. Let $r : \mathcal{C} \to \mathcal{D}$ be a retraction by ω-accessible functors and suppose $\mathcal{C}$ is continuous. Let s be an ω-accessible section of r. Because both commute with filtered colimits, we have $\varepsilon_{\mathcal{D}} \circ \mathcal{I}\mathrm{nd}(r) \simeq r \circ \varepsilon_{\mathcal{C}}$ and $s \circ \varepsilon_{\mathcal{D}} \simeq \varepsilon_{\mathcal{C}} \circ \mathcal{I}\mathrm{nd}(s)$. This means we get the following retract diagram:

$$\begin{tikzcd} \mathcal{I}\mathrm{nd}(\mathcal{D}) \arrow[r, "\mathcal{I}\mathrm{nd}(s)"] \arrow[d, "\varepsilon_{\mathcal{D}}"] & \mathcal{I}\mathrm{nd}(\mathcal{C}) \arrow[r, "\mathcal{I}\mathrm{nd}(r)"] \arrow[d, "\varepsilon_{\mathcal{C}}"] & \mathcal{I}\mathrm{nd}(\mathcal{D}) \arrow[d, "\varepsilon_{\mathcal{D}}"] \\ \mathcal{D} \arrow[r, "s"] & \mathcal{C} \arrow[r, "r"] & \mathcal{D}. \end{tikzcd}$$

Let $\theta = \mathcal{I}\mathrm{nd}(r) \circ \beta_{\mathcal{C}} \circ s$. The functor θ is a good candidate to be the left adjoint to $\varepsilon_{\mathcal{D}}$. From the unit $\mathrm{Id} \simeq \varepsilon_{\mathcal{C}} \circ \beta_{\mathcal{C}}$ we get $u : \mathrm{Id} \simeq \varepsilon_{\mathcal{D}} \circ theta$. From the counit $\beta_{\mathcal{C}} \circ \varepsilon_{\mathcal{C}} \to \mathrm{Id}$ we also get a counit transformation $k : \theta \circ \varepsilon_{\mathcal{D}} \to \mathrm{Id}$. The map $k\theta \circ \theta u : \theta \to \theta$ is homotopic to the identity transformation. But $\varepsilon_{\mathcal{D}} k \circ u \varepsilon_{\mathcal{D}} : \varepsilon_{\mathcal{D}} \to \varepsilon_{\mathcal{D}}$ is *not* homotopic to the identity transformation, instead $\varepsilon_{\mathcal{D}} k$ is only idempotent.

The ∞-category $[\mathcal{D}, \mathcal{I}\mathrm{nd}(\mathcal{D})]$ has all filtered colimits; thus idempotents split [**?**, Cor. 4.4.5.16]. Let $\theta \xrightarrow{\tau} \beta \xrightarrow{\sigma} \theta$ be such a splitting. We get a new counit map $k' = k \circ (\sigma \varepsilon_{\mathcal{D}}) : \beta \varepsilon_{\mathcal{C}} \to \mathrm{Id}$ and a new unit map $u' = (\varepsilon_{\mathcal{D}} \tau) \circ u : \mathrm{Id} \simeq \varepsilon_{\mathcal{D}} \beta$. This time $\varepsilon_{\mathcal{D}} k' \circ u' \varepsilon_{\mathcal{D}}$ is homotopic to the unit transformation, as well as $k' \beta \circ \beta u'$. So β is a left adjoint to $\varepsilon_{\mathcal{D}}$, hence $\mathcal{D}$ is a continuous ∞-category. □

Proposition 2.5.4. *The ∞-category $\mathcal{I}\mathrm{nd}(\mathcal{D})$ is continuous for any ∞-category $\mathcal{D}$.*

Proof. Let us denote a generic object of $\mathcal{I}\mathrm{nd}(\mathcal{D})$ as "colim_I"d_i and a generic object of $\mathcal{I}\mathrm{nd}(\mathcal{I}\mathrm{nd}(\mathcal{D}))$ as "$\mathrm{colim}^{(2)}_I$"" $\mathrm{colim}^{(1)}_{J_i}$"$d_{ij}$. Then, the functor $\alpha : \mathcal{I}\mathrm{nd}(\mathcal{D}) \to \mathcal{I}\mathrm{nd}(\mathcal{I}\mathrm{nd}(\mathcal{D}))$ is given by sending "colim_I"d_i to "$\mathrm{colim}^{(1)}_I$"d_i. The functor ε is given by sending "$\mathrm{colim}^{(2)}_I$"" $\mathrm{colim}^{(1)}_{J_i}$"$d_{ij}$ to "$\mathrm{colim}_I \, \mathrm{colim}_{J_i}$"$d_{ij}$.

Then, the left adjoint β is given by sending "colim_I"d_i to "$\mathrm{colim}^{(2)}_I$"d_i (i.e. $\beta = \mathcal{I}\mathrm{nd}(\alpha)$). This is proven by the following canonical isomorphisms

$$\begin{aligned} \left[\text{“}\operatorname*{colim}^{(2)}_{I}\text{”} d_i, \text{“}\operatorname*{colim}^{(2)}_{J}\text{””} \operatorname*{colim}^{(1)}_{K_j}\text{”} a_{jk} \right] &= \lim_I \operatorname*{colim}_J \operatorname*{colim}_{K_j} [d_i, a_{jk}] \\ &= \left[\text{“}\operatorname*{colim}_{I}\text{”} d_i, \text{“}\operatorname*{colim}_{J} \operatorname*{colim}_{K_j}\text{”} a_{jk} \right]. \end{aligned}$$

□

Proposition 2.5.5. *A ∞-category is continuous if and only if it is a retract by ω-accessible functors of an ∞-category $\mathcal{I}\mathrm{nd}(\mathcal{D})$*

Proof. Let $\mathcal{C}$ be a continuous ∞-category. The adjunction $\beta : \mathcal{I}\mathrm{nd}(\mathcal{C}) \rightleftarrows \mathcal{C} : \varepsilon$ is such that $\varepsilon\beta = \mathrm{Id}$. This describes $\mathcal{C}$ as a retract if $\mathcal{I}\mathrm{nd}(\mathcal{C})$ by ω-accessible (even cocontinuous) functors and proves the necessary condition. The sufficient condition is a consequence of Propositions 2.5.4 and 2.5.3. □

We introduce the following full sub-∞-categories of $\mathcal{CONT}_{\mathrm{filt}}$.

- The ∞-category $\mathcal{I}\mathrm{nd}_{\mathrm{filt}}$ spanned by ∞-categories $\mathcal{I}\mathrm{nd}(D)$ for a small ∞-category D.
- The ∞-category $\mathcal{C}\mathrm{ont}_{\mathrm{filt}}$ spanned by retracts of ∞-categories $\mathcal{I}\mathrm{nd}(D)$ for a small ∞-category D. We shall call such continuous ∞-categories *accessible*.
- The ∞-category $\mathcal{P}\mathrm{rsh}_{\mathrm{filt}}$ spanned by ∞-categories $\mathcal{P}(D)$ for a small ∞-category D.
- The ∞-category $\mathcal{PC}\mathrm{ont}_{\mathrm{filt}}$ spanned by retracts of ∞-categories $\mathcal{P}(D)$ for a small ∞-category D. We shall call such continuous ∞-categories *presentable*.

We have the following inclusions

$$\begin{tikzcd}
\mathcal{P}\mathrm{rsh}_{\mathrm{filt}} \arrow[r, hook] \arrow[d, hook] & \mathcal{I}\mathrm{nd}_{\mathrm{filt}} \arrow[r, hook] \arrow[d, hook] & \mathcal{IND}_{\mathrm{filt}} \arrow[d, hook] \\
\mathcal{PC}\mathrm{ont}_{\mathrm{filt}} \arrow[r, hook] & \mathcal{C}\mathrm{ont}_{\mathrm{filt}} \arrow[r, hook] & \mathcal{CONT}_{\mathrm{filt}} .
\end{tikzcd}$$

Proposition 2.5.6. *The ∞-category $\mathcal{C}\mathrm{ont}_{\mathrm{filt}}$, $\mathcal{PC}\mathrm{ont}_{\mathrm{filt}}$ and $\mathcal{IND}_{\mathrm{filt}}$ are respectfully the idempotent completion of $\mathcal{I}\mathrm{nd}_{\mathrm{filt}}$, $\mathcal{P}\mathrm{rsh}_{\mathrm{filt}}$ and $\mathcal{CONT}_{\mathrm{filt}}$.*

Proof. By definition $\mathcal{C}\mathrm{ont}_{\mathrm{filt}}$ is the idempotent completion of $\mathcal{I}\mathrm{nd}_{\mathrm{filt}}$ but *within* $\mathcal{CAT}_{\mathrm{filt}}$. The statement is then a consequence of Lemma 2.5.7 below. The proof is similar for the others. □

Lemma 2.5.7. *Let $j : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor with $\mathcal{D}$ an idempotent complete ∞-category, then the idempotent completion $\mathcal{C}_{\mathrm{idem}}$ is equivalent to the full subcategory of $\mathcal{D}$ spanned by retracts of objects in $\mathcal{C}$.*

Proof. We have a factorisation $\mathcal{C} \to \mathcal{C}_{\mathrm{idem}} \xrightarrow{i} \mathcal{D}$. Let x and y in $\mathcal{C}_{\mathrm{idem}}$ be retracts of objects x' and y' in $\mathcal{C}$. Then $[x,y]$ is a retract in $\mathcal{S}$ of $[x',y']$. Similarly, $[ix,iy]$ is a retract of $[jx',jy']$. By the isomorphism $[x',y'] = [jx',jy']$, the two retractions correspond to the same idempotent and are isomorphic. □

Remark 2.5.8. We do not assume our (normal) ∞-categories to be locally small, i.e. to have small hom spaces. But when D is small, $\mathcal{I}\mathrm{nd}(D)$ and $\mathcal{P}(D)$ are locally small. Because small spaces are stable by retract, a retract of a locally small ∞-category is again locally small. This proves that all objects of $\mathcal{PC}\mathrm{ont}_{\mathrm{filt}}$ are locally small. In particular, they can be filtered by small *full* sub-categories.

For an accessible continuous ∞-category $\mathcal{C}$, we shall say that a full subcategory $i : D \hookrightarrow \mathcal{C}$ *generates* $\mathcal{C}$ *by filtered colimits* if $\mathcal{C}$ is a localisation of $\mathcal{I}\mathrm{nd}(D)$. The inclusion i induces a fully faithful functor $i_! = \mathcal{I}\mathrm{nd}(i) : \mathcal{I}\mathrm{nd}(D) \hookrightarrow \mathcal{I}\mathrm{nd}(\mathcal{C})$ with a right adjoint $i^* : \mathcal{I}\mathrm{nd}(\mathcal{C}) \to \mathcal{I}\mathrm{nd}(D)$. If we assume further that D is generating $\mathcal{C}$ by filtered colimits, we get a fully faithful functor functor $\alpha' : \mathcal{C} \to \mathcal{I}\mathrm{nd}(D)$ and a left adjoint $\varepsilon' : \mathcal{I}\mathrm{nd}(D) \to \mathcal{C}$. Altogether, we have a diagram

$$\begin{tikzcd}
\mathcal{I}\mathrm{nd}(\mathcal{C}) \arrow[r, "\varepsilon" description] \arrow[d, shift right, "i^*"] & \mathcal{C} \arrow[l, shift right=2, "\beta"'] \arrow[l, shift left=2, "\alpha"] \arrow[d, equal] \\
\mathcal{I}\mathrm{nd}(D) \arrow[r, "\varepsilon'" description] \arrow[u, shift right, "i_!"'] & \mathcal{C} \arrow[l, shift right=2, dashed, "\beta'"'] \arrow[l, shift left=2, "\alpha'"]
\end{tikzcd}$$

where $\alpha' = i^*\alpha$ and $\varepsilon' = \varepsilon i_!$. The next result is going to prove that for some well chosen D the functor β factors by some $\beta' : \mathcal{C} \to \mathcal{I}\mathrm{nd}(D)$.

Proposition 2.5.9. *For $\mathcal{C}$ be an accessible continuous ∞-category, there exists $D \subset \mathcal{C}$ a full sub-category generating $\mathcal{C}$ by filtered colimits such that the functor $\beta : \mathcal{C} \to \mathcal{I}\mathrm{nd}(\mathcal{C})$ factors as $i_!\beta'$ where $\beta' : \mathcal{C} \to \mathcal{I}\mathrm{nd}(D)$ is left adjoint to ε'. Moreover, if $\mathcal{C}$ is presentable, D can be chosen stable by finite colimits.*

Proof. Let D_0 be a small full subcategory generating $\mathcal{C}$ by filtered colimits. Then for each object d of D_0, $\beta(d)$ can be written as the colimit in $\mathcal{I}\mathrm{nd}(\mathcal{C})$ of a filtered diagram $x^{(d)}$ in $\mathcal{C}$. Let D be the full subcategory of $\mathcal{C}$ containing D_0 and all the objects of the diagrams $x^{(d)}$ for all d. The ∞-category D generates $\mathcal{C}$ by filtered colimits since it contains D_0. The functor $\beta : \mathcal{C} \to \mathcal{I}\mathrm{nd}(\mathcal{C})$ restricted to D_0 takes its values in $\mathcal{I}\mathrm{nd}(D)$. Using the facts that β is cocontinuous, $\mathcal{C}$ is generated by filtered colimits by D and that $\mathcal{I}\mathrm{nd}(D)$ is stable under filtered colimits in $\mathcal{I}\mathrm{nd}(\mathcal{C})$, we get that $\beta : \mathcal{C} \to \mathcal{I}\mathrm{nd}(\mathcal{C})$ take in fact its values in $\mathcal{I}\mathrm{nd}(D) \subset \mathcal{I}\mathrm{nd}(\mathcal{C})$. We call $\beta' : \mathcal{C} \to \mathcal{I}\mathrm{nd}(D)$ the resulting functor. This proves the existence of D as expected.

If $\mathcal{C}$ is presentable, we can chose D as the full subcategory of $\mathcal{C}$ containing D_0, all the objects of the diagrams $x^{(d)}$ for all d, and stable by finite colimits. This proves the latest assertion.

We prove now that β' is left adjoint to ε'. By construction of β', we have $\beta = i_!\beta$. Then, the adjunction is a consequence of the equivalences between

$$\begin{array}{lll} \text{maps in } \mathcal{I}\mathrm{nd}(D) & \beta'(x) \to y & \\ \text{maps in } \mathcal{I}\mathrm{nd}(\mathcal{C}) & \beta(x) = i_!\beta'(x) \to i_!y & (i_! \text{ fully faithful}) \\ \text{and maps in } \mathcal{C} & x \to \varepsilon i_!(y) = \varepsilon'(y). & \end{array}$$

□

Definition 2.5.10. For an accessible continuous ∞-category $\mathcal{C}$ we shall call *standard presentation* the data of a small full subcategory $D \subset \mathcal{C}$ as in Proposition 2.5.9. Moreover if $\mathcal{C}$ is continuous and presentable, we shall always assume that the ∞-category D of a standard presentation is stable by finite colimits.

Given a standard presentation, we have a triple adjunction

$$\begin{tikzcd} \mathcal{I}\mathrm{nd}(D) \arrow[r, "\varepsilon" description] & \mathcal{C} \arrow[l, shift right=2, "\beta"'] \arrow[l, shift left=2, "\alpha"] \end{tikzcd}$$

where both α and β are fully faithful. The interest of such a presentation for a presentable continuous ∞-category, is that both functors ε and β are cocontinuous, so $\mathcal{C}$ is in fact a retract by cocontinuous functors of an ∞-category $\mathcal{I}\mathrm{nd}(D)$. We shall use this remark in Section 4.

2.6 Injective and quasi-injective ∞-topoi

In this section we introduce a number of important classes of ∞-topoi which have the nice characteristic to be completely determined by their ∞-category of points. A diagram organizing the ∞-categories presented here is given at the end.

For two ∞-topoi $\mathbf{X}$ and $\mathbf{Y}$, we denote by $\mathrm{Map}(\mathbf{X}, \mathbf{Y})$ the *space* of morphisms between them, i.e. the internal groupoid (or core) of $[\mathcal{S}\mathrm{h}(\mathbf{Y}), \mathcal{S}\mathrm{h}(\mathbf{X})]^{\mathrm{lex}}_{\mathrm{cc}}$.

Definition 2.6.1. An ∞-topos $\mathbf{X}$ is *injective* if for every sub-∞-topos $\mathbf{Y} \to \mathbf{Z}$, the composition morphism $\mathrm{Map}(\mathbf{Z}, \mathbf{X}) \to \mathrm{Map}(\mathbf{Y}, \mathbf{X})$ has a section. Dually, the ∞-logos corresponding to an injective ∞-topos will be called *projective*. In other terms, this says that there always exists a lift for a diagram of ∞-topoi

$$\begin{tikzcd} \mathbf{Y} \arrow[r] \arrow[d, hook] & \mathbf{X} \\ \mathbf{Z}. \arrow[ur, dashed] & \end{tikzcd}$$

Remark 2.6.2. This notion of injective ∞-topos corresponds to the notion of weakly injective ∞-topos defined in [Joh02, C.4.3.1]. The equivalences with the other definitions of injectivity can be proven the same way as in 1-topos theory.

Proposition 2.6.3. *All affine ∞-topoi are injective. Furthermore, an ∞-topos is injective if and only if it is a retract in* $\mathcal{T}\mathrm{opos}$ *of an affine ∞-topos.*

Proof. We start by proving that affine ∞-topoi are injective. Let $\mathcal{F} = \mathcal{S}\mathrm{h}(\mathbf{Y})$ and $\mathcal{G} = \mathcal{S}\mathrm{h}(\mathbf{Z})$ be two ∞-logoi and $f : \mathbf{Y} \to \mathbf{Z}$ be an immersion of ∞-topoi. For an ∞-category $\mathcal{C}$, we denote by $\mathcal{C}^{(\mathrm{core})}$ its internal ∞-groupoid. Thanks to the universal property of affine ∞-topoi, we have the following equivalences $\mathrm{Hom}\big(\mathbf{Y}, \mathbf{A}^D\big) \simeq \mathrm{core}([D, \mathcal{F}])$ and $\mathrm{Hom}\big(\mathbf{Z}, \mathbf{A}^D\big) \simeq \mathrm{core}([D, \mathcal{G}])$. Then the reflective localisation f^* gives the desired reflective localisation $(f^*)^D$.

Let $\mathbf{X}$ be an injective ∞-topos, then by definition, there exists an immersion $\mathbf{X} \to \mathbf{A}^D$ with D a small ∞-category. Because $\mathbf{X}$ is injective, this morphism must have a retraction. Reciprocally, we need only to prove that a retract of an injective ∞-topos is still injective. Let $r : \mathbf{X} \to \mathbf{X}'$ be a retraction in $\mathcal{T}\mathrm{opos}$ with $\mathbf{X}$ injective and $s : \mathbf{X}' \to \mathbf{X}$ a section. Let $i : \mathbf{Y} \hookrightarrow \mathbf{Z}$ be an immersion and $f : \mathbf{Y} \to \mathbf{X}'$ be any map. Then $sf : \mathbf{Y} \to \mathbf{X}$ can be extended in $g : \mathbf{Z} \to \mathbf{X}$ because $\mathbf{X}$ is injective. Then $rg : \mathbf{Z} \to \mathbf{X}$ extends f. □

We shall see in Corollary 3.1.3 that all quasi-injective topoi are exponentiable. In the meantime, we can prove this.

Proposition 2.6.4. *Let $\mathbf{I}$ be an injective ∞-topos and $\mathbf{X}$ an exponentiable ∞-topos, then $\mathbf{I}^{\mathbf{X}}$ is an injective ∞-topos.*

Proof. By adjunction the problem of finding a lifting

$$\begin{tikzcd} \mathbf{Y} \arrow[r] \arrow[d, hook] & \mathbf{I}^{\mathbf{X}} \\ \mathbf{Z} \arrow[ur, dashed] & \end{tikzcd}$$

is equivalent to find a lifting

$$\begin{tikzcd} \mathbf{X} \times \mathbf{Y} \arrow[r] \arrow[d] & \mathbf{I} \\ \mathbf{X} \times \mathbf{Z} \arrow[ur, dashed] & \end{tikzcd}$$

Since $\mathbf{I}$ is injective, the lemma will be proven if we show that the functor $\mathbf{X} \times -$ preserve inclusions of ∞-topoi. This is the content of Proposition 2.4.9. □

By replacing the free ∞-logoi by presheaves ∞-categories in Proposition 2.6.3, we obtain the notion of quasi-injective ∞-topoi.

Definition 2.6.5. An ∞-topos $\mathbf{X}$ shall be called *quasi-injective* if it is a retract of a quasi-affine ∞-topos in $\mathcal{T}\mathrm{opos}$. Dually, the ∞-logos corresponding to a quasi-injective ∞-topos will be called *quasi-projective*. We denote by $\mathcal{Q}\mathcal{I}\mathrm{nj}$ and $\mathcal{Q}\mathcal{P}\mathrm{roj}$ the ∞-categories of quasi-injective ∞-topoi and quasi-projective ∞-logoi.

Proposition 2.6.6. *The ∞-category* $\mathcal{I}\mathrm{nj}$ *and* $\mathcal{Q}\mathcal{I}\mathrm{nj}$ *are respectfully the idempotent completion of* $\mathcal{A}\mathrm{ff}$ *and* $\mathcal{Q}\mathcal{A}\mathrm{ff}$.

Proof. Proposition 2.6.3 says that $\mathcal{I}\mathrm{nj}$ is the idempotent completion of $\mathcal{A}\mathrm{ff}$ but within $\mathcal{T}\mathrm{opos}$. Similarly, $\mathcal{Q}\mathcal{I}\mathrm{nj}$ is by definition the idempotent completion of $\mathcal{Q}\mathcal{A}\mathrm{ff}$ within $\mathcal{T}\mathrm{opos}$. The ∞-category $\mathcal{T}\mathrm{opos}$ is idempotent complete by [?, Prop. 6.3.2.3]. The statement is then a consequence of Lemma 2.5.7. □

2.7 Lean ∞-topoi

We prove in this section that quasi-injective ∞-topoi have the particular property of being completely characterised by their ∞-categories of points. We call *lean* the ∞-topoi with such a property.

Definition 2.7.1 (Points and models)**.** Let $\mathbf{1} = \mathbf{A}^0$ be the terminal ∞-topos, a *point* of an ∞-topos $\mathbf{X}$ is defined as a morphism of ∞-topoi $\mathbf{1} \to \mathbf{X}$. Following the convention of Remark 2.2.4, the ∞-category of points of an ∞-topos $\mathbf{X}$ is defined to be

$$\mathcal{P}\mathrm{t}(\mathbf{X}) := [\mathbf{1}, \mathbf{X}] = [\mathcal{S}\mathrm{h}(\mathbf{X}), \mathcal{S}]^{\mathrm{lex}}_{\mathrm{cc}} .$$

Dually, given an ∞-logos $\mathcal{E}$, we shall say that a functor in $[\mathcal{E}, \mathcal{S}]^{\mathrm{lex}}_{\mathrm{cc}}$ is a *model* of the ∞-logos $\mathcal{E}$. We shall denote the ∞-category of models of $\mathcal{E}$ by $\mathcal{M}\mathrm{od}(\mathcal{E})$. We have $\mathcal{M}\mathrm{od}(\mathcal{S}\mathrm{h}(\mathbf{X})) = \mathcal{P}\mathrm{t}(\mathbf{X})$.

For an ∞-logos $\mathcal{E}$, the inclusion $\mathcal{M}\mathrm{od}(\mathcal{E}) \subset [\mathcal{E}, \mathcal{S}]$ induces a canonical evaluation functor

$$\mathrm{ev} : \mathcal{E} \times \mathcal{M}\mathrm{od}(\mathcal{E}) \to \mathcal{S}$$

sending a sheaf and a point to the *stalk* of the sheaf at this point. This functor defines what we shall call the *stalk functor*

$$\mathrm{Stalk} : \mathcal{E} \to [\mathcal{M}\mathrm{od}(\mathcal{E}), \mathcal{S}]$$

sending a sheaf to the diagram of its stalks.

Because filtered colimits commute with finite limits in $\mathcal{S}$, the ∞-category of models of an ∞-logos is always complete for filtered colimits. Moreover, any ∞-logoi morphism $\mathcal{E} \to \mathcal{F}$ induces a functor $\mathcal{M}\mathrm{od}(\mathcal{F}) \to \mathcal{M}\mathrm{od}(\mathcal{E})$ which commutes to filtered colimits. In consequence, the stalk functor $\mathrm{Stalk} : \mathcal{E} \to [\mathcal{M}\mathrm{od}(\mathcal{E}), \mathcal{S}]$ takes its values in the full subcategory $[\mathcal{M}\mathrm{od}(\mathcal{E}), \mathcal{S}]_{\mathrm{filt}}$ of functors preserving filtered colimits. We shall in fact define the *stalk functor* as being the functor

$$\mathrm{Stalk} : \mathcal{E} \to [\mathcal{M}\mathrm{od}(\mathcal{E}), \mathcal{S}]_{\mathrm{filt}} .$$

This functor is not faithful in general, and this leads to some definitions.

Definition 2.7.2 (Lean ∞-topos)**.**

(i) A ∞-topos $\mathbf{X}$ is said to *have enough points* and, dually, an ∞-logos $\mathcal{E}$ is said to *have enough models*, if the stalk functor is conservative.

(ii) A ∞-topos $\mathbf{X}$ and its corresponding ∞-logos $\mathcal{E}$ are said to be *lean*, if the stalk functor induces an equivalence $\mathcal{E} \simeq [\mathcal{M}\mathrm{od}(\mathcal{E}), \mathcal{S}]_{\mathrm{filt}}$.

Let $\mathcal{C}$ be a ∞-category and $\mathcal{C}'$ its idempotent completion. The universal property of this completion is given by the equivalence of ∞-categories $[\mathcal{C}, \mathcal{D}] \simeq [\mathcal{C}', \mathcal{D}]$ where $\mathcal{D}$ is any idempotent complete ∞-category [**?**, Prop. 5.1.4.9]. In particular, if $\alpha : f \to g$ be a natural transformation between two functors $f, g : \mathcal{C} \to \mathcal{D}$, it extends naturally to a natural transformation $\alpha' : f' \to g'$ between functors $\mathcal{C}' \to \mathcal{D}$.

Lemma 2.7.3. *The map α is an isomorphism if and only if its extension α' is.*

Proof. The equivalence of ∞-categories $[\mathcal{C}, \mathcal{D}] = [\mathcal{C}', \mathcal{D}]$ sends α to α'. □

Lemma 2.7.4. *A retract of a lean ∞-topos is lean.*

Proof. The stalk functor $\mathrm{Stalk} : \mathcal{E} \to [\mathcal{M}\mathrm{od}(\mathcal{E}), \mathcal{S}]_{\mathrm{filt}}$ is a natural transformation of functor of $\mathcal{E}$. Restricted to lean ∞-logos, Stalk is an isomorphism. The statement is then a consequence of Lemma 2.7.3. □

Proposition 2.7.5. *Quasi-injective ∞-topoi are lean.*

Proof. Because of Lemma 2.7.4, it is enough to prove that quasi-affine ∞-topoi are lean. Let $\mathcal{E} = [C, \mathcal{S}]$ be a quasi-free ∞-logos, we have $\mathcal{M}\mathrm{od}(\mathcal{E}) = \mathcal{I}\mathrm{nd}(C)$. The result follows from the isomorphisms

$$[\mathcal{M}\mathrm{od}(\mathcal{E}), \mathcal{S}]_{\mathrm{filt}} = [\mathcal{I}\mathrm{nd}(C), \mathcal{S}] = [C, \mathcal{S}] = \mathcal{E}.$$

□

Remark 2.7.6. The notions of topological space and that of ∞-topos with enough points can be compared by the fact that the *set* of points of a space is enhanced into a ∞-*category* of points for the ∞-topos. From this point of view, lean ∞-topoi play a role analog to that of discrete topological spaces. More precisely since topological spaces can have in fact a *poset* of points, lean ∞-topoi are analog of Alexandroff spaces. However, in opposition to discrete spaces or Alexandroff spaces, lean ∞-topoi are quite frequent in ∞-topoi theory. Proposition 2.7.5 produces a large class of example of lean ∞-topoi which include in particular affine ∞-topoi, presheaf ∞-topoi and classifying ∞-topoi of algebraic theories.

Remark 2.7.7. We do not know any example of a lean ∞-topos which is not quasi-injective. Are there any?

Recall that $\mathcal{C}\mathrm{ont}_{\mathrm{filt}}$ and $\mathcal{P}\mathcal{C}\mathrm{ont}_{\mathrm{filt}}$ are the ∞-categories of accessible and presentable continuous ∞-categories with ω-accessible functors.

Corollary 2.7.8. *The functors*

$$[-,\mathcal{S}]_{\mathrm{filt}} : (\mathcal{CAT}_{\mathrm{filt}})^{op} \to \mathcal{CAT} \qquad \textit{and} \qquad \mathcal{M}\mathrm{od} : \mathcal{L}\mathrm{ogos} \to \mathcal{CAT}^{op}_{\mathrm{filt}}$$

provide equivalences of ∞-categories

$$\mathcal{Q}\mathcal{I}\mathrm{nj} \simeq \mathcal{C}\mathrm{ont}_{\mathrm{filt}} \qquad \textit{and} \qquad \mathcal{I}\mathrm{nj} \simeq \mathcal{P}\mathcal{C}\mathrm{ont}_{\mathrm{filt}}\,.$$

Proof. Up to an idempotent completion (Proposition 2.5.6 and Proposition 2.6.6) , it is sufficient to prove the equivalences

$$\mathcal{Q}\mathcal{A}\mathrm{ff} \simeq \mathcal{I}\mathrm{nd}_{\mathrm{filt}} \qquad \text{and} \qquad \mathcal{A}\mathrm{ff} \simeq \mathcal{P}\mathrm{rsh}_{\mathrm{filt}}\,.$$

We showed in Proposition 2.7.5 that $[-,\mathcal{S}]_{\mathrm{filt}}$ takes values in $\mathcal{Q}\mathcal{A}\mathrm{ff}$ when $\mathcal{C} = \mathcal{I}\mathrm{nd}(C)$. But we need to prove that an ω-accessible functor $f : \mathcal{I}\mathrm{nd}(C) \to \mathcal{I}\mathrm{nd}(D)$ is send to an ∞-logoi morphism $g : \mathcal{P}(D^{op}) \to \mathcal{P}(C^{op})$. This is a consequence of the sequence of equivalences between

ω-accessible functors	$\mathcal{I}\mathrm{nd}(C) \to \mathcal{I}\mathrm{nd}(D)$	
functors	$C \to \mathcal{I}\mathrm{nd}(D) = [D^{op},\mathcal{S}]^{\mathrm{flat}}$	
functors	$C \times D^{op} \to \mathcal{S}$	flat in the second variable
flat functors	$D^{op} \to [C,\mathcal{S}] = \mathcal{P}(C^{op})$	
cc lex functors	$\mathcal{P}(D^{op}) \to \mathcal{P}(C^{op})$.	

The fact that $\mathcal{P}\mathrm{t}$ provide an inverse is a straightforward computation. The proof of $\mathcal{A}\mathrm{ff} \simeq \mathcal{P}\mathrm{res}_{\mathrm{filt}}$ is similar. ☐

Example 2.7.9. Corollary 2.7.8 give a recipe to construct many lean ∞-topoi/logoi, it is sufficient to apply $[-,\mathcal{S}]_{\mathrm{filt}}$ to any ∞-category $\mathcal{C} = \mathcal{I}\mathrm{nd}(D)$. Here is a few examples.

(a) For $\mathcal{S} = \mathcal{I}\mathrm{nd}(\mathcal{F}\mathrm{in})$, we get the ∞-logos $\mathcal{S}[X] = [\mathcal{F}\mathrm{in},\mathcal{S}]$ classifying objects.

(b) For $\mathcal{S}^\bullet = \mathcal{I}\mathrm{nd}(\mathcal{F}\mathrm{in}^\bullet)$, we get the ∞-logos $[\mathcal{F}\mathrm{in}^\bullet,\mathcal{S}]$ classifying pointed objects.

(c) For $\mathcal{S}_{>n} = \mathcal{I}\mathrm{nd}(\mathcal{F}\mathrm{in}_{>n})$, the ∞-category of n-connected spaces, we get the ∞-logos $[\mathcal{F}\mathrm{in}_{>n},\mathcal{S}]$ classifying n-connected objects.

(d) For $\mathcal{S}\mathrm{p} = \mathcal{I}\mathrm{nd}(\mathcal{F}\mathrm{in}\mathcal{S}\mathrm{p})$, we get the ∞-logos $[\mathcal{F}\mathrm{in}\mathcal{S}\mathrm{p},\mathcal{S}]$ classifying spectra.

As a consequence of Corollary 2.7.8, we have, for any accessible continuous ∞-category $\mathcal{C}$, the reconstruction formula

$$\mathcal{C} = [[\mathcal{C},\mathcal{S}]_{\mathrm{filt}}\,,\mathcal{S}]^{\mathrm{lex}}_{\mathrm{cc}}\,.$$

The following lemma is going to generalize this to an equivalence

$$\mathcal{C} \otimes \mathcal{E} = [[\mathcal{C},\mathcal{S}]_{\mathrm{filt}}\,,\mathcal{E}]^{\mathrm{lex}}_{\mathrm{cc}}\,. \tag{main}$$

where $\mathcal{C}$ continuous and presentable and $\mathcal{E}$ is any ∞-logos. It will be the key lemma in the proof of the characterization of exponentiable ∞-topoi.

Before this, we need a comment on the naturality of the formula (main) with respect to $\mathcal{C}$. The left hand side is natural for cocontinuous functors, but the right hand side is natural for the more general ω-accessible functors. So both can be viewed as natural in cocontinuous functors.

Lemma 2.7.10. *Let $\mathcal{C}$ be a continuous and cocomplete ∞-category and $\mathcal{E}$ an ∞-logos, then there is a canonical isomorphism*

$$\mathcal{C} \otimes \mathcal{E} = [[\mathcal{C}, \mathcal{S}]_{\mathrm{filt}}, \mathcal{E}]^{\mathrm{lex}}_{\mathrm{cc}}.$$

Moreover, this isomorphism is natural in $\mathcal{C}$ for cocontinuous functors.

Proof. From Corollary 2.7.8 we know that the right hand part makes sense and is natural in $\mathcal{C}$. By Lemmas 2.5.7 and 2.7.3, it is sufficient to prove it for $\mathcal{C} = \mathcal{I}\mathrm{nd}(C)$ where C is a finitely cocomplete ∞-category. In this case, we have $[\mathcal{I}\mathrm{nd}(C), \mathcal{S}]_{\mathrm{filt}} = [C, \mathcal{S}] = \mathcal{P}(C^{op})$. Then the conclusion follows by the canonical equivalences

$$\mathcal{I}\mathrm{nd}(C) \otimes \mathcal{E} = \left[\mathcal{I}\mathrm{nd}(C)^{op}, \mathcal{E}\right]^{\mathrm{c}} = [C^{op}, \mathcal{E}]^{\mathrm{lex}} = [\mathcal{P}(C^{op}), \mathcal{E}]^{\mathrm{lex}}_{\mathrm{cc}}.$$

where the last equivalence follows from Theorem 2.1.4.

The naturality with respect to cocontinuous functors, it is clear for $\mathcal{C} = \mathcal{I}\mathrm{nd}(C)$. Then, to extend it to all presentable continuous ∞-categories we need to use Proposition 2.5.9 stating that such an ∞-category can be written as a retract of some $\mathcal{I}\mathrm{nd}(C)$ by cocontinuous functors. ☐

Remark 2.7.11. Let $\mathbf{X}$ be the topos such that $\mathcal{E} = \mathcal{S}\mathrm{h}(\mathbf{X})$ and let $\mathbf{Y}$ be the topos such that $\mathcal{S}\mathrm{h}(\mathbf{Y}) = [\mathcal{C}, \mathcal{S}]_{\mathrm{filt}}$, we have $\mathcal{C} = \mathcal{P}\mathrm{t}(\mathbf{Y})$. Recall that $\mathcal{C} \otimes \mathcal{E} = [\mathcal{E}^{op}, \mathcal{C}]^{\mathrm{c}} = [\mathcal{E}^{op}, \mathcal{P}\mathrm{t}(\mathbf{Y})]^{\mathrm{c}}$. Lemma 2.7.10 can be understood by saying that a morphism of topos $\mathbf{X} \to \mathbf{Y}$) is the same thing as a sheaf on $\mathbf{X}$ with values in $\mathcal{P}\mathrm{t}(\mathbf{Y})$. $\mathbf{Y}$ is a particular lean topos. Is the same property true for arbitrary lean topoi ?

It is convenient to introduce specific notations for quasi-affine and quasi-injective ∞-topoi.

Notation 2.7.12. For C a small ∞-category, the ∞-topos corresponding to the quasi-affine ∞-logos $\mathcal{P}(C^{op}) = [C, \mathcal{S}]$ will be denoted by $\mathbf{B}C$. The ∞-category of points of such an ∞-topos is $\mathcal{I}\mathrm{nd}(C)$. Similarly, for $\mathcal{C}$ a continuous ∞-category, the quasi-injective ∞-topos corresponding to the ∞-logos $[\mathcal{C}, \mathcal{S}]_{\mathrm{filt}}$ will be denoted by $\mathbf{B}\mathcal{C}$. The ∞-category of points of such an ∞-topos is of course $\mathcal{C}$. The use of Roman or Calligraphic fonts should help to decide whether $\mathbf{B}-$ is applied to a small or a continuous ∞-category.

We finish this section by recapitulating in a diagram the different kind of ∞-topoi that we have defined so far. We have the following subcategories of lean ∞-topoi:

$$\begin{tikzcd} \mathcal{A}\mathrm{ff} \arrow[r, hook] \arrow[d, hook, "\text{id. comp.}"'] & \mathcal{Q}\mathcal{A}\mathrm{ff} \arrow[d, hook, "\text{id. comp.}"] \\ \mathcal{I}\mathrm{nj} \arrow[r, hook] & \mathcal{Q}\mathcal{I}\mathrm{nj} \end{tikzcd}$$

where the bottom row is the idempotent completion of the top one. These ∞-categories are dual to the following subcategories of $\mathcal{L}\mathrm{ogos}$:

$$\begin{tikzcd} \mathcal{F}\mathrm{ree} \arrow[r, hook] \arrow[d, hook, "\text{id. comp.}"'] & \mathcal{Q}\mathcal{F}\mathrm{ree} \arrow[d, hook, "\text{id. comp.}"] \\ \mathcal{P}\mathrm{roj} \arrow[r, hook] & \mathcal{Q}\mathcal{P}\mathrm{roj} \end{tikzcd}$$

where $\mathcal{P}\mathrm{roj}$ (resp. $\mathcal{Q}\mathcal{P}\mathrm{roj}$) is the ∞-category of projective (resp. quasi-projective) ∞-logoi (see Definition 2.6.5). Passing to the ∞-categories of points/models, the previous ∞-categories are equivalent to subcategories of $\mathcal{CAT}_{\mathrm{filt}}$

$$\begin{tikzcd} \mathcal{P}\mathrm{rsh}_{\mathrm{filt}} \arrow[r, hook] \arrow[d, hook, "\text{id. comp.}"'] & \mathcal{I}\mathrm{nd}_{\mathrm{filt}} \arrow[d, hook, "\text{id. comp.}"] \\ \mathcal{P}\mathcal{C}\mathrm{ont}_{\mathrm{filt}} \arrow[r, hook] & \mathcal{C}\mathrm{ont}_{\mathrm{filt}}. \end{tikzcd}$$

3 Exponentiable ∞-topoi

In this section we prove our main result establishing that exponentiable ∞-topoi are precisely those whose ∞-logos is continuous. This result is an ∞-version of the theorem of Johnstone and Joyal [JJ82, Theorem 4.10].

Definition 3.0.1. Let $\mathbf{X}$ be an ∞-topos, we will say that $\mathbf{X}$ is *exponentiable* if the functor $\mathbf{Y} \mapsto \mathbf{Y} \times \mathbf{X}$ has a right adjoint. For an ∞-topos $\mathbf{Y}$ we will say that the particular exponential $\mathbf{Y}^{\mathbf{X}}$ exists if there exists an ∞-topos $\mathbf{Y}^{\mathbf{X}}$ and a morphism of ∞-topoi $\mathbf{X} \times \mathbf{Y}^{\mathbf{X}} \to \mathbf{Y}$ such that the induced map $\mathrm{Map}(\mathbf{Z}, \mathbf{Y}^{\mathbf{X}}) \to \mathrm{Map}(\mathbf{Z} \times \mathbf{X}, \mathbf{Y})$ is an isomorphism in $\mathcal{S}$ for every $\mathbf{Z}$ in $\mathcal{T}$opos.

Remark 3.0.2. By proposition 5.2.2.12 in HTT [?], an ∞-topos $\mathbf{X}$ is exponentiable if and only if for any $\mathbf{Y}$ in $\mathcal{T}$opos, the particular exponential $\mathbf{Y}^{\mathbf{X}}$ exists.

3.1 Tensor and cotensor of ∞-topoi by ∞-categories

The ∞-category of ∞-logoi is proven complete and cocomplete in [?, Prop. 6.3.2.3 & Cor. 6.3.4.7]. The following statement is a way to say that the $(\infty, 2)$-category of ∞-logoi is 2-complete and 2-cocomplete.

Theorem 3.1.1. *The natural enrichment of the ∞-category of ∞-logoi over ∞-categories is tensored and cotensored over small ∞-categories.*

(a) If C is a small ∞-category, the cotensor of an ∞-logos $\mathcal{E}$ by C is given by the diagram ∞-category $\mathcal{E}^C = [C, \mathcal{E}]$, which is also $\mathcal{P}(C^{op}) \otimes \mathcal{E}$ (viewed as a sum of ∞-logoi).

(b) For a free ∞-logos $\mathcal{S}[D]$, the tensor $C \odot \mathcal{S}[D]$ is given by the free ∞-logos $\mathcal{S}[C \times D]$.

Proof. Let us consider the following functor:

$$\begin{aligned} \mathcal{L}\mathrm{ogos}^{op} \times \mathcal{L}\mathrm{ogos} \times \mathcal{C}\mathrm{at} &\to \mathcal{CAT} \\ (\mathcal{F}, \mathcal{E}, C) &\longmapsto \left[C, [\mathcal{F}, \mathcal{E}]^{\mathrm{lex}}_{\mathrm{cc}}\right]. \end{aligned}$$

By definition, this functor is representable (in $\mathcal{CAT}$) in the third variable by the ∞-category $\mathrm{Hom}(\mathcal{F}, \mathcal{E})$. The ∞-category $\mathcal{L}$ogos will be cotensored (tensored) over $\mathcal{C}$at if the functor is representable in the first (resp. second) variable.

(a) The existence of the cotensor is a consequence of the canonical equivalence of ∞-categories $[\mathcal{F}, \mathcal{E}]^C = \left[\mathcal{F}, \mathcal{E}^C\right]$ and the fact that limits and colimits are computed termwise in $\mathcal{E}^C = [C, \mathcal{E}]$. The fact that $\mathcal{E}^C = \mathcal{P}(C^{op}) \otimes \mathcal{E}$ is the sum of ∞-logos is Proposition 2.4.8.

(b) For two small ∞-categories C and D, the isomorphism $C \odot \mathcal{S}[D] = \mathcal{S}[C \times D]$ results from the following equivalences between

cc lex functors	$\mathcal{S}[D] \to \mathcal{E}^C$,
functors	$D \to \mathcal{E}^C$,
functors	$C \times D \to \mathcal{E}$,
and cc lex functors	$\mathcal{S}[C \times D] \to \mathcal{E}$.

To prove that tensors exist in general, we use the fact that any ∞-logos $\mathcal{E}$ is a colimit of a diagram of free ∞-logoi and the fact that if $C \odot -$ exists, being left adjoint to the cotensor $(-)^C$, it has to commute with all colimits. If $\mathcal{E} = \mathrm{colim}_i \, \mathcal{S}[D_i]$, we can then define $C \odot \mathcal{E} := \mathrm{colim}_i \, \mathcal{S}[C \times D_i]$. ∎

Remark 3.1.2. As $(\infty, 2)$-categories, we explained in Remark 2.2.4 our choice of the convention $\mathcal{T}\mathrm{opos} = \mathcal{L}\mathrm{ogos}^{1op}$. The choice not to oppose the 2-cells provide the following simple relation of compatibility: for any ∞-topos $\mathbf{X}$ with corresponding ∞-logos $\mathcal{E}$, the tensor $C \odot \mathbf{X}$ corresponds to the cotensor $\mathcal{E}^C$ and the cotensor $\mathbf{X}^C$ with the tensor $C \odot \mathcal{E}$.

By duality, we get the following result in $\mathcal{T}\mathrm{opos}$. Recall $\mathbf{B}C$ from Notation 2.7.12.

Corollary 3.1.3. *The ∞-category $\mathcal{T}\mathrm{opos}$ is tensored and cotensored over $\mathcal{C}\mathrm{at}$.*

(a) The tensor of an ∞-topos $\mathbf{X}$ by an ∞-category C is $\mathbf{B}C \times \mathbf{X}$.

(b) The cotensor of $\mathbf{X}$ by C is the exponential $\mathbf{X}^{\mathbf{B}C}$. In particular, quasi-affine ∞-topoi are exponentiable.

Proof. (a) Let $\mathbf{X}$ be an ∞-topos with corresponding ∞-logos $\mathcal{E}$. The tensor $C \odot \mathbf{X}$ correspond to the cotensor $\mathcal{E}^C = \mathcal{P}(C^{op}) \otimes \mathcal{E}$. Geometrically the sum of ∞-logoi $\mathcal{P}(C^{op}) \otimes \mathcal{E}$ corresponds to the product of ∞-topoi $\mathbf{B}C \times \mathbf{X}$.

(b) By computation of the tensor as $\mathbf{B}C \times \mathbf{X}$, it is clear that the cotensor is $\mathbf{X}^C = \mathbf{X}^{\mathbf{B}C}$. The fact that this exists for all $\mathbf{X}$ proves that $\mathbf{B}C$ is exponentiable. □

Corollary 3.1.4. *Let $\mathbf{A}$ be the line ∞-topos and C be an ∞-category, we have canonical isomorphisms between the following objects:*

(a) the cotensor of $\mathbf{A}$ by C,

(b) the exponential $\mathbf{A}^{\mathbf{B}C}$, and

(c) the affine ∞-topos $\mathbf{A}^C$.

Proof. By Corollary 3.1.3 we need only to prove (a) $\Leftrightarrow$ (c). Then, by Theorem 3.1.1, the ∞-logos of the cotensor of $\mathbf{A}$ by C is $C \odot \mathcal{S}[X] = \mathcal{S}[C]$, which is the ∞-logos corresponding to $\mathbf{A}^C$. □

3.2 Exponentiability theorem

We now prove the main theorem of this work.

Theorem 3.2.1 (Exponentiable ∞-topoi)**.** *Let $\mathbf{X}$ be an ∞-topos with corresponding ∞-logos $\mathcal{E}$, then $\mathbf{X}$ is exponentiable if and only if $\mathcal{E}$ is a continuous ∞-category.*

The proof of Theorem 3.2.1 will be a direct consequence of the following two propositions.

Proposition 3.2.2. *Let $\mathbf{X}$ be an ∞-topos, the following assertions are equivalent:*

(a) the ∞-topos $\mathbf{X}$ is exponentiable;

(b) the exponential $(\mathbf{A}^C)^{\mathbf{X}}$ exists for every affine ∞-topos $\mathbf{A}^C$;

(c) the exponential $\mathbf{A}^{\mathbf{X}}$ exists.

Proposition 3.2.3. *Let $\mathbf{X}$ be an ∞-topos with corresponding ∞-logos $\mathcal{E}$, then the exponential $\mathbf{A}^{\mathbf{X}}$ exists if and only if $\mathcal{E}$ is a continuous ∞-category.*

The rest of the section will be dedicated to the proofs of these propositions.

Proof of Proposition 3.2.2. Clearly, we have (a) $\Rightarrow$ (b) $\Rightarrow$ (c). We will finished the proof by proving (c) $\Rightarrow$ (b) and (b) $\Rightarrow$ (a).

(c) $\Rightarrow$ (b). Let C be an ∞-category, recall from Corollary 3.1.4 that for any ∞-topos $\mathbf{Y}$, the cotensor $\mathbf{Y}^C$ is the exponential $\mathbf{Y}^{\mathbf{B}C}$. The compatibility of iterated exponential gives canonical isomorphisms

$$(\mathbf{Y}^{\mathbf{X}})^C = (\mathbf{Y}^{\mathbf{X}})^{\mathbf{B}C} = (\mathbf{Y}^{\mathbf{B}C})^{\mathbf{X}} = (\mathbf{Y}^C)^{\mathbf{X}}.$$

Applied to $\mathbf{Y} = \mathbf{A}$, this gives $(\mathbf{A}^{\mathbf{X}})^C = (\mathbf{A}^C)^{\mathbf{X}}$. This proves the implication since cotensors always exist by Corollary 3.1.3.

(b) $\Rightarrow$ (a). Let $\mathbf{Y}$ be any ∞-topos, we have to prove that $\mathbf{Y}^{\mathbf{X}}$ exists. By Proposition 2.3.6, any ∞-topos is a pullback of a diagram $\mathbf{A}^{C_i}$ of affine ∞-topoi. The exponential $\mathbf{Y}^{\mathbf{X}}$, if it exists, has to be the limit of the diagram of $(\mathbf{A}^{C_i})^{\mathbf{X}}$. The existence is then a consequence of the existence of limits in the ∞-category of ∞-topoi [?, Cor. 6.3.4.7]. □

Lemma 3.2.4. *If* $\mathbf{X}$ *is an exponentiable* ∞*-topos, then the* ∞*-category* $\mathcal{S}\mathrm{h}(\mathbf{X})$ *is continuous.*

Proof. By Proposition 2.6.3 the ∞-topos $\mathbf{A}$ is injective and so is $\mathbf{A}^{\mathbf{X}}$ by Proposition 2.6.4. The result follows from Corollary 2.7.8 and the equivalence of ∞-categories $\mathrm{pt}(\mathbf{A}^{\mathbf{X}}) \simeq \mathcal{S}\mathrm{h}(\mathbf{X})$. □

We arrive now at the proof of Proposition 3.2.3. The proof uses the clever trick of Lemma 2.7.10. A more concrete construction associated to a standard presentation will be given in Section 4.1.

Proof of Proposition 3.2.3. The necessary condition is a consequence of Proposition 3.2.2 and Lemma 3.2.4. Reciprocally, if $\mathcal{E}$ is continuous and cocomplete, Corollary 2.7.8 and Notation 2.7.12 say that it is always the ∞-category of points of quasi-affine ∞-topos $\mathbf{B}\mathcal{E}$. We need to prove that the corresponding ∞-topos satisfies the universal property of $\mathbf{A}^{\mathbf{X}}$. This is a consequence of the sequence of equivalences between

$$
\begin{array}{lll}
\text{topoi morphisms} & \mathbf{X}\times\mathbf{Y}\to\mathbf{A} & \\
\text{logoi morphisms} & \mathcal{S}[X]\to\mathcal{E}\otimes\mathcal{F} & \\
\text{objects of} & \mathcal{E}\otimes\mathcal{F} & \\
\text{cc lex functors} & [\mathcal{E},\mathcal{S}]_{\mathrm{filt}}\to\mathcal{F} & \text{by Lemma 2.7.10} \\
\text{topoi morphisms} & \mathbf{Y}\to\mathbf{A}^{\mathbf{X}}. &
\end{array}
$$

In other terms, $\mathbf{A}^{\mathbf{X}}$ is indeed the ∞-topos corresponding to the ∞-logos $[\mathcal{E},\mathcal{S}]_{\mathrm{filt}}$. □

Remark 3.2.5. This proof simplifies a bit in case the ∞-logos $\mathcal{E}$ has the stronger property to be a dualizable cocomplete ∞-category, that is a retract by cocontinuous functors of an ∞-category of presheaves over a small ∞-category (see Theorem 5.1.3). Let $\mathcal{E}^\vee = [\mathcal{E},\mathcal{S}]_{\mathrm{cc}}$ be the dual of $\mathcal{E}$. Recall that if $\mathcal{D}$ is a dualizable ∞-category, we have $(\mathcal{D}^\vee)^\vee = \mathcal{D}$ and, for any other cocomplete ∞-category $\mathcal{F}$, $\mathcal{D}^\vee\otimes\mathcal{F} = [\mathcal{D},\mathcal{F}]_{\mathrm{cc}}$. Then we have the following sequence of equivalences between

$$
\begin{array}{ll}
\text{logoi morphisms} & \mathcal{S}[X]\to\mathcal{E}\otimes\mathcal{F} \\
\text{objects of} & \mathcal{E}\otimes\mathcal{F} = (\mathcal{E}^\vee)^\vee\otimes\mathcal{F} = [\mathcal{E}^\vee,\mathcal{F}]_{\mathrm{cc}} \\
\text{cc functors} & \mathcal{E}^\vee\to\mathcal{F} \\
\text{cc lex functors} & \mathcal{S}\mathrm{ym}(\mathcal{E}^\vee)\to\mathcal{F}
\end{array}
$$

where $\mathcal{S}\mathrm{ym}(-)$ is the *symmetric* ∞*-logos* functor, defined as the left adjoint to the forgetful functor $\mathcal{L}\mathrm{ogos}\to\mathcal{P}\mathrm{res}_{\mathrm{cc}}$. (The construction of $\mathcal{S}\mathrm{ym}(-)$ will be the matter of another work [?], generalizing to ∞-topoi the results of [BC95].) This proves that $\mathbf{A}^{\mathbf{X}}$ is the ∞-topos corresponding to the symmetric ∞-logos $\mathcal{S}\mathrm{ym}(\mathcal{E}^\vee)$. This gives also a formula to compute the symmetric ∞-logos of a dualizable ∞-category $\mathcal{C}$

$$\mathcal{S}\mathrm{ym}(\mathcal{C}) = [\mathcal{C}^\vee,\mathcal{S}]_{\mathrm{filt}}\,.$$

This formula is to be compared with the case of commutative rings where, if a k-algebra A (corresponding to a scheme X) is dualizable as a k-module (i.e. retracts of finitetely generated free modules), then the algebra of functions on $\mathbf{A}^X$ is the symmetric algebra $\mathrm{Sym}(A^\vee)$. In commutative rings, this condition of dualizability exhaust the rings whose corresponding scheme can be exponentiated (see [NW17]), but in ∞-topoi, we have more exponentiable objects. Essentially, this difference can be understood by the fact that infinite sums do not exists in a ring, but they do in an ∞-logoi. If rings are augmented to rings of formal power series, then more exponentiable objects exist.

3.3 Examples of exponentiable ∞-topoi

Since any ∞-category $\mathcal{I}\mathrm{nd}(C)$ is continuous, we have a first class of examples of exponentiable topos.

Proposition 3.3.1. *If* $\mathbf{X}$ *be an* ∞*-topos such that* $\mathcal{S}\mathrm{h}(\mathbf{X}) = \mathcal{I}\mathrm{nd}(C)$ *for some small* ∞*-category* C*, then* $\mathbf{X}$ *is exponentiable.*

This includes the case of quasi-affine ∞-topoi (presheaves ∞-logoi), which we knew from Corollary 3.1.3.(b). In particular, any topos $\mathbf{X}$ such that $\mathcal{S}\mathrm{h}(\mathbf{X}) = \mathcal{S}_{/K}$ is exponentiable.

In general, an ∞-category $\mathcal{I}\mathrm{nd}(C)$ which is not an ∞-category of presheaves is not a topos, but there are some examples. This is the case of the ∞-logos $\mathcal{PS}\mathrm{p}$ of parametrized spectra [**?**, Hoy18], for which $\mathcal{PS}\mathrm{p} = \mathcal{I}\mathrm{nd}(\mathcal{F}\mathrm{in}\mathcal{PS}\mathrm{p})$ where $\mathcal{F}\mathrm{in}\mathcal{PS}\mathrm{p}$ is the ∞-category of finite parametrized spectra (bundles of finite spectra on finite spaces).

Corollary 3.3.2. *The topos corresponding to the ∞-logos $\mathcal{PS}\mathrm{p}$ is exponentiable.*

The construction of $\mathcal{PS}\mathrm{p}$ has been generalized in [Hoy18]. If $\mathcal{C}$ is any stable cocomplete ∞-category, then the ∞-category $\mathcal{PC}$ of objects of $\mathcal{C}$ parametrized by objects of $\mathcal{S}$ is an ∞-logos. Moreover, if $\mathcal{C} = \mathcal{I}\mathrm{nd}(C)$ for some small ∞-category C, we have $\mathcal{PC} = \mathcal{I}\mathrm{nd}(D)$ where D is the ∞-category of objects of C parametrized by finite spaces. Thus the ∞-topos corresponding to $\mathcal{PC}$ is exponentiable.

Another source of example of ∞-logoi of ind-objects is given by coherent spaces. Let X be a locally compact locale [Joh82, Ch. VII.4] and $\mathcal{O}(X)$ the corresponding frame. Then $\mathcal{O}(X)$ is continuous and a retract of $\mathcal{I}\mathrm{nd}(\mathcal{O}(X))$. As opposed to what happen with ∞-topoi, the poset $\mathcal{I}\mathrm{nd}(\mathcal{O}(X))$ is again a frame and $\varepsilon : \mathcal{I}\mathrm{nd}(\mathcal{O}(X)) \to \mathcal{O}(X)$ is a morphism of frames. Moreover, X is quasi-separated and quasi-compact if and only if $\beta : \mathcal{O}(X) \to \mathcal{I}\mathrm{nd}(\mathcal{O}(X))$ is also a morphism of frames. In the case where X is only quasi-separated, then β does not preserves the terminal object, but $\mathcal{O}(X)$ is still a retract in frames of $\mathcal{I}\mathrm{nd}(\mathcal{O}(X))_{/\beta(1)} = \mathcal{I}\mathrm{nd}\big(\mathcal{O}(X)_{/\beta(1)}\big)$, where $\mathcal{O}(X)_{/\beta(1)} = \mathcal{O}_c(X)$ is the sub-poset of $\mathcal{O}(X)$ generated by relatively compact open subspaces. Let Y be the coherent locale corresponding to $\mathcal{I}\mathrm{nd}\big(\mathcal{O}(X)_{/\beta(1)}\big)$. Then X is a retract of Y in the category of locales, and therefore in the ∞-category of ∞-topoi.

For a coherent locale Y such that $\mathcal{O}(Y) = \mathcal{I}\mathrm{nd}(C)$, we have always $\mathcal{S}\mathrm{h}(Y) = \mathcal{I}\mathrm{nd}(D)$ where D is the smallest full sub-∞-category of $\mathcal{S}\mathrm{h}(Y)$ containing C and stable by finite colimits. We deduce that the topos of Y is exponentiable and so is any retract.

Corollary 3.3.3. *Let X be a locally quasi-compact and quasi-separated topological space (in particular a coherent space), then its associated ∞-topos is an exponentiable ∞-topos.*

Remark 3.3.4. Corollary 3.3.3 implies that the ∞-topoi associated to locally quasi-compact and Hausdorff topological spaces (in particular manifolds) are exponentiable.

Before to state the following proposition, we recall some definitions. An *etale extension* of an ∞-topos $\mathbf{X}$ is an etale map $\mathbf{Y} \to \mathbf{X}$, i.e. if $\mathcal{S}\mathrm{h}(\mathbf{Y}) \simeq \mathcal{S}\mathrm{h}(\mathbf{X})_{/X}$ for some object X. An *open subtopos* of $\mathbf{X}$ is sub-∞-topos $\mathbf{Y} \subset \mathbf{X}$ which is an etale extension. A morphisme of ∞-topoi $f : \mathbf{Y} \to \mathbf{X}$ is *proper* if it satisfies the stable right Beck-Chevalley condition ([**?**, Def.7.3.1.4]). An *closed subtopos* of $\mathbf{X}$ is sub-∞-topos $\mathbf{Y} \subset \mathbf{X}$ which is proper. A morphism of ∞-topoi $f : \mathbf{Y} \to \mathbf{X}$ is *locally contractible* if $f^* : \mathcal{S}\mathrm{h}(\mathbf{X}) \to \mathcal{S}\mathrm{h}(\mathbf{Y})$ has a left adjoint $f_!$ satisfying the projection formula $f_!(f^*X \times_{f^*Y} Z) = X \times_Y f_! Z$. A morphism of ∞-topoi $f : \mathbf{Y} \to \mathbf{X}$ has *trivial shape* if $f^* : \mathcal{S}\mathrm{h}(\mathbf{X}) \to \mathcal{S}\mathrm{h}(\mathbf{Y})$ is fully faithful.

Proposition 3.3.5. *The class of exponentiable ∞-topoi is stable by*

(a) *products;*

(b) *retracts;*

(c) *etale extensions: given an etale morphism $\mathbf{Y} \to \mathbf{X}$, if $\mathbf{X}$ is exponentiable, then so is $\mathbf{Y}$;*

(d) *open and closed subtopoi: given an open or closed immersion $\mathbf{Y} \hookrightarrow \mathbf{X}$, if $\mathbf{X}$ is exponentiable, then so is $\mathbf{Y}$;*

(e) *quotient with proper fiber of trivial shape: given a proper morphism of trivial shape $\mathbf{Y} \hookrightarrow \mathbf{X}$, if $\mathbf{Y}$ is exponentiable, then so is $\mathbf{X}$;*

(f) *quotient with locally contractible fibers of trivial shape: given a locally contractible morphism of trivial shape $\mathbf{Y} \hookrightarrow \mathbf{X}$, if $\mathbf{Y}$ is exponentiable, then so is $\mathbf{X}$.*

Proof. The properties (a) and (b) are direct consequences of the definition of exponentiable objects.
(c) Let $\mathcal{S}\mathrm{h}(\mathbf{Y}) = \mathcal{S}\mathrm{h}(\mathbf{X})_{/X}$, then given a retraction $\varepsilon : IndC \rightleftarrows \mathcal{S}\mathrm{h}(\mathbf{X}) : \beta$, $\mathcal{S}\mathrm{h}(\mathbf{X})_{/X}$ is a retract of $\mathcal{I}\mathrm{nd}(C)_{/\beta X} = \mathcal{I}\mathrm{nd}(C_{/\beta X})$.
(d) The case of open sub-∞-topoi is a consequence of (c). Let $\mathbf{Y} \hookrightarrow \mathbf{X}$ be a closed immersion, then by [**?**, Rem. 7.3.1.5], the direct image $f_* : \mathcal{S}\mathrm{h}(\mathbf{Y}) \to \mathcal{S}\mathrm{h}(\mathbf{X})$ commute with filtered colimits so that $\mathcal{S}\mathrm{h}(\mathbf{Y})$ is a retract of $\mathcal{S}\mathrm{h}(\mathbf{X})$ by ω-accessible functors. The result follows from Proposition 2.5.3.
(e) Let $\mathbf{Y} \to \mathbf{X}$ be a proper morphism with trivial shape, then using again the fact that the direct image $f_* : \mathcal{S}\mathrm{h}(\mathbf{Y}) \to \mathcal{S}\mathrm{h}(\mathbf{X})$ commute with filtered colimits, we get that so that $\mathcal{S}\mathrm{h}(\mathbf{X})$ is a retract of $\mathcal{S}\mathrm{h}(\mathbf{Y})$ by ω-accessible functors and we use Proposition 2.5.3.
(f) Let $\mathbf{Y} \to \mathbf{X}$ be a locally contractible morphism with trivial shape. The functors $f_!$ and f^* present $\mathcal{S}\mathrm{h}(\mathbf{X})$ as a retract of $\mathcal{S}\mathrm{h}(\mathbf{Y})$ by ω-accessible functors and we use Proposition 2.5.3. ☐

Example 3.3.6. We consider the topos $\mathbb{T}^2/\mathbb{R}$ which is the quotient (in $\mathcal{T}\mathrm{opos}$) of the torus $\mathbb{T}^2 = S^1 \times S^1$ by an irrational action of $\mathbb{R}$. Then, the quotient map $\mathbb{T}^2 \to \mathbb{T}^2/\mathbb{R}$ is locally contractible of trivial shape. We know from Remark 3.3.4 that the ∞-topos of any manifold is exponentiable. Then Proposition 3.3.5 proves that $\mathbb{T}^2/\mathbb{R}$ is also exponentiable.

4 Leray sheaves & geometric theories

In this section, we re-connect with the theory of wavy arrows of [JJ82]. We use them to provide an alternative description of sheaves on an exponentiable topos $\mathbf{X}$. This description has a close relationship with Leray's original definition of sheaves and we name it after him. We use it to provide a more concrete description of the ∞-logos $\mathcal{S}\mathrm{h}(\mathbf{A}^{\mathbf{X}})$ in Proposition 4.1.7. Then we give a number of sufficient conditions for such a description to be valid for sheaves with values in an ∞-category other than $\mathcal{S}$.

4.1 Leray sheaves of spaces

In this section, we re-connect with the theory of wavy arrows of [JJ82]. We use them to provide an alternative description of sheaves on an exponentiable topos $\mathbf{X}$. This is used to provide a more concrete description of the ∞-logos $\mathcal{S}\mathrm{h}(\mathbf{A}^{\mathbf{X}})$ in Proposition 4.1.7.

Definition 4.1.1. We shall say that an endofunctor $W : \mathcal{C} \to \mathcal{C}$ of an ∞-category $\mathcal{C}$ is an *idempotent monad* if it is equipped with a natural transformation $\varepsilon : W \to \mathrm{Id}_{\mathcal{C}}$ such that both $\varepsilon W : W^2 \to W$ and $W\varepsilon : W^2 \to W$ are isomorphisms of endofunctors. This data is equivalent to that of the coreflective subcategory of fixed points of W inside $\mathcal{C}$ by [**?**, Proposition 5.2.7.4].

Given an exponentiable ∞-topos $\mathbf{X}$, $\mathcal{S}\mathrm{h}(\mathbf{X})$ is a continuous ∞-category and we have a standard presentation:

$$\mathcal{I}\mathrm{nd}(D) \underset{\alpha}{\overset{\beta}{\leftleftarrows}}\!\!\!\xrightarrow{\ \varepsilon\ } \mathcal{S}\mathrm{h}(\mathbf{X}) \ .$$

We then obtain an *cocontinuous idempotent comonad* $W = \beta\varepsilon$ on $\mathcal{I}\mathrm{nd}(D)$ and an identification between $\mathcal{S}\mathrm{h}(\mathbf{X})$ and the ∞-category $\mathrm{Fix}(W)$ of fixed points of W in $\mathcal{I}\mathrm{nd}(D)$. This description is to be compared with the classical one where ∞-logoi are defined as localizations of presheaf ∞-categories:

$$\mathcal{P}(C) \underset{\alpha}{\overset{\varepsilon}{\rightleftarrows}} \mathcal{S}\mathrm{h}(\mathbf{X})$$

and sheaves are the fixed points of an *left exact idempotent monad* $M = \alpha\varepsilon$. In the case of a standard presentation, W is left adjoint to M and provide an alternative subcategory of $\mathcal{I}\mathrm{nd}(D)$ equivalent to $\mathcal{S}\mathrm{h}(\mathbf{X})$.

The idempotent comonad $W : \mathcal{I}\mathrm{nd}(D) \to \mathcal{I}\mathrm{nd}(D)$ is not left exact but it is cocontinuous. In particular, W is the left Kan extension of the functor $D \to \mathcal{I}\mathrm{nd}(D) \xrightarrow{W} \mathcal{I}\mathrm{nd}(D)$. Using $\mathcal{I}\mathrm{nd}(D) \subset [D^{op}, \mathcal{S}]$, we can associate to w a bimodule $D^{op} \times D \to \mathcal{S}$ such that $w(a,b) = [a, \beta\varepsilon(b)]_{\mathcal{I}\mathrm{nd}(D)}$. Because D is assumed complete under finite colimits, we have $\mathcal{I}\mathrm{nd}(D) = [D^{op}, \mathcal{S}]^{\mathrm{lex}}$ with the embedding $i : \mathcal{I}\mathrm{nd}(D) \subset [D^{op}, \mathcal{S}]$ commuting with filtered colimits.

Let $(w \otimes_{D^{op}} -)$ be the left Kan extension of $w : D \to \mathcal{P}(D)$ along the Yoneda embedding $D \to \mathcal{P}(D)$. That is for $F : D^{op} \to \mathcal{S}$ we have:

$$w \otimes_{D^{op}} F = \int^{b \in D^{op}} w(-, b) \times F(b)\,.$$

The following lemma says that $w \otimes_{D^{op}} F = W(F)$. This implies that the values of $W(F)$ can be computed pointwise in $\mathcal{S}$.

Lemma 4.1.2. *The canonical map $w \otimes_{D^{op}} F \to W(F)$ is an isomorphism.*

Proof. Direct from Lemma 4.1.3 below. □

Lemma 4.1.3. *Given a functor $f : \mathcal{C} \to \mathcal{D}$, we have a commutative diagram*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook] \arrow[d, "f"'] & \mathcal{I}\mathrm{nd}(\mathcal{C}) \arrow[r, hook] \arrow[d, "\mathcal{I}\mathrm{nd}(f)"] & \mathcal{P}(\mathcal{C}) \arrow[d, "\mathcal{P}(f)"] \\
\mathcal{D} \arrow[r, hook] & \mathcal{I}\mathrm{nd}(\mathcal{D}) \arrow[r, hook] & \mathcal{P}(\mathcal{D})
\end{tikzcd}$$

Proof. An object of $\mathcal{P}(\mathcal{C})$ is in $\mathcal{I}\mathrm{nd}(\mathcal{C})$ if and only if it can be written as a filtered colimit of representable. Then its image by $\mathcal{P}(f)$ is filtered colimit of representable in $\mathcal{P}(\mathcal{D})$, hence an object of $\mathcal{I}\mathrm{nd}(\mathcal{D})$. □

Remark 4.1.4. An object of $w(a, b)$ is what is called a *wavy arrow* and denoted $a \rightsquigarrow b$ in [JJ82]. Using the notations of Appendix A.2, a wavy arrow is also an object of $\mathrm{Tot}\,(w)$. Proposition A.2.1 gives another way to write the formula for $w \otimes_{D^{op}} F$ (which is more reminiscent of the theory of $\mathcal{K}$-sheaves on locally compact spaces, or of that of exponentiable locales [Sco72, Joh82])

$$(w \otimes_{D^{op}} F)(a) = \operatorname*{colim}_{(a \rightsquigarrow b)^{op}} F(b)\,.$$

Without any further assumption on X, this colimit is not of a particular kind (filtered, sifted...). But it will be as soon as $w(-,-)$ has some exactness properties in its second variable (see Lemma 4.2.11).

Definition 4.1.5 (Leray sheaves)**.** Let D be an ∞-category with finite colimits and $W : \mathcal{I}\mathrm{nd}(D)$ a cocontinuous idempotent monad. Then, a functor $F : D^{op} \to \mathcal{S}$ which is

(i) left exact, and

(ii) such that $w \otimes_{D^{op}} F \simeq F$

shall be called a *Leray sheaf* on D. The ∞-category of Leray sheaves is denoted by $\mathcal{S}\mathrm{h}_{Leray}(D)$.

Proposition 4.1.6. *Let $\mathbf{X}$ be an exponentiable ∞-topos (i.e. such that $\mathcal{S}\mathrm{h}(\mathbf{X})$ is continuous) together with a standard presentation of $\mathcal{S}\mathrm{h}(\mathbf{X})$. Then, $\mathcal{S}\mathrm{h}(\mathbf{X})$ is canonically equivalent to $\mathcal{S}\mathrm{h}_{Leray}(D)$.*

Proof. By hypothesis, we have a colocalization $\varepsilon : \mathcal{I}\mathrm{nd}(D) \leftrightarrows \mathcal{S}\mathrm{h}(\mathbf{X}) : \beta$. In particular $\mathcal{S}\mathrm{h}(\mathbf{X})$ is equivalent to the full subcategory of $\mathcal{I}\mathrm{nd}(D)$ of functors such that units $F \simeq WF$. Then, the result is consequence of Lemma 4.1.2. □

We can now come back to the proof of Proposition 3.2.3 and give a more concrete description of $\mathcal{S}\mathrm{h}\big(\mathbf{A}^{\mathbf{X}}\big) = [\mathcal{S}\mathrm{h}(\mathbf{X}), \mathcal{S}]_{\mathrm{filt}}$. Given a standard presentation

$$\begin{tikzcd}
\mathcal{I}\mathrm{nd}(D) \arrow[r, "\varepsilon" description] & \mathcal{S}\mathrm{h}(\mathbf{X}) \arrow[l, bend right, "\beta"'] \arrow[l, bend left, "\alpha"]
\end{tikzcd}.$$

we are going to describe $[\mathcal{S}\mathrm{h}(\mathbf{X}),\mathcal{S}]_{\mathrm{filt}}$ as a retract of $\mathcal{P}(D^{op})$ in $\mathcal{L}$ogos. Because W has its values in ind-objects, the bimodule $w : D^{op}\times D \to \mathcal{S}$ is left exact in the first variable. The idempotent comonad structure of W can be rewritten in the following way: the bimodule w bears a bimodule map $w \to \mathrm{Map}_D$ inducing the following isomorphism (composition of wavy arrows):

$$\int^{c\in D^{op}} w(a,c)\times w(c,b) \simeq w(a,b)\ . \tag{1}$$

Let us denote by $\mathcal{W} : \mathcal{P}(D^{op}) \to \mathcal{P}(D^{op})$ the left Kan extension of $D^{op} \to [D,\mathcal{S}] = \mathcal{P}(D^{op})$. For $G : D \to \mathcal{S}$ it is defined by

$$\mathcal{W}(G) = G\otimes_{D^{op}} w = \int^{c\in D^{op}} G(c)\times w(c,-)\,.$$

In terms of Remark 4.1.4, $\mathcal{W}(G)(d) = \mathrm{colim}_{c\rightsquigarrow d}\, G(c)$. In particular, the left exactness of w in the first variable ensure that this colimit is indexed by a filtered ∞-category. This is in agreement with the fact that $\mathcal{W}$ is a left exact functor, see Lemma 4.2.11. (Beware that $-\otimes_{D^{op}} w$ is a different functor from the previous $w\otimes_{D^{op}} -$ since the coend are done on different variables of w. The position of w next to the $\otimes$ reveal which variable is used in the coend. The composition law (1) can be written $w\otimes_{D^{op}} w = w$, and this is consistent with the meaning of both functors $-\otimes_{D^{op}} w$ and $w\otimes_{D^{op}} -$.)

The functor $\mathcal{W} : \mathcal{P}(D^{op}) \to \mathcal{P}(D^{op})$ is cocontinuous and because w is left exact in the first variable, it is also left exact by Theorem 2.1.4. It also has the structure of an idempotent comonad (this is clear from the formula $w\otimes_{D^{op}} w = w$). We define $\mathcal{Q}$ as the ∞-logos of fixed points of $\mathcal{W}$. It is the full subcategory of $\mathcal{P}(D^{op})$ spanned by functors $G : D \to \mathcal{S}$ satisfying

$$G\otimes_{D^{op}} w \simeq G\,.$$

We shall prove that $\mathcal{Q} = [\mathcal{S}\mathrm{h}(\mathbf{X}),\mathcal{S}]_{\mathrm{filt}}$.

Proposition 4.1.7. *Given an exponentiable ∞-topos $\mathbf{X}$ and a standard presentation of $\mathcal{S}\mathrm{h}(\mathbf{X})$, we have a canonical isomorphism $\mathcal{Q} = \mathcal{S}\mathrm{h}\big(\mathbf{A}^{\mathbf{X}}\big)$.*

Proof. The ∞-logos $\mathcal{S}\mathrm{h}\big(\mathbf{A}^{\mathbf{X}}\big) = [\mathcal{S}\mathrm{h}(\mathbf{X}),\mathcal{S}]_{\mathrm{filt}}$ is quasi-projective and so is $\mathcal{Q}$ as a retract of a quasi-free ∞-logos. Therefore, by Corollary 2.7.8, they will be isomorphic if they have the same ∞-categories of models, i.e. if $\mathcal{S}\mathrm{h}(\mathbf{X}) \simeq [\mathcal{Q},\mathcal{S}]^{\mathrm{lex}}_{\mathrm{cc}}$. On one side, using Proposition 4.1.6 we know that $\mathcal{S}\mathrm{h}(\mathbf{X})$ can be described as the ∞-category of functors $\big[\mathcal{I}\mathrm{nd}(D)^{op},\mathcal{S}\big]^{\mathrm{c}} = [D^{op},\mathcal{S}]^{\mathrm{lex}}$ such that $w\otimes_{D^{op}} F = F$. On the other side, the ∞-category $[\mathcal{Q},\mathcal{S}]^{\mathrm{lex}}_{\mathrm{cc}}$ is equivalent, by definition of $\mathcal{Q}$, to the ∞-category of cocontinuous and left exact functors $F : \mathcal{P}(D^{op}) \to \mathcal{S}$ such that $F\circ\mathcal{W} \simeq F$. That is, for any $G : D^{op} \to \mathcal{S}$

$$G\otimes_{D^{op}} w\otimes_{D^{op}} F \simeq G\otimes_{D^{op}} F\,.$$

Because of the naturality in G, this gives back the same condition that $w\otimes_{D^{op}} F = F$ on the restriction $F : D^{op} \to \mathcal{S}$. ☐

Remark 4.1.8. This construction of $\mathcal{S}\mathrm{h}\big(\mathbf{A}^{\mathbf{X}}\big)$ is the one given in of [JJ82], but the above presentation goes around the use of a Grothendieck topology to describe $\mathcal{Q}$ as a quotient of the ∞-logos $\mathcal{P}(D^{op})$.

Corollary 4.1.9. *Let $\mathcal{F}$ be any ∞-logos, and $\mathbf{X}$ and exponentiable topos with a chosen standard presentation, then the ∞-category $\mathcal{S}\mathrm{h}(\mathbf{X})\otimes\mathcal{F}$ is canonically equivalent to the fixed points of $w\otimes_{D^{op}} -$ in $[D^{op},\mathcal{F}]^{\mathrm{lex}}$.*

Proof. By the universal property of $\mathcal{Q} = \mathcal{S}\mathrm{h}\big(\mathbf{A}^{\mathbf{X}}\big)$, we have $\mathcal{S}\mathrm{h}(\mathbf{X})\otimes\mathcal{F} = [\mathcal{Q},\mathcal{F}]^{\mathrm{lex}}_{\mathrm{cc}}$. The latter ∞-category is that of fixed points of $w\otimes_{D^{op}} -$ acting on $[D^{op},\mathcal{F}]^{\mathrm{lex}}$. ☐

Remark 4.1.10 (Comparison with $\mathcal{K}$-sheaves)**.** Let X be a locally quasi-compact Hausdorff topological space and $\mathcal{O}_c(X)$ the subposet of $\mathcal{O}(X)$ spanned by open subspaces V with a compact closure $\overline{V}$. Then there exists a standard presentation $\varepsilon : \mathcal{I}\mathrm{nd}(\mathcal{O}_c(X)) \rightleftarrows \mathcal{O}(X);\beta$ where β sends a open subspace U to the colimit of all V in $\mathcal{O}_c(X)$ such that $\overline{V}\subset U$. From there we get a standard presentation of $\mathcal{S}\mathrm{h}(X)$ where D is the finite colimit completion of $\mathcal{O}_c(X)$ in $\mathcal{S}\mathrm{h}(X)$. The computation of Leray sheaves describe them as functors $\mathcal{O}_c(X)^{op} \to \mathcal{S}$ such that

(i) $F(0) = 1$

(ii) $F(U \cup V) = F(U) \times_{F(U \cap V)} F(V)$ for any U and V in $\mathcal{O}_c(X)$

(iii) $F(U) = \operatorname{colim}_{U \ll U'} F(U')$ where $U \ll U'$ means that there exists a compact subspace K of X such that $U \subset K \subset U'$.

Let $\mathcal{K}(X)$ be the poset of compact subspaces of X. In [?, Theorem 7.3.4.9], Lurie prove that the ∞-category of sheaves on X is equivalent the the ∞-category of functors $F : \mathcal{K}(X)^{op} \to \mathcal{S}$ such that

(i) $F(0) = 1$

(ii) $F(K \cup K') = F(K) \times_{F(K \cap K')} F(K')$ for any K and K' in $\mathcal{K}(X)$

(iii) $F(K) = \operatorname{colim}_{K \ll K'} F(K')$ where $K \ll K'$ means that there exists an open subset U such that $K \subset U \subset K'$.

We have an adjunction $\mathrm{cl} : \mathcal{O}_c(X) \rightleftarrows \mathcal{K}(X) : \mathrm{int}$ between the closure and interior functors. We claim that this adjunction induces an equivalence between Leray sheaves and $\mathcal{K}$-sheaves and that this equivalence is the one induced by their identification as $\mathcal{S}\mathrm{h}(X)$. This justifies the idea that, for an exponentiable topos $\mathbf{X}$, the ∞-category D of a standard presentation of $\mathcal{S}\mathrm{h}(\mathbf{X})$, plays the role of some "compact spaces" over $\mathbf{X}$. However, we do not know how to make this idea into a more precise statement.

4.2 Leray sheaves with values in $\mathcal{C}$

We now turn to the problem of describing sheaves with values in another ∞-category than $\mathcal{S}$, for example $\mathcal{C} = \mathcal{S}\mathrm{p}$ the ∞-category of spectra, or $\mathcal{C}\mathrm{at}$ the ∞-category of small ∞-categories. Several definitions exists for sheaves with values in an ∞-category $\mathcal{C}$, we shall consider only two of them.

(i) If $\mathcal{C}$ is cocomplete, we can define $\mathcal{S}\mathrm{h}(\mathbf{X}, \mathcal{C}) = \mathcal{S}\mathrm{h}(\mathbf{X}) \otimes \mathcal{C}$.

(ii) If $\mathcal{C}$ is complete, we can define $\mathcal{S}\mathrm{h}(\mathbf{X}, \mathcal{C}) = \left[\mathcal{S}\mathrm{h}(\mathbf{X})^{op}, \mathcal{C}\right]^{\mathrm{c}}$.

These two descriptions coincide whenever $\mathcal{C}$ is presentable (since then we can use $\mathcal{B} \otimes \mathcal{C} = [\mathcal{B}^{op}, \mathcal{C}]^{\mathrm{c}}$) and we are going to restrict ourselves to this situation.

If $\mathbf{X}$ is exponentiable, we saw that any standard presentation gives an equivalence

$$\mathcal{S}\mathrm{h}(\mathbf{X}) = \text{fixed points of } W \text{ in } \mathcal{I}\mathrm{nd}(D) = [D^{op}, \mathcal{S}]^{\mathrm{lex}}.$$

The two above descriptions suggest two generalisations of this characterization:

classical description	Leray description
$\mathcal{S}\mathrm{h}(\mathbf{X}) \otimes \mathcal{C}$	fixed points of $W \otimes \mathrm{Id}_{\mathcal{C}}$ in $[D^{op}, \mathcal{S}]^{\mathrm{lex}} \otimes \mathcal{C}$
$\left[\mathcal{S}\mathrm{h}(\mathbf{X})^{op}, \mathcal{C}\right]^{\mathrm{c}}$	fixed points of $(w \otimes_{D^{op}} -)$ in $[D^{op}, \mathcal{C}]^{\mathrm{lex}}$

In the second characterization, we are using the fact that $\mathcal{C}$ being cocomplete, the coend $w \otimes_{D^{op}} -$ is defined on functors $D^{op} \to \mathcal{C}$. However, it is not clear a priori that it preserves the full subcategory of left exact functors. A second problem is the coincidence of the two idempotent comonads $W \otimes \mathrm{Id}_{\mathcal{C}}$ and $(w \otimes_{D^{op}} -)$ under the identification

$$[D^{op}, \mathcal{S}]^{\mathrm{lex}} \otimes \mathcal{C} = \left[\mathcal{I}\mathrm{nd}(D)^{op}, \mathcal{S}\right]^{\mathrm{c}} \otimes \mathcal{C} = \mathcal{I}\mathrm{nd}(D) \otimes \mathcal{C} = \left[\mathcal{I}\mathrm{nd}(D)^{op}, \mathcal{C}\right]^{\mathrm{c}} = [D^{op}, \mathcal{C}]^{\mathrm{lex}}.$$

Both these problems will need further assumption on $\mathcal{E}$ and $\mathcal{C}$.

Definition 4.2.1 (Leray sheaves)**.** Let D be an ∞-category with finite colimits and $W : \mathcal{I}\mathrm{nd}(D)$ a cocontinuous idempotent monad. Then, a functor $F : D^{op} \to \mathcal{C}$ which is

(i) left exact, and

(ii) such that $w \otimes_{D^{op}} F \simeq F$ (in $[D^{op}, \mathcal{C}]$)

shall be called a *Leray sheaf on D with values in $\mathcal{C}$*. The ∞-category of Leray sheaves is denoted by $\mathcal{S}\mathrm{h}_{Leray}(D, \mathcal{C})$.

A first strategy to prove the existence of an equivalence $\mathcal{S}\mathrm{h}(\mathbf{X}, \mathcal{C}) \simeq \mathcal{S}\mathrm{h}_{Leray}(D, \mathcal{C})$ associated to a standard presentation is to take advantage of the fact that we now this is true when $\mathcal{C} = \mathcal{F}$ is an ∞-logos by Corollary 4.1.9. It is in fact frequent that $\mathcal{C}$ has a functor to an ∞-logos $\mathcal{C} \to \mathcal{F}$ which creates finite limits and some colimits. For example, the inclusion of spectra in parametrized spectra $\mathcal{S}\mathrm{p} \hookrightarrow \mathcal{P}\mathcal{S}\mathrm{p}$ creates all limits and contractible colimits, or the Segal inclusion of $\mathcal{C}\mathrm{at} \hookrightarrow \mathcal{S}^{\Delta^{op}}$ creates all limits and filtered colimits, or, if $\mathcal{C}$ is an ∞-category of algebras over some operad or Lawvere theory the $\mathcal{C} \to \mathcal{S}$ creates all limits and sifted colimits. In this context, we have a cartesian square

$$\begin{tikzcd}
{[D^{op}, \mathcal{C}]^{\mathrm{lex}}} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & {[D^{op}, \mathcal{C}]} \arrow[d] \\
{[D^{op}, \mathcal{F}]^{\mathrm{lex}}} \arrow[r, hook] & {[D^{op}, \mathcal{F}]}
\end{tikzcd}$$

Thus, the action of $w \otimes_{D^{op}} -$ on $[D^{op}, \mathcal{C}]$ will preserve the full sub-∞-category of left exact functors as soon as the functor $[D^{op}, \mathcal{C}] \to [D^{op}, \mathcal{F}]$ commute with the action of $w \otimes_{D^{op}} -$. A sufficient condition is that the colimit computing the values of $w \otimes_{D^{op}} -$ is of the type created by the functor $\mathcal{C} \to \mathcal{F}$. The colimit $\mathrm{colim}_{(c \rightsquigarrow d)^{op}} -$ is filtered (resp. sifted or contractible) if and only if $w(c, -) : D \to \mathcal{S}$ preserves finite limits (resp. finite product or the empty limit). This suggests the following definition.

Definition 4.2.2. Let Λ be a class of finite ∞-categories. A continuous logos $\mathcal{E}$ shall be said to have a *presentation with Λ-limits* if $\beta : \mathcal{E} \to \mathcal{I}\mathrm{nd}(\mathcal{E})$ preserves them.

The following lemma says that this condition can be tested on a standard presentation.

Lemma 4.2.3. *Let I be a finite ∞-category. If $\mathcal{C}$ is continuous and presentable, then $\beta : \mathcal{C} \to \mathcal{I}\mathrm{nd}(\mathcal{C})$ preserves limits of I-diagrams if and only if there exists a standard presentation such that D is stable by limits of I-diagrams and $\beta' : \mathcal{C} \to \mathcal{I}\mathrm{nd}(D)$ preserves limits of I-diagrams.*

Proof. By construction of β' in Proposition 2.5.9, it is enough to prove that $i_! = \mathcal{I}\mathrm{nd}(i) : \mathcal{I}\mathrm{nd}(D) \to \mathcal{I}\mathrm{nd}(\mathcal{C})$ preserves limits of I-diagrams. We prove first that finite limits exists in $\mathcal{I}\mathrm{nd}(D)$. This is because the ∞-category $[D^{op}, \mathcal{S}]$ has finite limits and the full sub-∞-category $\mathcal{I}\mathrm{nd}(D) = [D^{op}, \mathcal{S}]^{\mathrm{lex}} \subset \left[D^{op}, \overline{\mathcal{S}}\right]$ is stable by finite limits.

Next, we consider the following diagram:

$$\begin{tikzcd}
D \arrow[r, hook, "I\text{-cont}"'] \arrow[d, "i"', "I\text{-cont}"] & \mathcal{I}\mathrm{nd}(D) \arrow[r, hook, "\mathrm{lex}"'] \arrow[d, dashed, "\mathcal{I}\mathrm{nd}(i)"] & \mathcal{P}(D) \arrow[r, hook, "\mathrm{lex}"'] & \widehat{\mathcal{P}}(D) \arrow[d, "\widehat{\mathcal{P}}(i)"', "I\text{-cont}"] \\
\mathcal{C} \arrow[r, hook] \arrow[rrr, bend right, "\mathrm{lex}"'] & \mathcal{I}\mathrm{nd}(\mathcal{C}) \arrow[rr, hook] & & \widehat{\mathcal{P}}(\mathcal{C}) .
\end{tikzcd}$$

The commutation of the diagram is given by Lemma 4.1.3. When D is small, finite limits exists in $\mathcal{I}\mathrm{nd}(D)$ because $[D^{op}, \mathcal{S}]$ has finite limits and the full sub-∞-category $\mathcal{I}\mathrm{nd}(D) = [D^{op}, \mathcal{S}]^{\mathrm{lex}} \subset \left[D^{op}, \overline{\mathcal{S}}\right]$ is stable by finite limits. By Lemma 2.1.5, when D has limits of I-diagrams, the map $\widehat{\mathcal{P}}(i)$ preserves these limits. Then, the map $\mathcal{I}\mathrm{nd}(i)$ preserves limits of I-diagrams as a restriction of $\widehat{\mathcal{P}}(i)$ to a full subcategory stable by limits of I-diagrams. ☐

Example 4.2.4. Any presheaf ∞-logos $\mathcal{E} = \mathcal{P}(C)$ has a standard presentation with finite limits since $\beta = \mathrm{Id}_{\mathcal{E}}$. This is also the case of the ∞-logos $\mathcal{PS}\mathrm{p} = \mathcal{I}\mathrm{nd}(\mathcal{F}\mathrm{in}\mathcal{PS}\mathrm{p})$ of parametrized spectra.

Remark 4.2.5. The condition for a continuous logos to have a presentation with all finite limits is an analog for ∞-topoi of the condition for a topological space X to be

(i) locally quasi-compact (existence of β), and

(ii) quasi-compact ($\beta(1) = 1$) and quasi-separated (β preserves finite intersections).

The condition to have a presentation with finite products only is another generalisation of these condition. The condition to have a presentation with terminal object only removes the quasi-separation hypothesis.

4.2.1 Values in $\mathcal{S}\mathrm{p}$

Recall from [**?**, Hoy18] that the ∞-category $\mathcal{PS}\mathrm{p}$ of parametrized spectra is an ∞-logos. There exist a canonical fibration $\mathcal{PS}\mathrm{p} \to \mathcal{S}$ and the canoncial inclusion $\mathcal{S}\mathrm{p} \to \mathcal{PS}\mathrm{p}$ is the inclusion of the fiber at 1. Therefore, the inclusion $\mathcal{S}\mathrm{p} \to \mathcal{PS}\mathrm{p}$ preserves (thus creates) all limits and contractible colimits. We can then apply the previous reasonning to this inclusion.

Theorem 4.2.6. *If $\mathbf{X}$ is an exponentiable topos such that $\mathcal{S}\mathrm{h}(\mathbf{X})$ has a standard presentation with terminal object, then there exists an equivalence $\mathcal{S}\mathrm{h}(\mathbf{X}, \mathcal{S}\mathrm{p}) \simeq \mathcal{S}\mathrm{h}_{Leray}(D, \mathcal{C})$.*

Proof. If β preserve the terminal object then the ∞-category $\mathrm{Tot}\,(w)_{c/}$ is contractible. We can then apply the strategy explained before Definition 4.2.2. □

Remark 4.2.7. More generally, the same proof works for sheaves with values in any *locus* in the sense of [Hoy18].

4.2.2 Values in algebras

Let $\mathcal{C}$ be an ∞-category of algebras over some colored operad or colored Lawvere theory. Let I be the ∞-category of colors, we have a forgetful functor $\mathcal{C} \to \mathcal{S}^I$ with values in an ∞-logos. This functor creates all limits and sifted colimits.

Theorem 4.2.8. *Let $\mathcal{C}$ be an ∞-category of algebras over some colored operad or colored Lawvere theory. If $\mathbf{X}$ is an exponentiable topos such that $\mathcal{S}\mathrm{h}(\mathbf{X})$ has a standard presentation with finite products, then there exists an equivalence $\mathcal{S}\mathrm{h}(\mathbf{X}, \mathcal{S}\mathrm{p}) \simeq \mathcal{S}\mathrm{h}_{Leray}(D, \mathcal{C})$.*

Proof. If β preserve finite products the ∞-category $\mathrm{Tot}\,(w)_{c/}$ is sifted. We can then apply the strategy explained before Definition 4.2.2. □

4.2.3 Values in $\mathcal{C}$ with left exact filtered colimits

We arrive now at the most general situation. We are going to change the strategy and not assume anymore that $\mathcal{C}$ has a nice forgetful functor to an ∞-logos. We are only going to assume that filtered colimits commute with finite limits in $\mathcal{C}$. To compensate, we need to assume the stronger condition that $\mathbf{X}$ has a presentation with all finite limits. The proof of the result is going to be more involved.

Theorem 4.2.9. *Let $\mathbf{X}$ be an exponentiable topos such that $\mathcal{S}\mathrm{h}(\mathbf{X})$ has a standard presentation with finite limits, and let $\mathcal{C}$ be an ∞-category where filtered colimits commute with finite limits. Then, for any standard presentation of $\mathcal{S}\mathrm{h}(\mathbf{X})$ where D has finite limits, there exists an isomorphism $\mathcal{S}\mathrm{h}(\mathbf{X}, \mathcal{C}) = \mathcal{S}\mathrm{h}_{Leray}(D, \mathcal{C})$.*

Proof. We need to solve the two problems diagnosed before. The first problem is the fact that the endofunctor $w \otimes_{D^{op}} -$ of $[D^{op}, \mathcal{C}]$ preserves left exact functor. This is the content of Lemma 4.2.11 below. The second problem is the coincidence of the two endofunctors $W \otimes \mathrm{Id}_{\mathcal{C}}$ and $(w \otimes_{D^{op}} -)$ under the identification $[D^{op}, \mathcal{S}]^{\mathrm{lex}} \otimes \mathcal{C} = [D^{op}, \mathcal{C}]^{\mathrm{lex}}$. We are going to prove this by proving that both functor have the same right adjoint. The right adjoint to $W \otimes \mathrm{Id}_{\mathcal{C}}$ is given on $[D^{op}, \mathcal{C}]^{\mathrm{lex}}$ by the pre-composition with W. More precisely, for functors $F : D^{op} \to \mathcal{C}$ and $G : D^{op} \to \mathcal{S}$, the continuous functor $F' : \mathcal{I}\mathrm{nd}(D)^{op} \to \mathcal{C}$ corresponding to F is the coend $F'(G) = [G, F]_{D^{op}} = \int_{d \in D^{op}} [G(d), F(d)]$ (see Appendix A.3). Then, the right adjoint to $W \otimes \mathrm{Id}_{\mathcal{C}}$ given by precomposition of F' with $W : \mathcal{I}\mathrm{nd}(D) \to \mathcal{I}\mathrm{nd}(D)$. Explicitely, we have

$$(F' \circ W)(G) = [w \otimes_{D^{op}} G, F]_{D^{op}} = [G, [w, F]_{D^{op}}]_{D^{op}} .$$

where we have used Proposition A.3.4. So the right adjoint is simply $F \mapsto [w, F]_{D^{op}}$ which is also right adjoint to $F \mapsto w \otimes_{D^{op}} F$ by Proposition A.3.4 again. ☐

Remark 4.2.10. When $\mathbf{X}$ is exponentiable we have then two descriptions of sheaves with values in $\mathcal{C}$:

(i) $\mathcal{S}\mathrm{h}(\mathbf{X}, \mathcal{C}) = \left[\mathcal{S}\mathrm{h}(\mathcal{C})^{op}, \mathcal{C}\right]^{\mathrm{c}}$;

(ii) $\mathcal{S}\mathrm{h}(\mathbf{X}, \mathcal{C}) =$ fixed points of $(w \otimes_{D^{op}} -)$ in $[D^{op}, \mathcal{C}]^{\mathrm{lex}}$.

These two definitions go along with the classical intuitions for the notion of sheaf: a sheaf on a space X is either

(i) some specific function on the open subspaces of X (Cartan's definition);

(ii) some specific function on the closed (or compact) subspaces of X (Leray's original definition);

The ∞-category D can be thought as an abstraction of the poset of compact subspaces. Make such a statement more precise in the context of ∞-topoi is an open problem.

We assume that D and $\mathcal{C}$ have finite limits. We use notations of Appendix A.

Lemma 4.2.11. *Let $\mathcal{C}$ be an ∞-category where filtered colimits commute with finite limits. If D has finite limits and $w : D^{op} \times D \to \mathcal{S}$ is left exact in the second variable (i.e. if $\mathrm{Tot}\,(w)_{c/}$ is filtered) then the endofunctor $w \otimes_{D^{op}} -$ on $[D^{op}, \mathcal{C}]$ preserves left exact functors.*

Proof. We need to prove that $(w \otimes_{D^{op}} F)(\mathrm{colim}_i\, c_i) = \lim_i\, (w \otimes_{D^{op}} F)(c_i)$ for any finite diagram $I \to D$. It is enough to prove this for the empty diagram and pushouts. The case of the empty diagram is straightforward because the initial object of D is strict. The case of pushouts is proven by the following canonical isomorphisms:

$$\begin{aligned}
(w \otimes_{D^{op}} F)(\mathrm{colim}_i\, c_i) &= \mathrm{colim}_{\mathrm{colim}_i c_i \rightsquigarrow d \in \left(\mathrm{Tot}(w)_{\mathrm{colim}\, c_i/}\right)^{op}} F(d) && \text{where the colimit is filtered}\\
&= \mathrm{colim}_{c_\bullet \rightsquigarrow d_\bullet \in \left(\mathrm{Tot}(w^I)_{c_\bullet/}\right)^{op}} F(\mathrm{colim}_i\, d_i) && \text{Lemma 4.2.12}\\
&= \mathrm{colim}_{(c_\bullet \rightsquigarrow d_\bullet)^{op}} \lim_i F(d_i) && \text{left exactness of } F\\
&= \lim_i \mathrm{colim}_{(c_\bullet \rightsquigarrow d_\bullet)^{op}} F(d_\bullet)_i && \text{Lemma 4.2.13 and hypothesis on } \mathcal{C}\\
&= \lim_i \mathrm{colim}_{(c_i \rightsquigarrow d_i)^{op}} F(d_i) && \text{Lemma 4.2.14}\\
&= \lim_i\, (w \otimes_{D^{op}} F)(c_i)\,.
\end{aligned}$$

☐

Let $\varepsilon : \mathcal{I}\mathrm{nd}(D) \leftrightarrows \mathcal{E} : \beta$ be a standard presentation. Recall that by definition D has finite colimits and that the bimodule $w : D^{op} \times D \to \mathcal{S}$ is left exact in the first variable. We are also assuming now that D has finite limits and that w is left exact in the second varaible. For two finite diagrams $c_\bullet, d_\bullet : I \to D$, we define the space of wavy arrows from c to d to be the end (see Appendix A.3)

$$w^I(c, d) = \int_{i \in I} w(c_i, d_i).$$

These spaces define a functor $w^I : \left(D^I\right)^{op} \times D^I \to \mathcal{S}$ whose total ∞-category is denoted by $\mathrm{Tot}\left(w^I\right)$.

Lemma 4.2.12. *For any diagram $c_\bullet : I \to D$, the adjunction* $\operatorname{colim} : D^I \rightleftarrows D : \operatorname{cst}$ *induces an adjunction* $\operatorname{colim} : \operatorname{Tot}\left(w^I\right)_{c_\bullet/} \rightleftarrows \operatorname{Tot}(w)_{\operatorname{colim} c_i/} : \operatorname{cst}$. *In particular, the functor* $\operatorname{colim} : \operatorname{Tot}\left(w^I\right)_{c_\bullet/} \to \operatorname{Tot}(w)_{\operatorname{colim} c_i/}$ *is co-initial.*

Proof. We prove first that the functors $\operatorname{colim} : \operatorname{Tot}\left(w^I\right)_{c_\bullet/} \rightleftarrows \operatorname{Tot}(w)_{\operatorname{colim} c_i/} : \operatorname{cst}$ exist. By construction of $\operatorname{Tot}\left(w^I\right)_{c_\bullet/}$ and $\operatorname{Tot}(w)_{\operatorname{colim} c_i/}$ as total ∞-categories, the functor colim is equivalent to the existence of a natural transformation $w^I(c_\bullet, d_\bullet) \to w(\operatorname{colim} c_i, \operatorname{colim} d_i)$. This transformation is the composition of

$$w^I(c_\bullet, d_\bullet) \to w^I(c_\bullet, \operatorname{cst}(\operatorname{colim} d_i)) \to w(\operatorname{cst}(\operatorname{colim} c_i), \operatorname{cst}(\operatorname{colim} d_i))$$

where the first map is induced by the canonical map $d_\bullet \to \operatorname{cst}(\operatorname{colim} d_i)$ in D and the second map is a consequence of the left exactness of w in the first variable and of Remark A.3.3: for d in D, we have an isomorphism

$$w^I(c, \operatorname{cst}(d)) = \int_{i \in I} w(c_i, d) = \lim_i w(c_i, d) = w(\operatorname{colim} c_i, d)\,. \tag{2}$$

On the other side, the functor $\operatorname{cst} : \operatorname{Tot}(w)_{\operatorname{colim} c_i/} \to \operatorname{Tot}\left(w^I\right)_{c_\bullet/}$ is induced by the diagonal transformation

$$w(c, d) \to= w(c, d)^{|I|} = w^I(\operatorname{cst}(c), \operatorname{cst}(d))$$

where we have used Remark A.3.3 to compute $w^I(\operatorname{cst}(c), \operatorname{cst}(d))$.

We now prove the adjunction between these two functors. Let $d_\bullet$ be a I-diagram in D and e be an element of D. The space of maps in $\operatorname{Tot}(w)_{\operatorname{colim} c_i/}$ is computed as the fiber product

$$\begin{tikzcd}
{[\operatorname{colim}_i d_i, e]_{\operatorname{colim} c_i/}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & {[\operatorname{colim}_i d_i, e]} \arrow[d] \\
1 \arrow[r] & w(\operatorname{colim}_i c_i, e)
\end{tikzcd}$$

and that in $\operatorname{Tot}\left(w^I\right)_{c_\bullet/}$ by the fiber product

$$\begin{tikzcd}
{[d_\bullet, \operatorname{cst}(e)]_{c_\bullet/}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & {[d_\bullet, \operatorname{cst}(e)]} \arrow[d] \\
1 \arrow[r] & w^I(d_\bullet, \operatorname{cst}(e))
\end{tikzcd}$$

The adjunction $\operatorname{colim} : D^I \rightleftarrows D : \operatorname{cst}$ gives an isomorphism $[\operatorname{colim}_i d_i, e] = [d_\bullet, \operatorname{cst}(e)]$. Using this and equation (2) within the two fiber products, we deduce an isomorphism

$$\left[\operatorname{colim}_i d_i, e\right]_{\operatorname{colim} c_i/} = [d_\bullet, \operatorname{cst}(e)]_{c_\bullet/}$$

which is the expected adjunction $\operatorname{colim} : \operatorname{Tot}\left(w^I\right)_{c_\bullet/} \rightleftarrows \operatorname{Tot}(w)_{\operatorname{colim} c_i/} : \operatorname{cst}$.

The last part of the statement is an application of Lemma 2.1.1. □

Lemma 4.2.13. *The ∞-category $\operatorname{Tot}\left(w^I\right)_{c_\bullet/}$ is filtered.*

Proof. The ∞-category $\operatorname{Tot}\left(w^I\right)_{c_\bullet/}$ is the total ∞-category of the functor $w^I(c_\bullet, -) : D^I \to \mathcal{S}$. We need to show that this functor is left exact. Let $d^k_\bullet$ be a finite diagram in D^I, we have isomorphisms:

$$\begin{aligned}
w^I(c_\bullet, \lim_k d^k_\bullet) &= \int_{i\in I} w(c_i, \lim_k d^k_i) \\
&= \int_{i\in I} \lim_k w(c_i, d^k_i) && \text{left exactness of } w \text{ in second variable} \\
&= \lim_k \int_{i\in I} w(c_i, d^k_i) && \text{continuity of the end functor} \\
&= \lim_k \; w^I(c_\bullet, d^k_\bullet)\,.
\end{aligned}$$

□

Lemma 4.2.14. *Let $I = 1 \leftarrow 0 \rightarrow 2$ be the pushout ∞-category and $i \in I$. If w is left exact in the second variable, the functor $ev_i : \mathrm{Tot}\left(w^I\right)_{c_\bullet/} \to \mathrm{Tot}\left(w\right)_{c_i/}$ is co-initial*

Proof. By Lemma 2.1.1, it is enough to prove that ev_i has a right adjoint. Given a diagram $c_\bullet = c_1 \leftarrow c_0 \rightarrow c_2$ and a wavy arrow $c_0 \rightsquigarrow d$ in $w(c_0, d)$, we use the hypothesis that $w(c, 1) = 1$ for any c to construct an object

$$\begin{tikzcd}
c_1 \arrow[d, squiggly] & c_0 \arrow[l] \arrow[r] \arrow[d, squiggly] & c_2 \arrow[d, squiggly] \\
1 & d \arrow[l] \arrow[r] & 1
\end{tikzcd}$$

of $\mathrm{Tot}\left(w^I\right)_{c_\bullet/}$ which is terminal in the fiber of $ev_0 : \mathrm{Tot}\left(w^I\right)_{c_\bullet/} \to \mathrm{Tot}\left(w\right)_{c_0/}$ above $c_0 \to d$. This proves that ev_0 is co-initial. The proof is similar for $i = 1, 2$. □

4.3 Leray sheaves and geometric theories

In this section we give an interpretation of the exponentiability of a topos $\mathbf{X}$ by the statement that the sheaves on $\mathbf{X}$ do form a geometric theory. In order to make precise such a statement, we need to introduce a few notions. The following definitions provide an attempt to define *geometric higher logical theories* but only from the point of view of ∞-category theory, not from a syntactic point of view. The reformulation of the exponentiability property of an ∞-topos is given in Proposition 4.3.7.

Definition 4.3.1 (Joyal [Joy08]). A *geometric sketch* is the data of a small ∞-category G (*generators*) and a set of maps R (*relations*) in the free ∞-logos $\mathcal{S}[G] = \mathcal{P}\big(G^{\mathrm{lex}}\big)$. A *model* of a geometric sketch in an ∞-topos $\mathcal{E}$ is a functor $G \to \mathcal{E}$ such that the corresponding ∞-logos morphism $\mathcal{S}[G] \to \mathcal{E}$ sends maps of R to isomorphisms.

Remark 4.3.2. The definition is chosen in analogy with a presentation of a ring by generators and relations. The ∞-category $\mathcal{S}[G] = \mathcal{P}\big(G^{\mathrm{lex}}\big)$ is essentially generated by colimits and finite limits from the objects of G. Intuitively, the notion of a geometric theory is a way to impose equations on the objects of G involving finite limits and arbitrary colimits.

We denote the ∞-category of models of (G, R) in $\mathcal{E}$ by $\mathcal{M}\mathrm{od}(G, R\,;\,\mathcal{E})$. The models of a geometric sketch define a functor

$$\begin{aligned}
\mathcal{L}\mathrm{ogos} &\to \mathcal{CAT} \\
\mathcal{E} &\longmapsto \mathcal{M}\mathrm{od}(G, R\,;\,\mathcal{E})
\end{aligned}$$

This functor is representable by the ∞-logos $\mathcal{S}(G, R)$ defined as the left exact localisation of $\mathcal{S}[G]$ generated by R. The ∞-logos $\mathcal{S}(G, R)$ is called the *enveloping ∞-logos* of (G, R). The corresponding ∞-topos is called the *classifying ∞-topos* of (G, R).

Definition 4.3.3. An *abstract geometric theory* is the data of a functor

$$\mathbb{T} : \mathcal{L}\mathrm{ogos} \to \mathcal{CAT} .$$

We shall say that an abstract geometric theory *is a geometric theory* (or *has a classifying* ∞*-topos*, or *is sketchable*) if $\mathbb{T}$ is representable, i.e. if there exists an ∞-logos $\mathcal{S}(\mathbb{T})$ and a natural equivalence

$$\mathbb{T}(\mathcal{E}) \simeq [\mathcal{S}(\mathbb{T}), \mathcal{E}]^{\mathrm{lex}}_{\mathrm{cc}} .$$

A *classifying* ∞*-topos* for a geometric theory is an ∞-topos $\mathbf{X}$ such that there exists an isomorphism of functors

$$\mathbb{T}(-) \simeq [\mathcal{S}\mathrm{h}(\mathbf{X}), -]^{\mathrm{lex}}_{\mathrm{cc}} .$$

A *sketch for a geometric theory* is a geometric sketch (G, R) such that there exists an isomorphism of functors

$$\mathbb{T}(-) \simeq \mathcal{M}\mathrm{od}(G, R; -) .$$

Example 4.3.4. Recall the classical definition of a *family of sheaves on* $\mathbf{X}$ *parametrized by* $\mathbf{Y}$ as a sheaf on $\mathbf{X} \times \mathbf{Y}$. The ∞-category of such families is simply $\mathcal{S}\mathrm{h}(\mathbf{X} \times \mathbf{Y})$. We define the *abstract theory of sheaves on* $\mathbf{X}$ to be the functor

$$\begin{array}{c} \mathcal{L}\mathrm{ogos} \to \mathcal{CAT} \\ \mathcal{S}\mathrm{h}(\mathbf{Y}) \longmapsto \mathcal{S}\mathrm{h}(\mathbf{X} \times \mathbf{Y}) \end{array}$$

We shall say that sheaves in $\mathbf{X}$ are models of a geometric sketch if the geometric theory of sheaves on $\mathbf{X}$ is sketchable.

Example 4.3.5. For a cocomplete ∞-category $\mathcal{C}$, the functor of sheaves with values in $\mathcal{C}$ (in the sense of 4.2(i))

$$\begin{array}{c} \mathcal{L}\mathrm{ogos} \to \mathcal{CAT} \\ \mathcal{E} \longmapsto \mathcal{E} \otimes \mathcal{C} \end{array}$$

is an abstract geometric theory. This theory is not sketchable in general, but this is the case when $\mathcal{C}$ is finitely presentable (i.e. $\mathcal{C} = \mathcal{I}\mathrm{nd}(C)$ for C an ∞-category with finite colimits). The envelopping ∞-logos is simply $\mathcal{P}(C^{op})$. This is in particular the case when $\mathcal{C}$ is the ∞-category $\mathcal{C}\mathrm{at}$ of ∞-categories, the ∞-category $\mathcal{S}\mathrm{p}$ of spectra, the ∞-category of E_∞-rings (spectral or not)...

Example 4.3.6. Let $(\mathcal{G}, \mathcal{G}^{ad}, \tau)$ be a geometry à la Lurie [Lur11, Lur17a]. Recall that $\mathcal{G}$ is a lex category. For a finite diagram $X : I \to \mathcal{G}$ with limit X_0, let $\ell_X : \lim X_i \to X_0$ be the canonical map in $\mathcal{G}^{\mathrm{lex}}$. Let R be the class of covering sieves of the topology τ and all maps ℓ_X, then $(\mathcal{G}, R)$ is a sketch for the classifying ∞-topos for local $\mathcal{G}$-structures.

Proposition 4.3.7. *(a) Every* ∞*-topos is the envelopping* ∞*-topos of a geometric sketch.*

(b) A ∞*-topos* $\mathbf{X}$ *is exponentiable if and only if the theory of sheaves on* $\mathbf{X}$ *is sketchable.*

Proof. (a) is a reformulation of Proposition 2.3.3.
(b) By Proposition 3.2.2, $\mathbf{X}$ is exponentiable if and only if the exponential $\mathbf{A}^{\mathbf{X}}$ exists. (b) is then a reformulation of the universal property of $\mathbf{A}^{\mathbf{X}}$ using (a). □

Remark 4.3.8. Although very simple, the reformulation of exponentiability proposed by Proposition 4.3.7 says in a precise way that exponentiable ∞-topoi are precisely those ∞-topoi whose sheaves can be described as diagrams of spaces satisfying conditions involving only finite limits and arbitrary colimits. We can build explicitly such a presentation with the Leray description of sheaves. Let $\varepsilon : \mathcal{I}\mathrm{nd}(D) \rightleftarrows \mathcal{S}\mathrm{h}(\mathbf{X}) : \beta$ be a standard presentation of $\mathbf{X}$. We know that $\mathcal{S}\mathrm{h}(\mathbf{A}^{\mathbf{X}})$ is a left exact localization of $\mathcal{P}(D^{op})$ by a set of map R'. Let $G = (D^{op})^{\mathrm{lex}}$, then D^{op} is a localization of G and $\mathcal{P}(D^{op})$ is a localization of $\mathcal{S}[D^{op}]$. We let R'' be a set of maps generating this localization. Viewing R' as maps in $\mathcal{S}[D^{op}]$, the pair $(G, R' \sqcup R'')$ is a geometric sketch for $\mathcal{S}\mathrm{h}(\mathbf{A}^{\mathbf{X}})$.

5 Dualisability of the ∞-category of stable sheaves

In this section, we apply the characterisation of exponentiable ∞-topoi to prove that their ∞-category of sheaves of spectra are dualizable stable cocomplete ∞-categories although they are not dualizable as unstable cocomplete ∞-categories. We characterize the dualizable cocomplete ∞-categories as retract of presentable presheaves ∞-categories and the dualizable stable cocomplete ∞-categories as presentable continuous stable ∞-categories. The main result follows from the fact that the stabilisation of continuous ∞-category is again continuous.

5.1 Dualizable cocomplete ∞-categories

We start by recalling from [Lur17b, Ch. 4.6.1] the notion of dualizable objects in a symmetric monoidal ∞-category $(\mathcal{C}, \otimes, \mathbb{1})$. An object X of $\mathcal{C}$ is dualizable if there exists another object $X^\vee$ in C with two maps $\eta : \mathbb{1} \to X^\vee \otimes X$ and $\varepsilon : X \otimes X^\vee \to \mathbb{1}$. where $\mathbb{1}$ is the unit of $\mathcal{C}$, such that the composite maps:

$$X \xrightarrow{\mathrm{Id}\otimes\eta} X \otimes X^\vee \otimes X \xrightarrow{\varepsilon\otimes\mathrm{Id}} X$$

$$X^\vee \xrightarrow{\eta\otimes\mathrm{Id}} X^\vee \otimes X \otimes X^\vee \xrightarrow{\mathrm{Id}\otimes\varepsilon} X^\vee$$

are the identities on X and $X^\vee$ respectively.

Remark 5.1.1. In the case where $\mathcal{C}$ is a closed symmetric monoidal ∞-category, a dualizable object X has its dual given by $X^\vee = [X, \mathbb{1}]$ where $[-,-]$ is the internal hom associated to the monoidal structure and $\mathbb{1}$ is the monoidal unit. Under this identification, the map $\varepsilon : X \otimes X^\vee \to \mathbb{1}$ is the evaluation.

If X is dualizable, then for any other object Y, we have a canonical isomorphism $X^\vee \otimes Y = [X, Y]$. Under this identification, the map $\eta : X^\vee \otimes X$ correspond to the canonical map $\mathbb{1} \to [X, X]$ associated to the identity of X.

Lemma 5.1.2. *In a closed symmetric monoidal ∞-category, any retract of a dualizable object is dualizable.*

Proof. Let $r : X \to Y$ be a retraction with X a dualizable object and let $s : Y \to X$ be a section. Set $Y^\vee = [Y, \mathbb{1}]$ an let's show that $Y^\vee$ has the right property. Because $r : X \to Y$ is a retraction, the same is true for $s^\vee : X^\vee \to Y^\vee$. We are then supplied with maps $\eta_Y = (r \otimes s^\vee)\eta_X$ and $\varepsilon_Y = \varepsilon_X(r^\vee \otimes s)$. The composition $(\mathrm{Id}_Y \otimes \varepsilon_Y) \circ (\eta_Y \otimes \mathrm{Id}_X) : Y \to Y$ is then a retract of Id_X, hence to the identity itself. The same is true for the other composition. ☐

Theorem 5.1.3. *The dualizable objects of $\mathcal{CAT}_{\mathrm{cc}}$ are the retracts of ∞-categories of the form $\mathcal{P}(D)$ with D a small ∞-category.*

Proof. We prove first that $\mathcal{P}(D)$ is dualizable with dual $\mathcal{P}(D^{op})$. The map $\varepsilon : \mathcal{P}(D^{op}) \otimes \mathcal{P}(D) = \mathcal{P}(D^{op} \times D) \to \mathcal{S}$ is the left Kan extension of $[-,-]_D : D^{op} \times D \to \mathcal{S}$, i.e. the coend functor. The map $\eta : \mathcal{S} \to \mathcal{P}(D^{op} \times D)$ is the left Kan extension of $1 \to \mathcal{P}(D^{op} \times D)$ pointing the bimodule $[-,-]_D : D^{op} \times D \to \mathcal{S}$.

Let $F : D^{op} \to \mathcal{S}$ gives be a fixed presheaf. Using the formulas for coends of Appendix A, we get

$$(\mathrm{Id} \otimes \varepsilon)(\eta \otimes \mathrm{Id})(F)(c) = \int_{d \in D^{op}} F(d) \times [c, d] = F(c)\,.$$

This proves that $(\mathrm{Id} \otimes \varepsilon)(\eta \otimes \mathrm{Id}) = \mathrm{Id}$. The proof of $1(\eta \otimes \mathrm{Id})(\mathrm{Id} \otimes \varepsilon) = \mathrm{Id}$ is similar. Using Lemma 5.1.2 we have proven that all retracts of $\mathcal{P}(D)$ are dualizable objects.

Reciprocally, let $\mathcal{C}$ be a dualizable ∞-category. Using the construction of the tensor product of cocomplete categories recalled in Remark 2.4.4, we have a localization

$$\mathcal{P}(\mathcal{C}^\vee \times \mathcal{C}) \longrightarrow \mathcal{C}^\vee \otimes \mathcal{C} = [\mathcal{C}, \mathcal{C}]_{\mathrm{cc}}$$

Localization are essentially surjective maps and there exists a small diagram $(\gamma_\bullet, c_\bullet) : I \to \mathcal{C}^\vee \times \mathcal{C}$ such that $\operatorname{colim}_i \gamma_i \otimes c_i = \mathrm{Id}$ in $[\mathcal{C}, \mathcal{C}]_{\mathrm{cc}}$. This formula means that for any object c in $\mathcal{C}$ we have the decomposition

$$\operatorname*{colim}_i \gamma_i(c) \otimes c_i = c\,.$$

where $\gamma_i(c)$ is in $\mathcal{S}$ and $\gamma_i(c) \otimes c_i$ is the tensor of $\mathcal{C}$ over $\mathcal{S}$. We are going to use a different decomposition of the identity, in terms of a coend over I. By Yoneda lemma (Lemma A.1.4), we have $c_i = \int^{j\in I} [j,i] \otimes c_j$ and

$$c = \operatorname*{colim}_i \ \gamma_i(c) \otimes \left(\int^{j\in I} [j,i] \otimes c_j \right) = \int^{j\in I} \left(\operatorname*{colim}_i \gamma_i(c) \otimes [j,i] \right) \otimes c_j\,.$$

The objects $\beta^j = \operatorname{colim}_i \gamma_i(c) \otimes [j,i]$ define a diagram $\beta^\bullet : I^{op} \to \mathcal{C}^\vee$. Using $\mathcal{C}^\vee = [\mathcal{C}, \mathcal{S}]_{\mathrm{cc}}$, this diagram is equivalent to the data of a cocontinuous functor $S : \mathcal{C} \to \mathcal{P}(I)$. On the other side the functor $c_\bullet : I \longrightarrow \mathcal{C}$ extends to a cocontinuous functor $R : \mathcal{P}(I) \to \mathcal{C}$. The Section 5.1 says that $RS = \mathrm{Id}$. This proves that $\mathcal{C}$ is a cocontinuous retract of $\mathcal{P}(I)$. ☐

Remark 5.1.4. Let $Tw(I^\to) \coloneqq \mathrm{Tot}([-,-]_I)$ be the twisted arrow ∞-category of I. It comes equipped with two opfibrations (source and target) $\mathrm{s} : Tw(I^\to) \to I^{op}$ and $\mathrm{t} : Tw(I^\to) \to I$. The trick of the proof of Theorem 5.1.3 to replace the diagram $\gamma_\bullet : I \to \mathcal{C}^\vee$ by a diagram $\beta^\bullet : I^{op} \to \mathcal{C}^\vee$ can be understood as composing $\gamma_\bullet$ along the cofinal functor of $\mathrm{t} : Tw(I^\to) \to I$ and defining $\beta^\bullet$ as the left Kan extension along $\mathrm{s} : Tw(I^\to) \to I^{op}$.

$$\begin{tikzcd} Tw(I^\to) \arrow[r, "\mathrm{t}"] \arrow[d, "\mathrm{s}"] & I \arrow[r, "\gamma_\bullet"] & \mathcal{C}^\vee \\ I^{op} \arrow[urr, "\beta^\bullet"'] & & \end{tikzcd}$$

5.2 Dualizable stable ∞-categories

We recall some fact about cocomplete stable ∞-categories from [Lur17b, Ch. 1 & Sec. 4.8]. An ∞-category is pointed if it has an initial and a terminal object which are isomorphic. By [?, Cor 1.4.2.27], a pointed ∞-category is stable if and only if the suspension operator Σ is invertible. We shall use this as our definition of stability. We denote by $\mathcal{S}t\mathcal{CAT}_{\mathrm{cc}}$ the full sub-∞-category of $\mathcal{CAT}_{\mathrm{cc}}$ spanned by cocomplete stable ∞-categories and by $\mathcal{S}\mathrm{p}$ the ∞-category of spectra. For an ∞-category $\mathcal{C}$ with a terminal object, we denote $\mathcal{C}^\bullet$ the ∞-category of pointed objects in $\mathcal{C}$. When $\mathcal{C}$ is cocomplete, we have $\mathcal{C}^\bullet = \mathcal{C} \otimes \mathcal{S}^\bullet$ [Lur17b, Example 4.8.1.21]. The colimit $\operatorname{colim} \mathcal{C}^\bullet \xrightarrow{\Sigma} \mathcal{C}^\bullet \xrightarrow{\Sigma} \dots$ in $\mathcal{CAT}_{\mathrm{cc}}$ converges to a pointed ∞-category $\mathcal{S}\mathrm{p}(\mathcal{C})$ where the suspension is inversible. This is the stabilisation of $\mathcal{C}$. The functor $\mathcal{C} \mapsto \mathcal{S}\mathrm{p}(\mathcal{C})$ is left adjoint to the inclusion $\mathcal{S}t\mathcal{CAT}_{\mathrm{cc}} \hookrightarrow \mathcal{CAT}_{\mathrm{cc}}$. The ∞-category of spectra can be defined as $\mathcal{S}\mathrm{p} = \operatorname{colim} \mathcal{S}^\bullet \xrightarrow{\Sigma} \mathcal{S}^\bullet \xrightarrow{\Sigma} \dots$. The cocontinuity properties of the tensor of cocomplete ∞-categories gives the formula $\mathcal{S}\mathrm{p}(\mathcal{C}) = \mathcal{C} \otimes \mathcal{S}\mathrm{p}$ [Lur17b, Example 4.8.1.23].

Theorem 5.2.1 (Lurie [Lur17b])**.** *The inclusion $\mathcal{S}t\mathcal{CAT}_{\mathrm{cc}} \hookrightarrow \mathcal{CAT}_{\mathrm{cc}}$ has a left adjoint given by $\mathcal{S}\mathrm{p} \otimes -$. Moreover, $\mathcal{S}t\mathcal{CAT}_{\mathrm{cc}}$ is stable by the tensor product of $\mathcal{CAT}_{\mathrm{cc}}$ and $\mathcal{S}\mathrm{p}$ becomes the new unit.*

The proof of the following theorem is the same as that of Theorem 5.1.3.

Theorem 5.2.2. *The dualizable objects in $\mathcal{S}t\mathcal{CAT}_{\mathrm{cc}}$ are the retracts of ∞-categories $\mathcal{P}(D) \otimes \mathcal{S}\mathrm{p} = [D^{op}, \mathcal{S}\mathrm{p}]$, for D a small ∞-categories.*

Lemma 5.2.3. *For D a small ∞-category with finite colimits, the ∞-category $\mathcal{I}\mathrm{nd}(D) \otimes \mathcal{S}\mathrm{p}$ is a retract of $\mathcal{P}(D) \otimes \mathcal{S}\mathrm{p}$ by cocontinuous functors.*

Proof. We have $\mathcal{I}\mathrm{nd}(D) \otimes \mathcal{S}\mathrm{p} = [D^{op}, \mathcal{S}\mathrm{p}]^{\mathrm{lex}}$. Using that in a stable ∞-category both limits and colimits commute with finite limits, we get that the inclusion $[D^{op}, \mathcal{S}\mathrm{p}]^{\mathrm{lex}} \subset [D^{op}, \mathcal{S}\mathrm{p}]$ commutes with limits and colimits. In particular, it admits an left adjoint which describe $\mathcal{I}\mathrm{nd}(D) \otimes \mathcal{S}\mathrm{p}$ as a retract by cocontinuous functor of $\mathcal{P}(D) \otimes \mathcal{S}\mathrm{p}$. ☐

Lemma 5.2.4. *For D a small ∞-category, the ∞-categories $\mathcal{P}(D) \otimes \mathcal{S}\mathrm{p} = [D^{op}, \mathcal{S}\mathrm{p}]$ is continuous.*

Proof. We have $\mathcal{S}\mathrm{p} = \mathcal{I}\mathrm{nd}(\mathcal{F}\mathrm{in}\mathcal{S}\mathrm{p})$ where $\mathcal{F}\mathrm{in}\mathcal{S}\mathrm{p}$ is the ∞-category of finite spectra. Therefore, $\mathcal{S}\mathrm{p}$ is continuous ∞-category. Let D be a small ∞-category, we have

$$[D^{op}, \mathcal{S}\mathrm{p}] = [D^{op} \times \mathcal{F}\mathrm{in}\mathcal{S}\mathrm{p}^{op}, \mathcal{S}]^{-,\mathrm{lex}} = \left[(D^{op})^{\mathrm{lex}} \times \mathcal{F}\mathrm{in}\mathcal{S}\mathrm{p}^{op}, \mathcal{S}\right]^{\mathrm{lex},\,\mathrm{lex}} .$$

Using the fact that a functor $(D^{op})^{\mathrm{lex}} \times \mathcal{F}\mathrm{in}\mathcal{S}\mathrm{p} \to \mathcal{S}$ is left exact in each variable if and only if it is globally left exact, we have

$$[D^{op}, \mathcal{S}\mathrm{p}] = \mathcal{I}\mathrm{nd}(D_{\mathrm{rex}} \times \mathcal{F}\mathrm{in}\mathcal{S}\mathrm{p}) .$$

This proves that $[D^{op}, \mathcal{S}\mathrm{p}]$ is continuous. □

Corollary 5.2.5. *A cocomplete stable ∞-category is dualizable if and only if it is presentable and continuous.*

Proof. Using Theorem 5.2.2 and Lemma 5.2.4, we deduce that dualizable stable cocomplete ∞-categories are continuous.

Reciprocally, if $\mathcal{C}$ is presentable and continuous, we use a standard presentation to write $\mathcal{C}$ as a retract by cocontinuous functors of some $\mathcal{I}\mathrm{nd}(D)$ for a small ∞-category D with all colimits. Since $\mathcal{C}$ is stable we have $\mathcal{C} \otimes \mathcal{S}\mathrm{p} = \mathcal{C}$ and $\mathcal{C}$ is also a retract by cococontinuous functors of $\mathcal{I}\mathrm{nd}(D) \otimes \mathcal{S}\mathrm{p}$. The result will be proven if we show that $\mathcal{I}\mathrm{nd}(D) \otimes \mathcal{S}\mathrm{p}$ is dualizable. This is a consequence of Lemma 5.2.3. □

Let $\mathbf{X}$ be an ∞-topos, the ∞-category $\mathcal{S}\mathrm{h}(\mathbf{X})$ is not dualizable in general in $\mathcal{CAT}_{\mathrm{cc}}$: this would require that $\mathcal{P}(\mathcal{S}\mathrm{h}(\mathbf{X})) \to \mathcal{S}\mathrm{h}(\mathbf{X})$ to have a left adjoint, but it is only left exact by Theorem 2.1.4. When $\mathbf{X}$ is exponentiable, the functor $\mathcal{I}\mathrm{nd}(\mathcal{S}\mathrm{h}(\mathbf{X})) \to \mathcal{S}\mathrm{h}(\mathbf{X})$ has a left adjoint, and this is enough for the stabilization $\mathcal{S}\mathrm{h}(\mathbf{X}, \mathcal{S}\mathrm{p})$ to be dualizable as a stable ∞-category.

Lemma 5.2.6. *The stabilization $\mathcal{C} \otimes \mathcal{S}\mathrm{p}$ of a presentable continuous ∞-category $\mathcal{C}$ is presentable and continuous.*

Proof. Using a standard presentation of $\mathcal{C}$, we can describe $\mathcal{C} \otimes \mathcal{S}\mathrm{p}$ as a retract of $\mathcal{I}\mathrm{nd}(D) \otimes \mathcal{S}\mathrm{p}$ by cocontinuous functor:

$$\mathcal{I}\mathrm{nd}(D) \otimes \mathcal{S}\mathrm{p} \underset{\varepsilon \otimes \mathcal{S}\mathrm{p}}{\overset{\beta \otimes \mathcal{S}\mathrm{p}}{\leftrightarrows}} \mathcal{C} \otimes \mathcal{S}\mathrm{p} .$$

By Lemma 5.2.3, $\mathcal{I}\mathrm{nd}(D) \otimes \mathcal{S}\mathrm{p}$ is a retract of $\mathcal{P}(D) \otimes \mathcal{S}\mathrm{p}$ by cocontinuous functors. Since $\mathcal{P}(D) \otimes \mathcal{S}\mathrm{p}$ is continuous by Lemma 5.2.4 so are $\mathcal{I}\mathrm{nd}(D) \otimes \mathcal{S}\mathrm{p}$ and $\mathcal{C}2 \otimes \mathcal{S}\mathrm{p}$. □

Corollary 5.2.7. *The stabilisation functor*

$$\begin{aligned} \mathcal{S}\mathrm{p} \otimes - : \mathcal{T}\mathrm{opos}^{op} &\to \mathcal{S}\mathrm{t}\mathcal{CAT}_{\mathrm{cc}} \\ \mathbf{X} &\longmapsto \mathcal{S}\mathrm{h}(\mathbf{X}) \otimes \mathcal{S}\mathrm{p} \end{aligned}$$

is symmetric monoidal and sends exponentiable objects to dualizable objects.

Proof. The first statement is Proposition 2.4.8. The second is a direct consequence of Lemma 5.2.6. □

A Coends for ∞-categories

This appendix establishes a few basic formula of manipulation of coends. The theory of coends for ∞-categories has been developed in [Cra10, Gla16, GHN15]. We use a slightly different approach.

A.1 Definition and first properties

Definition A.1.1. The left Kan extension of the mapping space functor $[-,-] : D^{op} \times D \to \mathcal{S}$ along $D^{op} \times D \to \mathcal{P}(D^{op} \times D)$ is called the *coend functor* and is denoted:

$$\int^D : \mathcal{P}(D^{op} \times D) \to \mathcal{S}$$

$$M \longmapsto \int^{d \in D} M(d,d)\,.$$

We have $\mathcal{P}(D^{op} \times D) = \mathcal{P}(D^{op}) \otimes \mathcal{P}(D)$. When $M = G \otimes F$ for some G in $\mathcal{P}(D^{op})$ and F in $\mathcal{P}(D)$, we shall denote $\int^{d \in D} G \otimes F$ by $G \otimes_D F$.

Remark A.1.2. Since the embedding $D^{op} \times D \to \mathcal{P}(D^{op} \times D)$ is generating $\mathcal{P}(D^{op} \times D)$ by colimits, the functor $\int^D$ is cocontinuous.

For any cocomplete ∞-category $\mathcal{C}$, we have $[D \times D^{op}, \mathcal{C}] = \mathcal{P}(D^{op} \times D) \otimes \mathcal{C}$. We can use this to extend the coend functor to bimodules with values in $\mathcal{C}$:

$$\int^D \otimes \mathrm{Id}_{\mathcal{C}} : \mathcal{P}(D^{op} \times D) \otimes \mathcal{C} \longrightarrow \mathcal{C}\,.$$

We shall simply denote by $\int^D$ this functor.

Remark A.1.3. By definition of the coend, we have, for any (c', c) in $D^{op} \times D$, that

$$[c', -] \otimes_D [-, c] = \int^{d \in D} [c', d] \times [d, c] = [c', c]\,.$$

More generally, we have the following formula.

Lemma A.1.4 (Yoneda)**.** *Let $\mathcal{C}$ be a cocomplete ∞-category and $F : D \to \mathcal{C}$, $G : D^{op} \to \mathcal{C}$ two functors. For any c in D, we have*

$$G \otimes_D [-, c] = \int^{d \in D} G(d) \otimes [d, c] = G(c) \qquad \text{and}$$

$$[c, -] \otimes_D F = \int^{d \in D} [c, d] \otimes F(d) = F(c)$$

(where the symbol $\otimes$ is the natural action of $\mathcal{S}$ on $\mathcal{C}$).

Proof. We prove first the result when $\mathcal{C} = \mathcal{S}$. The functor $- \otimes_D - : \mathcal{P}(D^{op}) \times \mathcal{P}(D) \to \mathcal{S}$ is the composition of the cocontinuous functor $\int^D$ with the canonical map $\mathcal{P}(D^{op}) \times \mathcal{P}(D) \to \mathcal{P}(D^{op}) \otimes \mathcal{P}(D) \simeq \mathcal{P}(D^{op} \times D)$ which is cocontinuous in each variable. This proves that is $- \otimes_D -$ is cocontinuous in each variable. We can then apply the fact that $G = \mathrm{colim}_{d \to G \in D^{op}_{/G}} [d, -]$ and Remark A.1.3 to get

$$\begin{aligned} G \otimes_D [-, c] &= \operatorname*{colim}_{d \to G \in D_{/G}} \int^u [d, -] \times [-, c] \\ &= \operatorname*{colim}_{d \to G \in D_{/G}} [d, c] \\ &= G(c)\,. \end{aligned}$$

The proof is similar for the second formula.

We now prove the result for a general $\mathcal{C}$. Let X be an object of $\mathcal{C}$ and $G : D^{op} \to \mathcal{S}$, then $G \otimes X$ is an object of $\mathcal{P}(D) \otimes \mathcal{C} = [D^{op}, \mathcal{C}]$. By definition of $\int^C \otimes \mathrm{Id}_{\mathcal{C}}$ we have

$$\begin{aligned} (G \otimes X) \otimes_D [-, c] &= \left(\int^C G \otimes_D [-, c] \right) \otimes X \\ &= G(c) \otimes X \\ &= (G \otimes X)(c)\,. \end{aligned}$$

This proves the result for pure tensors. The general result follows by cocontinuity of $\int^C \otimes \mathrm{Id}_{\mathcal{C}}$ and the fact that pure tensors generate $\mathcal{P}(D)\otimes\mathcal{C}$ by colimits (see Remark 2.4.3). ☐

Remark A.1.5. More generaly, we have the formula

$$G\otimes_D F = \operatorname*{colim}_{d\to G\in D_{/G}} F(d)$$

In particular, when G is the constant functor with value 1, we get

$$1\otimes_D F = \operatorname*{colim}_{d\in D} F(d)\,.$$

If moreover F is constant with value X in $\mathcal{C}$, which is equivalent to $M = G\otimes F$ being a constant bimodule, we get $\int^{d:D} M(d,d) = \mathrm{colim}_{c\in D} X = |D|\otimes X$, where $|D| = \mathrm{colim}_D 1$ is the localization groupoid of D.

Lemma A.1.6 (Fubini)**.** *Let C and D be two small ∞-categories and $\mathcal{E}$ be a cocomplete ∞-category. For any functor $F : C^{op}\times C\times D^{op}\times D\to\mathcal{E}$ we have:*

$$\int^{c\in C}\int^{d\in D} F(c,c,d,d)\simeq\int^{d\in D}\int^{c\in C} F(c,c,d,d)\ \ .$$

Proof. The statement is equivalent to the commutation of the square

$$\begin{tikzcd}
{[C^{op}\times C\times D^{op}\times D,\mathcal{E}]} \arrow[r, "\int^D"] \arrow[d, "\int^C"] & \mathcal{P}(C\times C^{op})\otimes\mathcal{E} \arrow[d, "\int^C"] \\
\mathcal{P}(D\times D^{op})\otimes\mathcal{E} \arrow[r, "\int^D"] & \mathcal{E}
\end{tikzcd}$$

which is implied by the functoriality of the tensor $\otimes$ of cocomplete ∞-categories and the equivalences

$$[C^{op}\times C\times D^{op}\times D,\mathcal{E}]\simeq\mathcal{P}(C\times C^{op})\otimes[D^{op}\times D,\mathcal{E}]\simeq\mathcal{P}(D\times D^{op})\otimes[C^{op}\times C,\mathcal{E}]\,.$$

☐

Lemma A.1.7 (Hom formula)**.** *For a bimodule $M : D^{op}\times D\to\mathcal{C}$, the coend of $\int^C M$ is also given by the coend*

$$[-,-]_D\otimes_{D^{op}\times D} M = \int^{(c,d)\in D^{op}\times D}[c,d]\otimes M(c,d)\,.$$

Proof. We have

$$\begin{aligned}
\int^{(c,d)\in D^{op}\times D}[c,d]\otimes M(c,d) &= \int^{c\in D^{op}}\int^{d\in D}[c,d]\otimes M(c,d) && \text{Fubini}\\
&= \int^{c\in D^{op}} M(c,c) && \text{Yoneda}\,.
\end{aligned}$$

☐

Remark A.1.8. From Remark A.1.5 and Lemma A.1.7 one recovers the definition of [GHN15] of the coend of $M : D^{op}\times D\to\mathcal{C}$ as the colimit of the diagram $Tw(D^{\to})\to D^{op}\times D\to\mathcal{C}$.

A.2 Coend with a bimodule

We fix a bimodule $w : C^{op} \times D \to \mathcal{S}$. It classifies a left fibration $\mathrm{Tot}\,(w) \to C^{op} \times D$:

$$\begin{tikzcd} \mathrm{Tot}\,(w) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{S}^{\bullet} \arrow[d] \\ C^{op} \times D \arrow[r] & \mathcal{S} \end{tikzcd}$$

($\mathcal{S}^{\bullet}$ is the ∞-category of pointed spaces). The objects of $\mathrm{Tot}\,(w)$ are triplets (c, d, α) where α is in $w(c, d)$). Morphisms $(c, d, \alpha) \to (c', d', \alpha')$ are pairs $(u : c' \to c, v : d \to d')$ such that

$$\begin{tikzcd} c \arrow[r, "\alpha"] & d \arrow[d, "v"] \\ c' \arrow[u, "u"] \arrow[r, "\alpha'"] & d' . \end{tikzcd}$$

For any c in C and d in D, we define the ∞-categories $\mathrm{Tot}\,(w)_{/d}$ and $\mathrm{Tot}\,(w)_{c/}$ by the following pullbacks

$$\begin{tikzcd} \mathrm{Tot}\,(w)_{/d} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Tot}\,(w) \arrow[d] \\ C^{op} \times \{d\} \arrow[r] & C^{op} \times D \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{Tot}\,(w)_{c/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Tot}\,(w) \arrow[d] \\ \{c\} \times D \arrow[r] & C^{op} \times D . \end{tikzcd}$$

In other words, $(\mathrm{Tot}\,(w)_{/d})^{op}$ and $\mathrm{Tot}\,(w)_{c/}$ are the ∞-categories of elements of the functors $w(-, d) : C^{op} \to \mathcal{S}$ and $w(c, -) : D \to \mathcal{S}$.

Proposition A.2.1. *Let $\mathcal{C}$ be a cocomplete ∞-category and $F : D \to \mathcal{C}$, $G : C^{op} \to \mathcal{C}$ two functors. For any c in D, we have*

$$w(c, -) \otimes_{D^{op}} F = \int^{d \in D^{op}} w(c, d) \times F(d) = \underset{c \to d \in (\mathrm{Tot}(w)_{c/})^{op}}{\mathrm{colim}} F(d)$$

and for any $d \in D$, we have

$$G \otimes_{C^{op}} w(-, d) = \int^{c \in C^{op}} G(d) \times w(d, c) = \underset{c \to d \in \mathrm{Tot}(w)_{/d}}{\mathrm{colim}} G(c) .$$

Proof. Using the fact that $\mathrm{Tot}\,(w)_{c/}$ is the ∞-category of elements of the functor $w(c, -)$, i.e. that

$$D_{/w(c,-)} = \mathrm{Tot}\,(w)_{c/}$$

in $\mathcal{C}\mathrm{at}_{/D}$, we get

$$w(c, -) = \underset{(D_{/w(c,-)})^{op}}{\mathrm{colim}} [d, -] = \underset{(\mathrm{Tot}(w)_{c/})^{op}}{\mathrm{colim}} [d, -]$$

The first formula follows from Lemma A.1.4. The proof of the second is similar. $\square$

A.3 The adjunction coend-end

Let $\mathcal{C}$ be a complete ∞-category and D a small ∞-category. We define the end of a bimodule $M : D^{op} \times D \to \mathcal{C}$ with the formula adjoint to that of Lemma A.1.7.

Definition A.3.1. The *end* functor

$$\int_D : \mathcal{P}(D^{op} \times D) \otimes \mathcal{C} \to \mathcal{C}$$

is defined by the hom of bimodule

$$\int_{d\in D} M(d,d) := [[-,-]_D\,, M]_{D^{op}\times D}$$

where, for an ∞-category C, $[-,-]_C$ is the space of maps in the ∞-category C. The formula make sense since $[D^{op}\times D, \mathcal{C}]$ is cotensored over $\mathcal{S}$ because $\mathcal{C}$ is.

For two functors $G: D\to\mathcal{S}$ and $F: D\to\mathcal{C}$, we define also the *end* of G and F by

$$\int_{d\in D} [G,F]$$

where $[G,F](c,d) = [G(c), F(d)]$ is the cotensor $\mathcal{S}^{op}\times\mathcal{C}\to\mathcal{C}$.

Remark A.3.2. The opposite of the end functor is in fact a coend functor with values in the cocomplete ∞-category $\mathcal{C}^{op}$.

$$\left(\int_D\right)^{op} = \int^{D^{op}} : [D^{op}\times D, \mathcal{C}]^{op} = [D\times D^{op}, \mathcal{C}^{op}] \to \mathcal{C}^{op}\,.$$

We deduce a Yoneda formula

$$\int_{d\in D} [[c,d]_D\,, F(d)] = F(c)$$

and Fubini formula.

Also, for two functors $F, G: D\to\mathcal{S}$, the end $\int_{d\in D}[G,F]$ coincides with the space of maps in $[D,\mathcal{S}]$: indeed, on one side the space of maps satisfies

$$[G,F]_D = \left[\operatorname*{colim}_{D^{op}/G}[d,-]\,, F\right] = \lim_{D^{op}/G} F(d)$$

and on the other side, we have by Yoneda

$$\int_{d\in D}[G,F] = \lim_{D^{op}/G}\int_{d\in D}[[d,-]\,, F] = \lim_{D^{op}/G} F(d)\,.$$

Remark A.3.3. Dual to Remark A.1.5, if the bimodule $M: D^{op}\times D\to\mathcal{S}$ is constant in the second variable, the end of M coincides with its limit

$$\int_{d\in D} M(d) = \lim_{c\in D} M(c)\,.$$

And if M is constant in both variables, we have $\int_{d:D} M(d) = M^{|D|}$, where $|D| = \operatorname{colim}_D 1$ is the localization groupoid of D.

Proposition A.3.4 (Adjunction coend-end)**.** *Let $\mathcal{C}$ be a complete and cocomplete ∞-category. Given a bimodule $w: C^{op}\times D\to\mathcal{S}$, and two functors $F: D^{op}\to\mathcal{C}$ and $G: C^{op}\to\mathcal{C}$, there exists a natural isomorphism*

$$[w\otimes_{D^{op}} F, G]_{C^{op}} = [F, [w,G]_{C^{op}}]_D\,.$$

The formula is also valid when F has values in $\mathcal{S}$.

Proof. By the (co)continuity properties of the coend and the end, it is sufficient to prove this for $F = [-,d]\otimes X$. This is consequence of the Yoneda formula for coend and end.

$$\begin{aligned}
[w\otimes_{D^{op}}[-,d]\otimes X, G]_{C^{op}} &= [w(-,d)\otimes X, G]_{C^{op}}\\
&= [X, [w,G]_{C^{op}}(d)]_{\mathcal{C}}\\
&= \left[X, [[-,d]\,, [w,G]_{C^{op}}]_D\right]_{\mathcal{C}}\\
&= [[-,d]\otimes X, [w,G]_{C^{op}}]_D\,.
\end{aligned}$$

The last statement is proven by removing the X in the above computation. ∎

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