Topo-logie

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Abstract

We claim that Grothendieck topos theory is best understood from a dual algebraic point of view. We are using the term *logos* for the notion of topos dualized, i.e. for the category of sheaves on a topos. The category of topoi is here defined to be the opposite of that of logoi. A logos is a structure akin to commutative rings and we detail many analogies between the topos–logos duality and the duality between affine schemes and commutative rings.

Contents

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1 A walk in the garden of topology

This text is an introduction to topos theory. Our purpose is to sketch some of the intuitive ideas underlying the theory, not to give a systematic exposition of it. It may serve as a complement to the formal expositions that can be found in the literature. We are using examples to illustrate many ideas. We have tried to make the text both accessible to a reader unfamiliar with the theory and interesting for more familiar readers. Certain points of view presented here are non-standard, even among experts, and we believe they should be more widely known.

The rest of this introduction explains how to compare topoi with more classical notions of spaces. It is aimed to be a summary of the rest of the text, where the same ideas will be detailed. In accordance with the theme of this book, we have limited this text to present topoi as a kind of spatial object. Unfortunately, the important relation of topoi with logic will not be dealt with as it should here. We have only made a few remarks here and there. Doing more would have required a much longer text.

1.1 Topoi as spaces

From sheaves to topoi The notion of topos was invented by Grothendieck's school of algebraic geometry in the 1960s. The motivation was Grothendieck's program for solving the Weil conjectures. An important step was the constructions of étale cohomology and l-adic cohomology for schemes. The methods to do so relied heavily on sheaf theory as previously developed by Cartan and Serre after Leray's original work. A central notion was that of *étale sheaf*, a new notion of sheaf in two aspects:

– an étale sheaf was defined as a contravariant functor on a *category*, rather than on the partially ordered set of open subsets of a topological space;

– the sheaf condition was formulated in term of *covering families* that could be chosen quite arbitrarly.

A *site* was defined to be a category equipped with a notion of covering families. Grothendieck and his collaborators eventually realized that the most important properties of a site depended only on the structure of the associated category of sheaves, for which sites were merely presentations by generators and relations [5, IV.0.1]. This structure was baptized *topos*, and an axiomatization was obtained by Jean Giraud. The name was chosen because a number of classical topological constructions (pasting, localizing, coverings, étale maps, bundles, fundamental groups, etc.) could be generalized from categories of sheaves on topological spaces to these abstract categories of sheaves. As a result, new objects, such as the category Set^G of actions of a group G or presheaves categories $\mathcal{P}\mathrm{r}(C) = [C^{op}, \mathsf{Set}]$, could be thought as spatial objects. In the introduction of the chapter on topoi of [5], the authors wrote clearly their ambition for these new types of spaces:

> "Exactly as the term topos itself suggests, it seems reasonable and legitimate to the authors of the current Seminar to consider that the object of Topology is the study of topoi (and not merely topological spaces)."

It is the purpose of this text to explain how topoi can be thought as spaces. The following differences with topological spaces will be our starting point.

– The points of a topos are the objects of a *category* rather than the elements of a mere set. In particular, a central object of the theory is the topos $\mathbf{A}$ whose category of points $\mathcal{P}\mathrm{t}(\mathbf{A})$ is the category Set of sets.

– A topos $\mathbf{X}$ is not defined by means of a "topology" structure on its category of points $\mathcal{P}\mathrm{t}(\mathbf{X})$. Rather, it is defined by its *category of sheaves* $\mathcal{S}\mathrm{h}(\mathbf{X})$, which are the continuous functions on $\mathbf{X}$ with values in $\mathbf{A}$.

A category of points Recall that the set of points of a topological space can be enhanced into a pre-order by the specialization relation.[1] The morphisms in the category of points of a topos must also be thought

[1] For two points x and y of a space X, x is a specialization of y if any open subset containing x contains y, or, equivalently, if $\overline{x} \subset \overline{y}$, where $\overline{x}$ is the closure of $\{x\}$. This relation is a pre-order $x \le y$. A space X is called T_0 if this preorder is an order and T_1 if this preorder coincides with equality of points. Any Hausdorff space is T_1.

as specializations. Topological spaces with a non trivial specialization order are non-separated. Somehow, a topos with a non trivial category of points corresponds to an even more extreme case of non-separation since points can have several ways to be specializations of each other, or even be their own specializations!

We already mentioned that the theory contains a topos $\mathbf{A}$ whose category of points is the category Set of sets. Another example of a topos with a non trivial category of points is given by the topos $\mathbf{B}G$ such that $\mathcal{S}\mathrm{h}(\mathbf{B}G) = \mathsf{Set}^G$ is the category of actions of a discrete group G. The category of points of this topos is the group G viewed as a groupoid with one object. A necessary condition for a topos to be a topological space is that its category of points be a poset. Both $\mathbf{A}$ and $\mathbf{B}G$ are then examples of topoi that are not topological spaces.

Having a category of points will allow the existence of topoi whose points can be the category of groups, or the category of rings, or of local rings or many other algebraic structures. Topoi can be used to represent certain moduli spaces and this is an important source of topoi not corresponding to topological spaces. This relation to classifying spaces is also an important part of the relation with logic.

Let Topos be the category of topoi. Behind the fact of having a category of points is the more general fact that the collection of morphisms $\mathrm{Hom}_{\mathsf{Topos}}(\mathbf{Y},\mathbf{X})$ between two topoi naturally forms a category. For example, when $\mathbf{Y} = \mathbf{1}$ is the terminal topos, we get back $\mathcal{P}\mathrm{t}(\mathbf{X}) = \mathrm{Hom}_{\mathsf{Topos}}(\mathbf{1},\mathbf{X})$, and when $\mathbf{X} = \mathbf{B}G$, the category $\mathrm{Hom}_{\mathsf{Topos}}(\mathbf{Y},\mathbf{B}G)$ can be proved to be the groupoid of G-torsors over $\mathbf{Y}$. So categories of points go along with the fact that Topos is a 2-category.

The evolution of the collection of points from a set to a poset to a category, and even to an ∞-category in the case of ∞-topoi, is part of a hierarchy of spatial notions (summarized in Table 1) that we are going to present.

Table 1: Types of spaces and their points

Type of space	Top. space	Locale	Topos	∞-Topos
Points	a set	a pre-order	a category	an $(\infty,1)$-category

Locales and frames In opposition to topological spaces, the points of topoi have in fact a secondary role. Topological spaces are defined by the structure of a *topology* on their set of points, but topoi are not defined in such a way.[2] In fact, we shall see that topos theory allows the existence of nonempty topoi with an empty category of points.

To understand the continuity of definition between topological spaces and topoi, we will require the slight change of perspective on what is a topological space given by the theory of locales. This theory is based on the fact that most features of topological spaces depend not so much on their set of points but only on their poset of open subsets (which we shall call *open domains* to remove the reference to the set of points). The open domains of a topological space X form a poset $\mathcal{O}(X)$ with arbitrary unions, finite intersections, and a distributivity relation between them. Such an algebraic structure is called a *frame*. A continuous map $f : X \to Y$ induces a morphism of frames $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$, that is, a map preserving order, unions and finite intersections. The opposite of the category of frames is called the category of *locales*. The functor sending a topological space X to its frame $\mathcal{O}(X)$ produces a functor $\mathsf{Top} \to \mathsf{Locale}$. We shall see in Section 2.2.13 how this functor corresponds in a precise way to forget the data of the underlying set of points of the topological space. The theory of locales is sometimes called *point-free topology* for this reason.

The structure of a frame is akin to that of a commutative ring: the union plays the role of addition, the intersection that of multiplication and there is a distributivity relation between the two. The definition of a locale as an object of the opposite category of frames is akin to the definition of an affine scheme as an

[2] Topos theory has the notion of a Grothendieck topology on a category. It is unfortunate that the name suggests the notion of a topology on a set, but this is actually something of a completely different nature.

object of the opposite category of commutative rings. The analogy goes even further since the frame $\mathcal{O}(X)$ can be realized as the set of continuous functions from X to the *Sierpiński space* $\mathbf{S}$.[3] This space plays a role analoguous to that of the affine line $\mathbb{A}^1$ in algebraic geometry: $\mathbb{A}^1$ is dual to the free ring $\mathbb{Z}[x]$ on one generator and similarly $\mathbf{S}$ is dual to the free frame $\underline{2}[x]$ on one generator. This analogy shows that replacing topological spaces by locales is a way to define spaces as dual to some "algebras" of continuous functions.

Topoi & logoi Although this is not its classical presentation, we believe that topos theory is best understood similarly from a dual algebraic point of view. We shall use the term *logos* for the algebraic dual of a topos.[4] A logos is a category with (small) colimits and finite limits satisfying some compatibility relations akin to distributivity (see Section 3.3 for a detailed account on this idea). A morphism of logoi is a functor preserving colimits and finite limits. The category of topoi is defined to be the opposite of that of logoi (see Section 3.1 for a precise definition).

Table 2 presents the analogy of structure between the notions of logos, frame and commutative ring. The general idea of a duality between geometry and algebra goes back to Descartes in his Geometry where geometric objects are constructed by algebraic operations. The locale–frame and topos–logos dualities are instances of many dualities of this kind, as shown in Table 3.[5]

Table 2: Ring-like structures

Algebraic structure	Commutative ring	Frame	Logos
Addition	$(+,0)$	$(\bigvee,0)$	(colimits, initial object)
Product	$(\times,1)$	$(\wedge,1)$	(finite limits, terminal object)
Distrib.	$a(b+c)=ab+bc$	$a\wedge\bigvee b_i=\bigvee a\wedge b_i$	universality and effectivity of colimits
Initial algebra	$\mathbb{Z}$	$\underline{2}$	Set
Free algebra on one generator	$\mathbb{Z}[x]=\mathbb{Z}^{(\mathbb{N})}$	free frame $\underline{2}[x]=[\underline{2},\underline{2}]$	free logos $\mathsf{Set}[X]=[\mathsf{Fin},\mathsf{Set}]$
Corresponding geom. object	the affine line $\mathbb{A}^1$	the Sierpiński space $\mathbf{S}$	the topos classifying sets $\mathbf{A}$
General geom. objects	Affine schemes	Locales	Topos

[3]The Sierpiński space $\mathbf{S}$ is the topology on $\{0,1\}$ where $\{0\}$ is closed and $\{1\}$ is open. A continuous map $X\to\mathbf{S}$ is an open–closed partition of X. The correspondence $C^0(X,\mathbf{S})=\mathcal{O}(X)$ associates to an open domain its characteristic function.

[4]The formal dual of a topos has been introduced by several authors. S. Vickers called the notion a *geometric universe* in [40] and M. Bunge and J. Funk call them *topos frames* in [7]. Our choice of terminology is motivated by the play on the word topo-logy. It also resonates well with topos, and with the idea that a logos is a kind of logical doctrine.

In practice, the manipulation of topoi forces one to jump between the categories Topos (where the morphisms are called *geometric morphisms*) and $\mathsf{Topos}^{\mathsf{op}}$ (where the morphisms are called *inverse images* of geometric morphisms). It is a source of confusion that the same name of "topos" is used to refer to a spatial object and to the category of sheaves on this space. Rather than distinguishing the categories by different names for their morphisms, we have preferred to give different names for the objects.

[5]The structural analogy between topos/logos theory and affine schemes/commutative rings has been a folkloric knowledge among experts for a long time. However, this point of view is conspicuously absent from the main references of the theory. When it is mentioned in the literature, it is only as a small remark.

Table 3: Some dualities

Geometry	*Algebra*	*Dualizing object (gauge space* $\mathbf{A}$*)*
Stone spaces	Boolean algebras	the boolean values $\mathbb{2} = \{0,1\}$
compact Hausdorff spaces	commutative $\mathbb{C}^\star$-algebras	the complex numbers $\mathbb{C}$
affine schemes	commutative rings	the affine line $\mathbb{A}^1$
locales	frames	the Sierpiński space $\mathbf{S}$
topoi	logoi	the topos $\mathbf{A}$ of sets
∞-topoi	∞-logoi	the ∞-topos $\mathbf{A}_\infty$ of ∞-groupoids

Functions with values in sets The analogue in the theory of topoi of the Sierpiński space $\mathbf{S}$, and of the affine line $\mathbb{A}^1$, is the *topos of sets* $\mathbf{A}$ (also known as the *object classifier*). The corresponding logos is the functor category $\mathsf{Set}[X] := [\mathsf{Fin}, \mathsf{Set}]$ where Fin is the category of finite sets. We said that the category of points of $\mathbf{A}$ is the category of small sets. It is an object difficult to imagine geometrically, but, algebraically, it corresponds simply to the *free logos* on one generator and we shall see in Table 10 that it has many similarities with the ring of polynomials in one variable $\mathbb{Z}[x]$.

The functions on a topos with values in $\mathbf{A}$ correspond to *sheaves of sets*. The notion of sheaf on a topological space depends only on the frame of open domains and can be generalized to any locale. The category of sheaves of sets $\mathcal{S}\mathrm{h}(X)$ on a locale X is a logos. This provides a functor $\mathsf{Locale} \to \mathsf{Topos}$. This functor is fully faithful and the topoi in its image are called *localic*. It can be proved that $\mathcal{S}\mathrm{h}(X)$ is equivalent to the category of morphisms of topoi $X \to \mathbf{A}$. Intuitively, the function corresponding to a sheaf F sends a point of X to the stalk of F at this point.[6] More generally, we shall see in (Sheaves as functions) that the logos $\mathcal{S}\mathrm{h}(\mathbf{X})$ dual to a topos $\mathbf{X}$ can always be reconstructed as $\mathcal{S}\mathrm{h}(\mathbf{X}) = \mathrm{Hom}_{\mathsf{Topos}}(\mathbf{X}, \mathbf{A})$. The morphism $\chi_F : \mathbf{X} \to \mathbf{A}$ corresponding to a sheaf F in $\mathcal{S}\mathrm{h}(\mathbf{X})$ is called its *characteristic function*.

Finally, in the same way that locales are spatial objects defined by means of their frame of functions into the Sierpiński space, topoi can be described as those spatial objects that can be defined by means of their logos of functions into the topos of sets.

Étale domains Sheaves of sets have a nice geometric interpretation as *étale domains* (or local homeomorphisms). Given a topos $\mathbf{X}$ and an object F in the corresponding logos $\mathcal{S}\mathrm{h}(\mathbf{X})$, the slice category $\mathcal{S}\mathrm{h}(\mathbf{X})_{/F}$ is a logos and the pullback along $F \to 1$ defines a logos morphism $f^* : \mathcal{S}\mathrm{h}(\mathbf{X}) \to \mathcal{S}\mathrm{h}(\mathbf{X})_{/F}$. The corresponding morphism of topoi $\mathbf{X}_F \to \mathbf{X}$ is called *étale*. An étale domain of $\mathbf{X}$ is an étale morphism with codomain $\mathbf{X}$. We shall see in Section 3.2.6 that any morphism of topoi $F : \mathbf{X} \to \mathbf{A}$ corresponds uniquely to a morphism of topoi $\mathbf{X}_F \to \mathbf{X}$ (where $\mathcal{S}\mathrm{h}(\mathbf{X}_F) = \mathcal{S}\mathrm{h}(\mathbf{X})_{/F}$). This construction generalizes the construction of the *espace étalé* of a sheaf by Godement [12, II.1.2].

The Sierpiński space $\mathbf{S}$, when viewed as a topos, can be proved to be a subtopos of $\mathbf{A}$. At the level of points, the embedding $\mathbf{S} \hookrightarrow \mathbf{A}$ corresponds to the embedding of $\{\varnothing, 1\} \hookrightarrow \mathsf{Set}$. A particular kind of étale domain of a topos $\mathbf{X}$ are then the *open domains*: they are the one whose characteristic function takes values in $\mathbf{S}$. Intuitively, they are the sheaves whose stalks are either empty or a singleton. Table 4 summarizes the situation.

[6] This result a way to formalize the intuitive idea that a sheaf of sets on a space should be a continuous family of sets (the family of its stalks).

Table 4: Sheaves on a topos

Geometric interpretation	*Algebraic interpretation*
Étale domains $\mathbf{X}_F \to \mathbf{X}$	Functions $\mathbf{X} \to \mathbf{A}$ to the topos of sets
Open domains $\mathbf{X}_U \to \mathbf{X}$	Functions $\mathbf{X} \to \mathbf{S}$ to the Sierpiński subtopos $\mathbf{S} \subset \mathbf{A}$

To have or have not enough functions Behind the idea to capture the structure of a space X by some algebra of functions into some fixed space $\mathbf{A}$, there is the idea that $\mathbf{A}$ is a kind of basic block from which X can be built. We shall say that a space X has *enough functions into* $\mathbf{A}$ if X can be written as a subspace $X \hookrightarrow \mathbf{A}^N$ of some power of $\mathbf{A}$.[7]

This notion makes sense in a variety of contexts. For example, a locale X has always enough maps into the Sierpiński space $\mathbf{S}$: the canonical evaluation map $ev : X \times C^0(X, \mathbf{S}) \to \mathbf{S}$ define a morphism of locales $X \to \mathbf{S}^{C^0(X,\mathbf{S})}$ that can be proved to be an embedding. Not every space (or locale) has enough maps into $\mathbb{R}$, but topological manifolds do and can be written as subspaces in some $\mathbb{R}^N$.[8] In the setting of algebraic geometry, affine schemes are precisely defined as the subobjects of affine spaces $\mathbb{A}^N$, that is, they are defined so that they have enough functions with values in $\mathbb{A}^1$. The fact that not every scheme is affine (like projective spaces) says that not all schemes have enough functions with values in $\mathbb{A}^1$. Finally, topoi can be proved to have enough maps in the topos $\mathbf{A}$.[9] However, not every topos has enough maps to the Sierpiński topos $\mathbf{S}$; only the localic topoi do.

This idea of having enough functions to some "gauge space" $\mathbf{A}$ is fundamental for all the dualities of Table 3. One of the main ideas behind the definition of topoi is that the Sierpiński gauge is not always enough: some spatial objects (such as the topoi $\mathbf{A}$ or $\mathbf{B}G$, or bad quotients such as $\mathbb{R}/\mathbb{Q}$) do not have enough open domains to be faithfully reconstructed from them. One needs to choose a larger gauge than $\mathbf{S}$ to capture those spaces. Topoi can—and must—be understood as those spatial objects that can be reconstructed from the gauge given by $\mathbf{A}$, that is, spaces with enough étale domains.

Such a perspective on topoi raises the question of the existence of types of spaces beyond topoi, spaces that would not have enough étale domains. The answer is positive and it is one of the motivation for the introduction of ∞-topoi and stacks (see Section 4 and [1, 27]). For now, let us only say that ∞-topoi and ∞-logoi are higher categorical analogues of topoi and logoi where the role of the 1-category of sets is played by the ∞-category of ∞-groupoids. Table 5 summarizes different kinds of spaces.

To have or have not enough points The theory of locales is famous for providing nonempty locales that have an empty poset of points (we shall give examples in Section 2.2.7). A fortiori, there exist nonempty topoi without any points.

The classical intuition of topological spaces, rooted in the ambient physical space, does not make it easy to imagine non-separated spaces. But even more difficult is to imagine nonempty topoi or locales without any points. This seems to contradict all the common sense of topology. However, this phenomenon becomes understandable if we compare it with the more common fact of the existence of polynomial with no rational roots. We shall detail this a bit in Section 2.2.9.

A locale is said to have *enough points* if two open domains can be distinguished by the points they contain. A locale with enough points can be proved to be the same thing as a sober topological space. Similarly, a

[7] The proper definition is that X can be written as the limit of some diagram of maps between copies of $\mathbf{A}$, but the approximate definition will suffice for our purpose here.

[8] Since $\mathbb{R}$ is separated, non-separated spaces (like the Sierpiński space) cannot embed faithfully in some $\mathbb{R}^N$. The locales with enough maps to $\mathbb{R}$ are the completely regular ones [17, Chapter IV].

[9] This is somehow the meaning of the statement that any topos is a subtopos of a presheaf topos. For a more precise statement, see the examples in Section 3.2.3.

Table 5: Types of spaces – 1

Given a space X, maps $Y \to X$ which are	are continuous functions on X with values in	They are also called	They form	which is called a	A space with enough of them is called
open immersions	the Sierpiński space $\mathbf{S}$.	open domains.	a poset	frame.	a locale.
étale (local homeomorphisms)	the space $\mathbf{A}$ of sets.	étale domains, or sheaves.	a 1-category	logos.	a topos.
∞-étale	the space $\mathbf{A}_\infty$ of ∞-groupoids.	∞-sheaves, or stacks.	an ∞-category	∞-logos.	an ∞-topos.

topos is said to have enough points if two sheaves can be distinguished by the family of their stalks (see Section 3.2.10). Intuitively, a topos $\mathbf{X}$ (or a locale) with enough points can be equipped with a surjection $\coprod_E 1 \twoheadrightarrow \mathbf{X}$ from a union of points.[10] In practice, most topoi have enough points. This is the case of $\mathbf{A}$, of $\mathbf{B}G$, of bad quotients such as $\mathbb{R}/\mathbb{Q}$, of presheaves topoi, of Zariski or étale spectra of rings, and of topoi classifying models of algebraic theories. Moreover, since any topos can always be embedded in a presheaf topos, any topos is always a subtopos of a topos with enough points.

Are topoi really spaces? Our excursion in the topological side of topoi has led us to distinguish different kinds of spatial objects summarized in Table 6. The discovery that topology is richer than the simple study

Table 6: Types of spaces – 2

Space with	enough open domains	enough étale domains	enough higher étale domains	maybe not enough higher étale domains
enough points	topological space	topos with enough points	∞-topos with enough points	beyond...
maybe not enough points	locale	topos	∞-topos	

of topological spaces is extraordinary. But after all these considerations, it is difficult not to question what a space is. Since we have removed points and open domains—the two fundamental features on which the notion of topological space is classically based—as defining characteristics of spaces, what is left of the intuition of what a space should be? And why should we agree to consider these news objects as spaces?

The best answer that we can propose—and that we will develop in the rest of this text—is that the intuition of space is in fact forged in a set of specific operations on spaces (e.g., covering, pasting, quotienting, localizing, intersecting, crossing, deforming, direct image, inverse image, homotopy, (co)homology, etc.),

[10] The two problems of having enough points $1 \to X$ or enough functions $X \to \mathbf{A}$ are somehow dual. In both cases, the question is how much of X can be "reconstructed" from some "gauge" given by mapping *from* a given object (the point) or *to* a given object (the space of coordinates). An object has enough points if it admits a surjection from a union of points. An object has enough functions if it admits an embedding into a product of $\mathbf{A}$.

which leads to distinguishing some classes of spaces (compact, connected, contractible, etc.) and some classes of maps (open immersions, étale maps, submersions, proper maps, bundles, etc.). So far, all of these notions and the structural relations they have between them have been successfully generalized to topoi. Some of them, like quotienting or cohomology, have even gained more regular properties in the context of topoi. So, if all the tools, language, and structural relations of topology make sense for topoi, shouldn't the question rather be, how can we afford not to think them as spaces?

1.2 Other views

Topoi as categories of spaces We have sketched how a logos can be thought dually as a single spatial object. But there exists also the point of view where a logos is thought as a *category* of spatial objects.[11] This point of view is justified by the following example. The category M of manifolds does not have certain quotients (e.g., leaf spaces of foliations are not manifolds in general). So it could be useful to embed M into a larger category where quotients could behave better. This is, for example, the idea behind the notion of *diffeology* [16]. Another implementation is to consider the embedding $M \hookrightarrow \mathcal{S}\mathrm{h}(M)$ into sheaves of sets on M.[12] The embedding $M \hookrightarrow \mathcal{S}\mathrm{h}(M)$ suggests interpreting the objects of $\mathcal{S}\mathrm{h}(M)$ as some kind of generalized manifolds. This is the so-called *functor of points* approach to geometry [39]. Within $\mathcal{S}\mathrm{h}(M)$, "bad quotients" such as the irrational torus $\mathbb{T}^2_\alpha = \mathbb{T}^2/\mathbb{R}$ or even the more bizarre $\mathbb{R}/\mathbb{R}_{dis}$[13] do exist with nice properties. For example, it is possible to define a theory of fundamental groups for these objects and prove that $\pi_1(\mathbb{T}^2_\alpha) = \mathbb{Z}^2$ and $\pi_1(\mathbb{R}/\mathbb{R}_{dis}) = \mathbb{R}_{dis}$.

Other logoi exist in which to embed the category of manifolds M. Synthetic differential geometry uses sheaves on C^∞-rings [22, 29]). Schreiber's approach to geometrization of gauge theories in physics relies on the same ideas but with sheaves of ∞-groupoids [33]. The same idea has also been used in algebraic geometry (where it was actually invented), where the embedding $\{\text{Affine schemes}\} \hookrightarrow \mathcal{S}\mathrm{h}(\{\text{Affine schemes}\}, \text{étale})$ provides a nice setting in which to define several kinds of gluing of affine schemes (general schemes, algebraic spaces). This setting has been useful for dealing with algebraic groups and constructing moduli spaces, such as Hilbert schemes. When sheaves of sets are replaced by sheaves of ∞-groupoids, the embedding $\{\text{Affine schemes}\} \hookrightarrow \mathcal{S}\mathrm{h}_\infty(\{\text{Affine schemes}\}, \text{étale})$ provides a nice setting in which to define Deligne–Mumford and Artin stacks. A variation on this setting involving ∞-logoi is also at the foundation of derived geometry [1].

Topoi and logic The theory of topoi has a logical aspect, discovered by Lawvere and Tierney in the late 1960s, which has been developed into one of its most spectacular and fundamental features. A sheaf is intuitively a family of sets (the family of its stalks). Therefore, it should be clear enough that all the operations and language existing in the category of sets can be transported to sheaves with the idea that they are applied stalk-wise. This is the intuition behind the idea that a logos can be thought as a category of generalized sets.[14] From there, if $\mathbb{T}$ is a logical theory, the notion of model of $\mathbb{T}$ in sets can be extended into that of a model in the generalized sets/objects of a logos. This construction follows the spirit of the interpretation of propositional theories in frames of open domains of topological spaces (in fact, the latter can even be viewed as a particular case of the former). Logoi have provided a rich setting in which to interpret many features of logic, Table 7 gives a rough summary of some. The theory has notably led to independence proofs in set theory [26, VI.2].

[11] A logos $\mathcal{S}\mathrm{h}(X)$ can always be thought as a category of spaces étale over $\mathbf{X}$, but the interpretation we are talking about here is different.

[12] These two examples are actually related. The category Diff of diffeologies can be realized as a full subcategory of $\mathcal{S}\mathrm{h}(M)$, and the embedding $M \hookrightarrow \mathcal{S}\mathrm{h}(M)$ factors through Diff.

[13] The object $\mathbb{R}/\mathbb{R}_{dis}$ is the quotient of $\mathbb{R}$ by the discrete action of $\mathbb{R}$. Classically, it is a single point, but in $\mathcal{S}\mathrm{h}(M)$, a function from a manifold X to $\mathbb{R}/\mathbb{R}_{dis}$ is an equivalent class of families (U_i, f_i), where U_i is an open cover of X, and $f_i : U_i \to \mathbb{R}$ are functions such that the differences $f_i - f_j$ are constant functions on U_{ij}. In more intrinsic terms, a morphism $X \to \mathbb{R}/\mathbb{R}_{dis}$ is the same thing as a closed differential 1-form on X, i.e., it represents the functor $X \mapsto Z^1_{dR}(X, \mathbb{R})$. In the embeddings $M \hookrightarrow \text{Diff} \hookrightarrow \mathcal{S}\mathrm{h}(M)$, the object $\mathbb{R}/\mathbb{R}_{dis}$ is actually an example of a sheaf that is not a diffeology.

[14] The relation of this point of view with the previous one, where a logos is thought as a category of spatial objects, is the matter of Lawvere cohesion theory, central to Schreiber's geometrization of physics [33].

Table 7: Translation logic–logos

Logic	*Logos* $\mathcal{E}$
Terms and types	*Objects and morphisms*
types/sorts **S**	objects $[S]$
variable $s : S$	identity maps $[s] = [S] \xrightarrow{id} [S]$
context $s : S, t : T$ empty context	products $[S] \times [T]$ terminal object $[] = 1$
terms $f(s)$ of type T	maps $[f] : [S] \to [T]$
dependent types $T(s)$	object $[T] \to [S]$ in $\mathcal{E}_{/[S]}$
predicates (dependent booleans) $P(s)$	monomorphisms $[P] \rightarrowtail [S]$
propositions (booleans) p	subterminal object $[p] \rightarrowtail []$
Disjunctive operations	*Colimit constructions*
disjunction $P(s) \vee Q(s)$	union $[P] \cup [Q] \rightarrowtail [S]$
existential quantifier $\exists s\, f(s)$	image of a map $\mathrm{im}\,([f]) : \mathrm{Im}\,([f]) \to [S]$
dependent sums $\sum_{s:S} T(s)$	domain $[T]$ of the map $[T] \to [S]$ interpreting the dependent type $T(s)$
Conjunctive operations	*Limit constructions*
conjunction $P(s) \wedge Q(s)$	intersection $[P] \cap [Q] \rightarrowtail [S]$
implication $P(s) \Rightarrow Q(s)$	Heyting's right adjoint to $[P] \cap -$
universal quantifier $\forall s\, f(s)$	image by the right adjoint to base change of subobjects along $[S] \to []$
function type $S \to T$	internal hom $[T]^{[S]}$
dependent products $\prod_{x:S} T(x)$	image by the right adjoint to base change along $[S] \to []$
Specific types	*Specific objects*
the type of propositions	subobject classifier Ω
modalities on propositions	internal monads $j : \Omega \to \Omega$
the type of types	the object classifier/universe U (only in ∞-logoi)
modalities on types	internal monads $j : U \to U$ (only in ∞-logoi)

If all the constructions of set theory make sense in any logos, the fact that a sheaf is a *continuous* family of sets leads to some differences of behavior. Such differences are already present in the frame semantics of propositional logic, where the logic ceases to be boolean and instead become intuitionist in the sense of Heyting. The logos semantics of logical theories is a fortiori intuitionistic, but there are new features. For example, the fact that not all covering maps have a section says that the axiom of choice can be false.

The logical use of logoi has also modified the notion a bit. The preference of logic for finite operations has led to replace SGA original definition by the so-called *elementary* definition of Lawvere and Tierney. The consideration of internal homs and subobject classifier as being part of the structure of a logos has also led to considering notions of morphisms between logoi different than the original ones (morphisms of locally cartesian closed categories, logical morphisms). From this point of view, the logical notion of topos is not, strictly speaking, the same as the topological one.

Our priority in this chapter is to explain how topoi are spatial objects, and we will unfortunately not say much about the relationship with logic. We have only made a few remarks here and there about classifying topoi for some logical theories. We refer the reader to [19, 26] for a good treatment of classifying topoi and the intimate relationship between logoi and logic.

Higher topoi In the 1970s and 1980s, the construction of moduli spaces led geometers to enhance sheaves of sets into stacks, that is, sheaves valued in groupoids, which were objects of higher categories. Around the same time, it was gradually understood that the objects of algebraic topology (homotopy types, spectra, chain complexes, cobordisms, etc.) were also naturally objects of higher categories. Two types of higher categories have emerged from these considerations: ∞*-topoi* and *stable* ∞*-categories*. The first provides a setting for stacks, that is, sheaves in ∞-groupoids; the second provides a setting for stable homotopy theories, that is, sheaves of spectra.[15]

The theory of ∞-logoi is essentially similar to that of logoi, but with the replacement of the category Set of sets by the ∞-category $\mathcal{S}$ of ∞-groupoids, that is, homotopy types.[16] The category of points of an ∞-topos is an $(\infty, 1)$-category. This allows ∞-topoi to capture more spatial objects than topoi. For example, the analogue of the topos of sets $\mathbf{A}$ is the ∞-topos $\mathbf{A}_\infty$ whose points are ∞-groupoids. As for topoi, an ∞-topos $\mathbf{X}$ is defined dually by its ∞*-logos* $\mathcal{S}\mathrm{h}_\infty(\mathbf{X})$ of functions with values in $\mathbf{A}_\infty$ (see Section 4). Table 8 gives a few correspondences between notions of category and ∞-category theories.

Topos theory is actually having a tremendous renewal with the development of ∞-topos theory. In fact, we believe that, more than a simple higher categorical analog, the notion of ∞-topos is actually an achievement of that of topos. Indeed, the theory of ∞-topoi/logoi turns out to be somehow simpler and more powerful than topos theory:

– it simplifies the descent properties of logoi (see Section 4.2.1);

– it simplifies the treatment of both homotopy theory and homology theory of logoi (see Section 4.2.7 and Section 4.2.8);

– from a logical point of view, ∞-logoi provide a setting where quantification on objects is allowed[17] (see Section 4.2.6).

But also, it contains a number of features absent from the classical theory. A central one is the notion of ∞*-connected objects* (see Section 4.2.4). To explain this, recall that, according to Whitehead theorem, a homotopy type is contractible if and only if its homotopy groups are trivial. Roughly speaking, an object of an ∞-topos is ∞-connected if all its homotopy groups are trivial, but such an object need not be a terminal object.[18] Their existence has several important consequences:

– they limit the power of Grothendieck topologies (not every ∞-logos can be defined from a site; see Section 4.2.5)

[15] A third kind of ∞-category has also emerged, ∞-categories with duals, which provides the proper setting for cobordism theories and extended field theories [6, 24]. We shall not talk about these.

[16] Some motivations for the enhancement $\mathsf{Set} \hookrightarrow \mathcal{S}$ are explained in [1]. See also [31] for some material on ∞-groupoids.

[17] Logoi only provide a setting in which to quantify on arbitrary subobjects, a restriction that is arguably not natural.

[18] It is useful to compare them to nilpotent elements in a ring.

Table 8: correspondence lower/higher category theories

1-*Categories*	$(\infty, 1)$-*Categories*
Sets	∞-groupoids (homotopy types)
Property of equality $a = b$	*Structure* of the choice of an isomorphism (a homotopy) $\alpha : a \simeq b$
Presheaves of sets $\mathcal{P}r(C) = [C^{op}, \mathsf{Set}]$	Presheaves of ∞-groupoids $\mathcal{P}r_\infty(C) = [C^{op}, \mathcal{S}]$
Logos = left exact localizations of $\mathcal{P}r(C)$	∞-Logos = left exact localizations of $\mathcal{P}r_\infty(C)$
Topos of sets $\mathbf{A}$ dual to the free logos $\mathsf{Set}[X] = [\mathsf{Fin}, \mathsf{Set}]$	∞-Topos of ∞-groupoids $\mathbf{A}_\infty$ dual to the free ∞-logos $\mathcal{S}[X] = [\mathcal{S}_{\text{fin}}, \mathcal{S}]$
Abelian groups	Spectra (reduced homology theories), or chain complexes
Abelian categories	Stable ∞-categories

– they create unexpected links between unstable and stable homotopy theories (see Section 4.2.3).

– they give rise to a differential calculus for ∞-logoi related to Goodwillie theory.[19]

None of these properties have analogue nor can be seen in classical topos theory.

It is a good idea to compare the enhancement of Set into $\mathcal{S}$ to that of $\mathbb{R}$ into $\mathbb{C}$. This comparison illustrates both the simplification that is provided by ∞-groupoids (better regularity for some properties) and the new features that can appear (new objects, new methods, etc.), together with the price to pay to leave behind an ancient world of problems and points of view. As complex numbers, so do ∞-groupoids and ∞-logoi offer a new world, both in algebra and geometry. On the geometry side, the new features of ∞-topos theory push the notion of spatial object further than anyone had anticipated (the situation compares to the enhancement of varieties into schemes with their singularities and nilpotent functions). On the algebra side, the interpretation of Goodwillie calculus in ∞-logoi provide a new "topological calculus" where spectra play the role of infinitesimal thickening of the point. These elements of the theory, which are ongoing work of the authors and others, are unfortunately too recent to be part of this report. We mention them only to give a glance at the future of the notion of space.

Further reading About locale theory, good books are [17, 30]. The article [21] contains nice elements of the theory, not in the previous book. About topos theory, two very good books are [18, 26]. For the more experienced user, the two volumes of [19] are unavoidable. About ∞-topos theory, the note [32] contains essential ideas. The main references are [23] and also the appendix of [25]. For an approach closer to what we did here, some material is in [4]. About ∞-category theory, some ideas are explained in some chapters of this volume [1, 27, 31, 35]; otherwise, we refer to the books [8, 20, 23].

2 The locale–frame duality

The purpose of this section is to explain how topology, in parallel of being a theory of geometric objects, can also be understood as the study of some algebraic objects. To each space X is associated its frame

[19] This is an ongoing work of the authors and their collaborators [2, 3].

$\mathcal{O}(X)$ of open domains, which is the same thing as $C^0(X,\mathbf{S})$, the set of continuous functions with values in the Sierpiński space. The frame $\mathcal{O}(X)$ is a ring-like object, and many of the geometric constructions about topological spaces can be formulated algebraically in terms of $\mathcal{O}(X)$. This easy model of an algebraic approach to geometry is a useful step in understanding the definition of a topos.

2.1 From topological spaces to frames

The Sierpiński space $\mathbf{S}$ is defined as the topology on the set $\{0,1\}$ such that 0 is a closed point and 1 an open point. The space $\mathbf{S}$ has an order on its points such that $0 < 1$. This makes it into a poset object in the category Top. If I is a set, then the map $\bigvee : \mathbf{S}^I \to \mathbf{S}$ sending a family to its supremum is continuous for the product topology. Moreover, when I is finite, the map $\wedge : \mathbf{S}^I \to \mathbf{S}$ sending a family to its infimum is also continuous. This presents $\mathbf{S}$ as a topological poset with all suprema and finite infima.

If X is a topological space, a continuous function $f : X \to \mathbf{S}$ is the data of a partition of X into an open subset U (the inverse image of 1) and its closed complement (the inverse image of 0). We shall say that f is the *characteristic function* of U. The set $C^0(X,\mathbf{S})$ of characteristic functions inherits from $\mathbf{S}$ an order relation where $f \le g$ if $f(x) \le g(x)$ for all x in X. The resulting poset structure on $C^0(X,\mathbf{S})$ coincides with the poset $\mathcal{O}(X)$ of open subset of X ordered by inclusion. Moreover, $C^0(X,\mathbf{S}) = \mathcal{O}(X)$ inherits also the algebraic operations of $\mathbf{S}$ where they coincide with the union and finite intersection in $\mathcal{O}(X)$: $(\bigvee f_i)(x) = \bigvee(f_i(x))$ and $(\bigwedge f_i)(x) = \bigwedge(f_i(x))$. This simple construction says an important thing: the algebra of open subsets of a space X can be thought as an algebra of continuous functions on X with values in the Sierpiński space.

The algebraic structure of $\mathcal{O}(X)$ is that of a *frame*: that is, a poset

- with arbitrary suprema $(\bigvee, 0)$,
- finite infima $(\wedge, 1)$,
- satisfying a distributivity condition $a \wedge \bigvee b_i = \bigvee(a \wedge b_i)$.

Given two frames F and F', a morphism of frames $u^* : F \to F'$ is a morphism of posets preserving all suprema and finite infima. The collection of frame morphisms $F \to F'$ is naturally a poset. This makes the category Frame of frames into a 2-category.

There exists a functor

$$\begin{aligned}
\mathcal{O} = C^0(-,\mathbf{S}) : \mathsf{Top}^{\mathsf{op}} &\longrightarrow \mathsf{Frame} \\
X &\longmapsto \mathcal{O}(X) \\
f : X \to Y &\longmapsto f^* : \mathcal{O}(Y) \to \mathcal{O}(X).
\end{aligned}$$

The notion of *locale* is defined as an object of the category $\mathsf{Locale} = \mathsf{Frame}^{\mathsf{op}}$.[20] This permits us to write the previous functor $\mathcal{O}$ as a covariant functor $\mathsf{Top} \to \mathsf{Locale}$. If L is a locale, we denote by $\mathcal{O}(L)$ the corresponding frame. The objects of $\mathcal{O}(L)$ will be called the *open domains* of L. If $f : L \to L'$ is a morphism of locales, we shall denote by $f^* : \mathcal{O}(L') \to \mathcal{O}(L)$ the corresponding morphism of frames.

The functor $\mathsf{Top} \to \mathsf{Locale}$ is not faithful. If X is the indiscrete topology on a set E, then X and the one point space 1 have same image under $\mathcal{O}$. The spaces that can be faithfully represented in Frame are those spaces whose set of points can be reconstructed from the frame of open subsets. They are called *sober* spaces.[21] This functor is not essentially surjective either. A frame F is the frame of open subset of a topological space if and only if there exists an injective frame morphism $F \hookrightarrow P(E)$ into the the power set of a certain set E. We shall see an example of a frame admitting no such embedding in Section 2.2.7.(vii). We shall also see in Section 2.2.13 that the functor $\mathsf{Top} \to \mathsf{Locale}$ is in a very precise way the functor forgetting the data of the set of points.

[20] When Frame is viewed as a 2-category, the 2-category Locale is defined by reversing the direction of 1-arrows only.

[21] We shall not assume, as is sometimes the case when comparing topological spaces to locales, that our topological spaces are sober. We shall explain precisely in Section 2.2.13 how the two notions should be properly compared. We refer to the classical literature for more details on sober spaces [17, 30].

2.2 Elements of locale geometry and frame algebra

The idea is that a locale is a formal geometric dual to the algebraic structure of frame. In other words, locales are spatial objects defined by an abstract algebra of open subsets, without reference to a set of points. The fact that $\mathsf{Locale} = \mathsf{Frame}^{\mathsf{op}}$ is indeed a category of geometric objects is justified by the fact that a number of topological notions and constructions can be transferred along $\mathsf{Top} \to \mathsf{Locale}$. The mechanism is simple: take a topological notion, try to formulate it in terms of the frame of open subsets, then generalize it to any frame.

2.2.1 Punctual and empty locales Let 1 be the one point space and $\varnothing$ the empty space. It is easy to prove that $\mathcal{O}(1) = \underline{2} := \{0 < 1\}$ is the initial object of the category Frame and that $\mathcal{O}(\varnothing) = \underline{1} := \{0\}$ is the terminal object. The corresponding objects in Locale are also denoted by 1 and $\varnothing$ and play the role of the point and the empty space. They are in the image of $\mathsf{Top} \to \mathsf{Locale}$.

2.2.2 Free frames and affine locales The algebraic approach of topology that is locale theory distinguishes a class of topological objects corresponding to the freely generated algebraic objects. Given a poset P, there exists a notion of the free frame $\underline{2}[P]$ on P. The free frame on no generators ($P = \varnothing$) is $\underline{2} := \{0 < 1\}$. It is the initial object of the category Frame, the equivalent of $\mathbb{Z}$ in the category of commutative rings. The free frame on one generator x is $\underline{2}[x] := \{0 < x < 1\}$. It is the equivalent of $\mathbb{Z}[x]$ in the category of commutative rings.

More generally, the free frame on a poset P is constructed as follows: first, one constructs $P^{\wedge}$ the free completion of P for finite intersections, then one freely completes $P^{\wedge}$ for arbitrary unions into a poset $\underline{2}[P] := [(P^{\wedge})^{op}, \underline{2}]$. This last construction is made by taking presheaves with values in $\underline{2}$. The construction of $\underline{2}[P]$ is analogous to that of the free ring on a set E by constructing first the free commutative monoid $M(E)$ on E, and then the free abelian group $\mathbb{Z}.M(E)$ on $M(E)$ (see Section 3.4.1). A frame morphism $\underline{2}[P] \to F$ is then equivalent to the data of a poset morphism $P \to F$.

We shall call $\mathbf{S}^P$ the locale dual to the free frame $\underline{2}[P]$. By analogy with algebraic geometry, the locales $\mathbf{S}^P$ can be called *affine spaces*. The algebraic result that any frame is a quotient of a free frame translates geometrically into the statement that any locale L has an embedding $L \hookrightarrow \mathbf{S}^P$ for some poset P.

Examples of affine locales

(i) The punctual locale is affine $1 = \mathbf{S}^0$. The free frame $\underline{2}$ on no generators is isomorphic to the frame $\mathcal{O}(1)$.

(ii) (The Sierpiński locale) The Sierpiński space is faithfully encoded by its corresponding locale. The frame $\mathcal{O}(\mathbf{S})$ has three elements $\{0 < \{1\} < \{0,1\}\}$. It is isomorphic to the free frame on one generator $\underline{2}[x] := \{0 < x < 1\}$.

(iii) If E is a set, then the frame $\underline{2}[E]$ is the poset of open subsets of the product $\mathbf{S}^E$ of E copies of the Sierpiński space $\mathbf{S}$.

(iv) If P is a poset, the locale dual to $\underline{2}[P]$ is $\mathbf{S}^P$, the "P-power" of $\mathbf{S}$. Recall that the category Locale is enriched over posets. It is in fact also cotensored over posets, and $\mathbf{S}^P$ is the cotensor of the Sierpiński space by P. It has the universal property that a morphism of locales $X \to \mathbf{S}^P$ is equivalent to a morphism of posets $P \to \mathrm{Hom}_{\mathsf{Locale}}(X, \mathbf{S}) = \mathcal{O}(X)$.

2.2.3 Alexandrov locales Let P be a poset. There exists a construction, due to Alexandrov, of a non-separated topology on the set of elements of P such that the specialization order coincides with the order of P. The open subsets for this topology are the upward closed subsets of P, which can be also defined as order-preserving maps $P \to \underline{2}$. The Alexandrov locale of P is the locale $\mathrm{Alex}(P)$ defined by the frame $[P, \underline{2}]$ of poset morphisms from P to $\underline{2}$. There is a canonical map $P \to \mathcal{P}\mathrm{t}(\mathrm{Alex}(P))$ that is injective but not surjective in general.[22] This construction provides a functor $\mathrm{Alex} : \mathsf{Poset} \to \mathsf{Locale}$ that is left adjoint to

[22] The poset $\mathcal{P}\mathrm{t}(A_P)$ is the completion of P for filtered unions, also called the poset of ideals of P (see [17]).

the functor $\mathcal{P}\mathrm{t} : \mathsf{Locale} \to \mathsf{Poset}$. In other words, for a locale X, morphisms $\mathrm{Alex}(P) \to X$ are equivalent to morphisms of posets $P \to \mathcal{P}\mathrm{t}(X)$.

Examples of Alexandrov locales

(i) Any discrete space defines an Alexandrov locale. The open subsets of the discrete topology on a set E do form the frame $P(E) = [E, \underline{2}]$.

(ii) The Sierpiński space is the Alexandrov locale associated to $P = \underline{2} = \{0 < 1\}$, that is, $\mathcal{O}(\mathbf{S}) = \underline{2}[x] = [\underline{2}, \underline{2}]$.

(iii) Let $\underline{n}$ be the poset $\{0 < 1 < \cdots < n-1\}$. A morphism of locales $X \to \mathrm{Alex}(\underline{n})$ is equivalent to the data of a *stratification of depth* n, that is, a sequence $U_{n-1} \subset U_{n-2} \subset \ldots \subset U_0 = X$ of open domains of $\mathbf{X}$.

(iv) The poset $\left[\mathcal{O}(X)^{op}, \underline{2}\right]$ is an Alexandrov frame. The corresponding locale shall be denoted $\widehat{X}$. We shall see that there is an embedding $X \to \widehat{X}$ and that $\widehat{X}$ is a kind of compactification of X.

2.2.4 The poset of points A point of a topological space X is the same thing as a continuous map $x : 1 \to X$. Such a map defines a morphism of frames $x^* : \mathcal{O}(X) \to \underline{2}$. Intuitively, this morphism sends an open subset to 1 if and only if it contains the point. Then, a *point* of a locale L is defined as a morphism $x : 1 \to L$, or equivalently, as a frame morphism $x^* : \mathcal{O}(L) \to \underline{2}$. Since the frame morphisms do form posets, the collection $\mathcal{P}\mathrm{t}(L)$ of all the points is naturally a poset. For two points $x^*, y^* : \mathcal{O}(L) \to \underline{2}$, we shall say that x^* is a *specialization* of y^* when $x^* \le y^*$. Intuitively, this says that any open domain containing x contains also y.

Examples of points

(i) If X is a topological space and $\underline{X}$ the corresponding locale, there is a canonical map $\mathcal{P}\mathrm{t}(X) \to \mathcal{P}\mathrm{t}(\underline{X})$. This map is injective if and only if X is T_0-space and bijective if and only if X is a sober space.

(ii) For a locale L, let $|\mathcal{P}\mathrm{t}(L)|$ be the underlying set of $\mathcal{P}\mathrm{t}(L)$. There is a canonical morphism $\mathcal{O}(L) \to P(|\mathcal{P}\mathrm{t}(L)|)$ that sends an open domain U to the set of points it contains. This defines a natural topology on the set $|\mathcal{P}\mathrm{t}(L)|$. The corresponding functor $\mathsf{Locale} \to \mathsf{Top}$ is right adjoint to the functor $\mathsf{Top} \to \mathsf{Locale}$. The image of this functor is the category of sober spaces. The map $\mathcal{O}(L) \to P(|\mathcal{P}\mathrm{t}(L)|)$ is not injective in general, hence the functor $\mathsf{Locale} \to \mathsf{Top}$ is not fully faithful. When it is injective, the locale is said to have enough points; intuitively, this means that $\mathcal{O}(L)$ is the frame of open domains of a sober space.

(iii) The poset of points of $\widehat{X}$ is the poset of all filters in $\mathcal{O}(X)$. The embedding $X \to \widehat{X}$ send a point of X to the filter of its neighborhoods.

(iv) We shall see in the examples of sublocales that there exist nonempty locales with an empty poset of points.

2.2.5 Open domains Let U be an open subset of a topological space X; then we have a canonical isomorphism of frames $\mathcal{O}(U) = \mathcal{O}(X)_{/U}$ (the slice of $\mathcal{O}(X)$ over U), and the inclusion $U \subset X$ corresponds to the frame morphism $U \cap - : \mathcal{O}(X) \to \mathcal{O}(X)_{/U}$. More generally, for any locale L and any U in $\mathcal{O}(L)$, the map $U \cap - : \mathcal{O}(L) \to \mathcal{O}(L)_{/U}$ is a frame morphism called an *open quotient* of frames. A map $U \to L$ of locales is called an *open embedding* if the corresponding map of frames is an open quotient. The class of open embedding is compatible with the classical topological notion: if X is a topological space and $U \to X$ is an open embeddings in Locale, then U can be proved to be an open topological subspace of X.

Examples of open domains

(i) The inclusion $\{1\} \hookrightarrow \mathbf{S}$ is an open embedding.

(ii) It is, in fact, the universal open embedding. Given an open embedding $U \hookrightarrow X$ of a locale X, there exists a unique morphism of locales $\chi_U : X \to \mathbf{S}$ inducing a cartesian square

$$\begin{tikzcd} U \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & \{1\} \arrow[d, hook] \\ X \arrow[r, "\chi_U"] & \mathbf{S}. \end{tikzcd}$$

The morphism of frames $\underline{2}[x] \to \mathcal{O}(X)$ corresponding to the characteristic function $\chi_U : X \to \mathbf{S}$ is the unique frame morphisms sending x to U.

2.2.6 Closed embeddings Let $U \subset X$ be an open subset of a topological space X and Z its closed complement. There is a canonical isomorphism of frames $\mathcal{O}(Z) = \mathcal{O}(X)_{U/}$ (the coslice of $\mathcal{O}(X)$ under U, that is, the poset of opens domains containing U) and the inclusion $Z \subset X$ corresponds to the frame morphism $U \cup - : \mathcal{O}(X) \to \mathcal{O}(X)_{U/}$. In general, for any open domain U of a locale L, the map $U \cup - : \mathcal{O}(L) \to \mathcal{O}(L)_{U/}$ is a frame morphism called a *closed quotient* of frames. A map $U \to L$ of locales is called a *closed embedding* if the corresponding map of frames is a closed quotient.

Examples of closed embeddings

(i) The inclusion $\{0\} \hookrightarrow \mathbf{S}$ is an closed embedding.

(ii) It is, in fact, the universal closed embedding. Given a closed embedding $Z \to X$, there exists a unique morphism of locales $X \to \mathbf{S}$ inducing a cartesian square

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \{0\} \arrow[d] \\ X \arrow[r, "\chi_Z"] & \mathbf{S}. \end{tikzcd}$$

The morphism of frames $\underline{2}[x] \to \mathcal{O}(X)$ corresponding to the characteristic function $\chi_Z : X \to \mathbf{S}$ is the unique frame morphisms sending x to the open complement U of Z.

2.2.7 Sublocales & frame quotients Let $Y \subset X$ be an inclusion of topological spaces; then the corresponding frame morphism $\mathcal{O}(X) \to \mathcal{O}(Y)$ is surjective.[23] A morphism of frames is called a *quotient* if it is surjective. A morphism of locales $L' \to L$ is called an *embedding*, or a *sublocale*, if the corresponding map of frames is a quotient.

Quotients can be generated in several ways. For example, given any inequality $A \le B$ in F, there exists a unique quotient $F \to F /\!\!/ (A = B)$ forcing the inclusion to become an identity. This is the analogue for frames of the quotient of a commutative ring A by a relation $a = b$ for two elements a and b of A. Any quotient can be generated by forcing a set of inequalities to become equalities.[24]

For any frame quotient $q^* : F \to F'$, there exists a right adjoint $q_* : F' \to F$ that is injective (but this is only a poset morphism and not a frame morphism). Then the quotient is completely determined by the poset morphism $j : q_* q^* : F \to F$. Such morphisms are called *closure operators*, or *nuclei*, and they can be axiomatized by the properties $U \le j(U)$, $j(j(U)) = j(U)$, and $j(U \wedge V) = j(U) \wedge j(V)$. A closure operator defines a unique quotient $q^* : F \to F /\!\!/ (1 = j)$ such that $j = q_* q^*$. The poset $F /\!\!/ (1 = j)$ is defined as the elements of F such that $U = j(U)$; in other terms, it is forcing all the canonical inequalities $U \le j(U)$ to become identities. We refer to the literature for more details about those [17]. Table 9 compares the situation of quotients of frames and commutative rings.

If X is a topological space, not every sublocale is a topological subspace. This is one of the differences between topological spaces and the corresponding locale–the latter has more subobjects. We give an example below.

[23] For topological spaces, the reciprocal is true only if X is T_0-separated.

[24] In terms of category theory, a frame quotient $F \to F'$ is a *left-exact localization* of F. The quotient $F \to F /\!\!/ (A = B)$ is then the left-exact localization generated by forcing $A \le B$ to become an identity.

Table 9: Quotients of frames & rings

Comm. ring A	ideal $J \subseteq A$	generators a_i for J	projection $A \to A$ on a complement of J in A	quotient A/J
Frame F	the set J of inequalities $A \leq B$ that become equalities in the quotient	generating inequalities $A_i \leq B_i$	nucleus $j : F \to F$	quotient $F /\!\!/ (1 = j)$

Examples of sublocales

(i) Any open embedding of a locale X is an embedding. If U is the object of $\mathcal{O}(X)$ corresponding to the open embedding, the quotient $\mathcal{O}(X) \to \mathcal{O}(U) = \mathcal{O}(X)_{/U}$ is generated by forcing the inequality $U \leq 1$ to become an equality. The corresponding nucleus is $V \mapsto U \Rightarrow V$, where $U \Rightarrow V$ is Heyting implication ($U \Rightarrow -$ is right adjoint to $U \cap -$).

(ii) Any closed embedding of a locale X is an embedding. Let U be the corresponding object of $\mathcal{O}(X)$; the quotient $\mathcal{O}(X) \to \mathcal{O}(Z) = \mathcal{O}(X)_{U/}$ is generated by forcing the inequality $0 \leq U$ to become an equality. The corresponding nucleus is $V \mapsto U \cup V$.

The collection of all embeddings $L' \hookrightarrow L$ in a fixed locale L is a poset. It can be proved that the closed embedding $Z \hookrightarrow L$ is the maximal element of the poset of embeddings of L that is disjoint from $U \hookrightarrow L$. If X is a topological space, $Z \hookrightarrow X$ corresponds to the closed topological subspace which is the complement of U.

(iii) Recall the Alexandrov locale $\widehat{X}$ dual to the frame $\left[\mathcal{O}(X)^{op}, \underline{2}\right]$. There exists a unique frame morphism $\left[\mathcal{O}(X)^{op}, \underline{2}\right] \to \mathcal{O}(X)$ that is the identity on $\mathcal{O}(X) \hookrightarrow \left[\mathcal{O}(X)^{op}, \underline{2}\right]$. This frame morphism is surjective and defines the embedding $X \to \widehat{X}$ mentioned earlier.

(iv) The subposet $\left[\mathcal{O}(X)^{op}, \underline{2}\right]^{\text{lex}} \subset \left[\mathcal{O}(X)^{op}, \underline{2}\right]$ spanned by maps preserving finite infima is a frame, called the frame of *ideals* of the distributive lattice $\mathcal{O}(X)$. The dual locale shall be denoted X_{coh}. The previous frame quotient $\left[\mathcal{O}(X)^{op}, \underline{2}\right] \to \mathcal{O}(X)$ factors as $\left[\mathcal{O}(X)^{op}, \underline{2}\right] \to \left[\mathcal{O}(X)^{op}, \underline{2}\right]^{\text{lex}} \to \mathcal{O}(X)$. Dually, this defines embeddings $X \to X_{\text{coh}} \to \widehat{X}$. The locale X_{coh}, which is always spatial, is the so-called *coherent compactification* of X.

(v) If E is a set viewed as a discrete locale, the *Stone–Čech compactification* βE of E can be defined as a sublocale of $\widehat{E}$. Let $[P(E)^{op}, \underline{2}]^{\text{ultra}} \subset [P(E)^{op}, \underline{2}]$ be the subposet spanned by maps $F : P(E)^{op} \to \underline{2}$ such that, for any subset $A \subset E$ and any partition $A = A_0 \coprod A_1$, we have $F(A) = F(A_0) \wedge F(A_1)$. Then $[P(E)^{op}, \underline{2}]^{\text{ultra}}$ is the frame of open domains of βE. Recall that the points of $\widehat{E}$ are the filters of $P(E)$. The points of βE are the ultrafilters.

(vi) Let x be a point of $\mathbb{R}$ and U_x be the complement of $\{x\}$. The open quotient $\mathcal{O}(X) \to \mathcal{O}(U_x)$ is generated by forcing the inclusion $]x-\epsilon, x[\cup]x, x+\epsilon[\subset]x-\epsilon, x+\epsilon[$ to become an equality.

The corresponding closure operator j_x is the following. For an open subset $V \subset \mathbb{R}$, we denote by V' its closed complement. If x is an isolated point of V', then $V \cup \{x\}$ is open and $j_x(V) = V \cup \{x\}$. If not, then $j_x(V) = V$. Hence, the image in the inclusion $\mathcal{O}(U_x) \to \mathcal{O}(X)$ is spanned by the open subsets V such that x is not an isolated point in V'.

(vii) Let x_i be an arbitrary family of points of $\mathbb{R}$ and U_i be the complement of $\{x_i\}$. The formalism of frames lets us describe in a simple way the frame corresponding to the intersection of all the U_i: it is the intersection of all the frames $\mathcal{O}(U_i)$ in $\mathcal{O}(X)$. By the previous example, this intersection is spanned by the open subsets V of X whose closed complement V' admits none of the x_i as isolated points.

This example becomes fun if we let x_i be the family of *all* points of $\mathbb{R}$. First, the intersection of all the U_x for all x identifies to the subframe of $\mathcal{O}(X)$ spanned by open subset V whose closed complement is *perfect*, that is, has no isolated points. Since non trivial perfect subsets of $\mathbb{R}$ exist (e.g., closed intervals, Cantor sets), the resulting intersection is not trivial. Let $\mathbb{R}^\circ \subset \mathbb{R}$ be the corresponding sublocale of $\mathbb{R}$. Now the funny thing is that $\mathbb{R}^\circ$, even though it is not the empty locale, cannot have any points! Indeed, any such point would define a point of $\mathbb{R}$ through the inclusion $\mathbb{R}^\circ \subset \mathbb{R}$, but, by definition of $\mathbb{R}^\circ$, none of the points of $\mathbb{R}$ are in $\mathbb{R}^\circ$.

This is our first example of a locale without any points; we will see another one later. We shall call *thin* a subset of $\mathbb{R}$ with empty interior. Intuitively, a property is true on the locale $\mathbb{R}^\circ$ if it is true outside of a thin and perfect subset of numbers. The frame $\mathcal{O}(\mathbb{R}^\circ)$ is also an example of a frame without any injective frame morphism into a power set $P(E)$ (since any element of the set E would then be a point). This example can be generalized to any Hausdorff space.

2.2.8 Generators, relations and classifying locales The algebraic notion of frame offers the means to define certain spaces by the data of generators and relations for their frame. This fact is useful for constructing spaces classifying certains subsets of a given space. Let $\underline{2}[E]$ be the free frame on a set E. A point of $\underline{2}[E] \to \underline{2}$ is the same thing as a map $E \to \underline{2}$, which is a subset of E. From this point of view, the locale $\mathbf{S}^E$ is the classifying space of subsets of E.[25] If we impose relations on the free frame $\underline{2}[E]$, this corresponds to building a subspace of $\mathbf{S}^E$, which is to impose some constraints on the kind of subsets of E corresponding to the points. If $E = A \times B$, we can, for example, extract the subsets that are the graphs of functions $A \to B$. We shall denote by $[a \mapsto b]$ an element (a,b) in $A \times B$. The notation is chosen to suggest that this corresponds to the condition "a is sent to b". The relations to impose on $\underline{2}[A \times B]$ to classify graphs of functions are given by the following inequalities, which have to be forced to become equalities:

- (existence of image) for any a: $\bigvee_b [a \mapsto b] \leq 1$,
- (unicity of image) for any a and $b \neq b'$: $0 \leq [a \mapsto b] \wedge [a \mapsto b']$.

The frame classifying functions $A \to B$ is then the left-exact localization of $\underline{2}[A \times B]$ generated by those maps. To classify surjections or injections, we need to add the following further relations:

- (surjectivity) for any b: $\bigvee_a [a \mapsto b] \leq 1$;
- (injectivity) for any b and $a \neq a'$: $0 \leq [a \mapsto b] \wedge [a' \mapsto b]$.

One of the most intriguing facts about locales is that, when A is infinite and B is not empty, it can be proved that the sublocale of $\mathbf{S}^{A \times B}$ classifying surjections is never empty [21]. In particular, when $A = \mathbb{N}$ and $B = P(\mathbb{N}) \simeq \mathbb{R}$, there exists a nonempty locale of surjections $\mathbb{N} \to \mathbb{R}$. This produces another example of a locale without points since any point would construct an actual surjection $\mathbb{N} \to \mathbb{R}$ in set theory. There is also a non trivial locale $\mathrm{Bij}(\mathbb{N}, \mathbb{R})$ classifying bijections between $\mathbb{N}$ and $\mathbb{R}$. From the point of view of this locale, the cardinals of $\mathbb{N}$ and $\mathbb{R}$ are then the same. More generally, any two infinite cardinals can be forced to be the same similarly. This kind of locale is useful in interpreting logical constructions, such as Cohen forcing [26].

2.2.9 Locales without points We mentioned a couple of examples of nonempty locales without any points. Another amusing example is given in [5, IV.7.4]. If $K = [0,1]$ is the real interval equipped with a measure μ, the poset of measurable subsets of K is not a frame, but the poset of classes of measurable subsets of K up to null sets is. Since it is clearly non trivial, it defines a nonempty locale K_μ. The points of this frame correspond to points of K with non zero measure. If μ is the Lebesgue measure, no such points exist, and K_μ has no points.

These phenomena of locales without points can be nicely explained with the analogy of frame theory with commutative algebra. Let P be a polynomial in $\mathbb{Q}[x]$ and $A = \mathbb{Q}[x]/P$ the quotient ring. A root of P in $\mathbb{Q}$ is a ring morphism $A \to \mathbb{Q}$. Geometrically, such objects are called *rational points* of $Spec(A)$. Now

[25] More precisely, if we define a family of subsets of E parameterized by a locale L as the data of a subobject of the trivial bundle $L \times E \to L$, then such data is equivalent to that of a morphism of locales $L \to S^E$.

if $P = x^2 + 1$, it does not have any root in $\mathbb{Q}$, and the corresponding scheme does not have enough rational points. To produce roots of P or points of $Spec(A)$, we need to take an extension of $\mathbb{Q}$.

The situation is similar for locales. The points of a locale X are defined as frame morphisms $\mathcal{O}(X) \to \underline{2}$. Given a presentation of $\mathcal{O}(X)$ by generators and relations, finding a point corresponds to interpreting the generators as 0 or 1 such that the relations are fulfilled. This might not be possible. However, this might become possible if the generators are interpreted as elements of larger frame than $\underline{2}$.

A locale is said to have *enough points* if two open domains can be distinguished by the points they contain. Recall that the set of points $|\mathcal{P}\mathrm{t}(L)|$ of a locale L has a canonical topology. Then a locale has enough points precisely when the map $\mathcal{O}(L) \to P(E)$ is injective. A locale with enough points can be proved to be the same thing as a sober topological space. The affine locale $\mathbf{S}^P$ have enough points. Since any locale is a sublocale of some $\mathbf{S}^P$, any locale is a sublocale of a locale with enough points.

2.2.10 Product of locales and tensor products of frames The product of two locales $X \times Y$ corresponds dually to a tensor product $\mathcal{O}(X) \otimes \mathcal{O}(Y)$ of their corresponding frames [21]. This tensor product is defined similarly to that of commutative rings and abelian groups.[26] Recall that a frame is in particular a sup-lattice, that is, a poset with arbitrary suprema. Sup-lattices play for frames the role played by abelian groups for commutative rings (see Table 18). A morphism of sup-lattices is defined to be a map preserving arbitrary suprema. For three sup-lattices A, B, C, a poset map $A \times B \to C$ is called *bilinear* if it preserves suprema in both variables. Then, it can be proved that such bilinear maps are equivalent to morphisms of sup-lattices $A \otimes B \to C$ for some sup-lattice $A \otimes B$ called the *tensor product* of A and B. There exists a canonical bilinear map $A \times B \to A \otimes B$.

Here are some properties of this tensor product. The unit is the poset $\underline{2}$. If P is a poset, the poset $[P^{op}, \underline{2}]$ is a sup-lattice,[27] and for two posets P and Q, we have $[P^{op}, \underline{2}] \otimes [Q^{op}, \underline{2}] = [(P \times Q)^{op}, \underline{2}]$. In other terms, the functor $\mathrm{Alex} : \mathsf{Poset} \to \mathsf{Locale}$ preserves products.

In the same way that the tensor product $A \otimes B$ of two commutative rings is a commutative ring, the tensor product of two frames $F \otimes G$ is a frame. Moreover, $A \otimes B$ is actually the sum of A and B in the category of commutative rings, and so is $F \otimes G$ the sum of F and G in the category of frames. Dually, the tensor operation corresponds to the cartesian product of locales. The canonical functor $\mathsf{Top} \to \mathsf{Locale}$ does not preserve cartesian products,[28] but products of locally compact spaces are preserved.

2.2.11 Surjections If $X \to Y$ is a surjective map of topological spaces, the morphism of frames $\mathcal{O}(Y) \to \mathcal{O}(X)$ is injective. The reciprocal is not true, since surjective continuous maps need also to be surjective on the set of points. A morphism of locales $L' \to L$ is called a *surjective* if the corresponding morphism of frames is injective. If X is a topological space, then for any quotient $X \to L$ in Locale, there exists a surjective map $X \to Y$ in Top whose image under $\mathsf{Top} \to \mathsf{Locale}$ is $X \to L$.

Examples of surjections

(i) Let X be a topological space and E its set of points. The canonical inclusion $\mathcal{O}(X) \subset P(E)$ is a frame morphism corresponding to a surjection $E \to X$, where E is viewed as a discrete locale. We shall see in Section 2.2.13 that the data of this surjection is precisely the difference between locales and topological spaces.

(ii) (Open covers) A collection $U_i \to L$ is an *open covering* if the resulting map $\coprod_i U_i \to L$ is surjective. This is equivalent to the condition that $\bigvee_i U_i = 1$ in $\mathcal{O}(L)$.

(iii) (Image factorization) Let $L' \to L$ be a map of locales; there exists a unique factorization $L' \to M \to L$ such that $L' \to M$ is a surjection and $M \to L$ is an embedding. This factorization is constructed dually

[26] Recall that the coproduct of two commutative rings A and B is given by the tensor product $A \otimes B$ of the underlying abelian groups. This tensor product is defined by the universal property that maps of abelian groups $A \otimes B \to C$ are equivalent to bilinear maps $A \times B \to C$.

[27] We shall see in Section 3.4.1 that it is in fact the free sup-lattice generated by P.

[28] $\mathbb{Q}^2$ is not the same computed in Top or in Locale (see [17, II.2.14]).

by defining $\mathcal{O}(M)$ as the image of the frame map $\mathcal{O}(L) \to \mathcal{O}(L')$.

2.2.12 Compact locales A space X is compact if, for any directed union U_i of open subsets of X, the condition $X = \bigcup U_i$ implies that $X = U_i$ for some i. This property is a way to say that the maximal object 1 of the frame $\mathcal{O}(X)$ is *finitary*, or equivalently, that the poset morphism $\mathrm{Hom}_{\mathcal{O}(X)}(1,-) : \mathcal{O}(X) \to \underline{2}$ (the "global sections") preserves directed unions. Then, a locale L is called *compact* if the maximal object 1 of $\mathcal{O}(L)$ is finitary.

Examples of compact locales

(i) Any compact topological space is compact when viewed as a locale.

(ii) A frame $[P, \underline{2}]$ is dual to a compact locale if and only if the poset P is *filtering* (for any pair x, y of objects of P there exist $z \leq x$ and $z \leq y$). This is true in particular if P has a minimal element.

(iii) For X a locale or a topological space, the Alexandrov locale $\widehat{X}$ dual to the frame $\left[\mathcal{O}(X)^{op}, \underline{2}\right]$ is compact. This justifies the remark that it is a kind of compactification of X.

(iv) The coherent compactification X_{coh} of X, dual to the frame $\left[\mathcal{O}(X)^{op}, \underline{2}\right]^{\mathrm{lex}}$, is also compact.

2.2.13 From locales to topological spaces We explained that the functor $\mathsf{Top} \to \mathsf{Locale}$ is not fully faithful, that is, that different spaces can have the same frame of open domains. Nonetheless, it is possible to reconstruct the category Top from Locale. For any set E, the power set $P(E)$ is a frame. A locale is called *discrete* if the corresponding frame is isomorphic to some $P(E)$. A locale L is said to have *enough points* if there exists a surjective morphism $E \to L$ from some discrete locale E. A choice of a set of points for a locale with enough points is a choice of such a surjection. Let X be a topological space and X_{dis} the discrete topology on the same set. The canonical embedding $\mathcal{O}(X) \subset P(X)$ is a frame morphism corresponding to a localic surjection $X_{dis} \to X$, that is, a topological space defines a locale together with a choice of a set of points.

Let $\mathsf{Locale}^{\to}$ be the category whose objects are the morphisms of locales. The category of topological spaces is equivalent to the full subcategory of $\mathsf{Locale}^{\to}$ spanned by maps $E \to L$, which are surjections with a discrete domain E. From this point of view, the functor $\mathsf{Top} \to \mathsf{Locale}$ is nothing but the functor sending a surjection $E \to L$ to L, that is, the functor forgetting the set of points. The image of this functor is the full subcategory of locales with enough points.

This simple result has two consequences. First, it should make clear the difference between the so-called *point-set topology* and *point-free topology*: topological spaces are locales with the extra structure of a fixed set of points. The second point is that the entire theory of topological spaces can be formulated in terms of the theory of locales, so the latter is in fact the most general one.

2.2.14 Concluding remarks Many other topological notions can be generalized to locales, such as connectedness, separation, pasting, or local homeomorphisms. Our purpose here was only to give a glance at the possibility of doing *point-free topology*, that is, topology without the prescription of a set of points. This step of forgetting the set of points is an essential one in the direction of the notion of topos. We refer to [17, 30] for a study of locales.

There are actually reasons to prefer the broader generality of locales to topological spaces. The most obvious reason is the nice duality $\mathsf{Locale} = \mathsf{Frame}^{\mathsf{op}}$, that is the fact that the spatial notion of locale can be equivalently manipulated in algebraic terms.[29] Another aspect is that the theory of locales is fundamentally constructive. For example, the proof that a product of a compact Hausdorff topological spaces is compact (Tychonov's theorem) depends on the axiom of choice, but not the proof that a product of compact Hausdorff locales is compact.

[29] The difference between topological spaces and locales is akin to that between algebraic varieties (over a non algebraically closed field) and schemes. The former have a prescription on the nature of their points that prevents them from being dual to some type of algebras, but the latter are designed to be perfectly dual to an algebraic structure; in particular, they can have no point in the sense of the former (rational points).

3 The topos–logos duality

We have explained how the theory of topological spaces could be reformulated in terms of locale theory, a notion of spatial object dual to the algebraic structure of frame. The notion of topos can be similarly presented as dual to the algebraic notion of *logos*. We start in Section 3.1 by giving a first definition of logoi and topoi which is useful to give examples and play with them. Then, Section 3.2 defines a number of topological notions for topoi (and the corresponding algebraic notions for logoi) with the purpose of convincing the reader that topoi are indeed spatial objects. Finally, Section 3.3 has the purpose to explain Giraud and Lawvere definitions of logoi and topoi and their relation with a distributivity condition between limits and colimits in a logos. The explanation is given from the point of view of descent theory, aka the art of pasting. Section 3.3 is a more technical section that can be skipped at a first reading.

3.1 First definition and examples

Essentially, a logos is a category with colimits, finite limits, and a compatibility relation between them akin to distributivity. However, the precise formulation of this property demands the introduction of several concepts and will be postponed until Section 3.3. We shall start here with the simplest, albeit not the most intuitive, definition of a logos. Nonetheless, it is convenient to introduce many examples to play with. The definitions by Giraud and Lawvere axioms will be given in Section 3.3.

We need a couple of preliminary notions. A *reflective localization* is a functor $L : \mathcal{E} \to \mathcal{F}$ admitting a fully faithful right adjoint. In particular, it is a co-continuous functor. A *left-exact localization* is a reflective localization L that preserves finite limits.

A *logos* is a category $\mathcal{E}$ that can be presented as a left-exact localization of a presheaf category $\mathcal{P}\mathrm{r}(C) := [C^{op}, \mathsf{Set}]$ on a small category C. A *morphism of logoi* $f^* : \mathcal{E} \to \mathcal{F}$ is a functor preserving (small) colimits and finite limits. The category of logoi will be denoted Logos. It is a 2-category if we take into account the natural transformations $f^* \to g^*$ between the morphisms.[30] A *topos* is defined to be an object of the category $\mathsf{Logos}^{\mathsf{op}}$. The category of topoi is defined as

$$\mathsf{Topos} = \mathsf{Logos}^{\mathsf{op}}.^{31}$$

We shall not use the classical terminology of *geometric morphisms* to refer to the morphisms in Topos, but simply talk about topos morphisms. If $\mathbf{X}$ is a topos, we shall denote by $\mathcal{S}\mathrm{h}(\mathbf{X})$ the corresponding logos. The objects of $\mathcal{S}\mathrm{h}(\mathbf{X})$ are called the *sheaves on* $\mathbf{X}$. For $u : \mathbf{Y} \to \mathbf{X}$ a topos morphism, we denote by $u^* : \mathcal{S}\mathrm{h}(\mathbf{X}) \to \mathcal{S}\mathrm{h}(\mathbf{Y})$ the corresponding logos morphism.

$$\begin{tikzcd} \mathsf{Logos}^{\mathsf{op}} \arrow[r, bend left, "\text{dual}"] & \mathsf{Topos} \arrow[l, bend left, "\mathcal{S}\mathrm{h}"] \end{tikzcd}$$

Given F in $\mathcal{S}\mathrm{h}(\mathbf{X})$, the object u^*F in $\mathcal{S}\mathrm{h}(\mathbf{Y})$ is called the *pullback*, or *base change of F along u*. A logos $\mathcal{E}$ always has a terminal object 1; a map $1 \to F$ in $\mathcal{E}$ shall be called a *global section of F*. This geometric vocabulary will be justified in Section 3.2.6.

3.1.1 Sheaves on a locale The example motivating the definition of a logos is the category of sheaves of sets on a space. Let X be a topological space; the category $\mathcal{S}\mathrm{h}(X)$ of sheaves on X is a reflective subcategory of $\mathcal{P}\mathrm{r}(\mathcal{O}(X)) = \left[\mathcal{O}(X)^{op}, \mathsf{Set}\right]$. The localization $\mathcal{P}\mathrm{r}(\mathcal{O}(X)) \to \mathcal{S}\mathrm{h}(X)$ is the *sheafification functor* that happens to be left-exact (we shall explain why below). Therefore, $\mathcal{S}\mathrm{h}(X)$ is a logos. The corresponding topos will be denoted simply by X. The construction of $\mathcal{S}\mathrm{h}(X)$ depends only on the frame

[30] Precisely, the category of morphisms of logoi is the full subcategory $[\mathcal{E}, \mathcal{F}]^{\mathrm{lex}}_{\mathrm{cc}} \subset [\mathcal{E}, \mathcal{F}]$ spanned by functors preserving colimits and finite limits.

[31] When Logos is viewed as a 2-category, Topos is defined by reversing the direction of 1-arrows only. This definition of 2-cells in Topos is in accordance with most of the references but not with the original convention of [5].

$\mathcal{O}(X)$ and is therefore defined for any locale X. This produces a functor

$$\begin{aligned} \mathcal{S}\mathrm{h} : \mathsf{Locale}^{\mathsf{op}} &\longrightarrow \mathsf{Logos} \\ X &\longmapsto \mathcal{S}\mathrm{h}(X) \\ f : X \to Y &\longmapsto f^* : \mathcal{S}\mathrm{h}(Y) \to \mathcal{S}\mathrm{h}(X). \end{aligned}$$

or equivalently a functor $\mathsf{Locale} \to \mathsf{Topos}$. This functor is faithful, and the topoi in the image of this functor are called *localic*. We shall see later the definition of the open domains of a topos, and that the open domain of localic topos reconstructs the frame of open of the corresponding locale.

The fact that the sheafification functor $\mathcal{P}\mathrm{r}(\mathcal{O}(X)) \to \mathcal{S}\mathrm{h}(X)$ is left-exact can be seen using the construction by Godement of this functor [12, II.1.2]. Let X be a topological space and $\mathcal{E}\mathrm{t}(X)$ be the full subcategory of $\mathsf{Top}_{/X}$ spanned by *local homeomorphisms*, or *étale maps*, $u : Y \to X$. Any such map $Y \to X$ defines a presheaf of local sections on X, which happens to be a sheaf. This produces a functor $\mathcal{E}\mathrm{t}(X) \to \mathcal{S}\mathrm{h}(X)$, which is an equivalence of categories. To prove this, Godement constructs a functor $\mathcal{P}\mathrm{r}(\mathcal{O}(X)) \to \mathcal{E}\mathrm{t}(X)$, which is the left adjoint to the functor $\mathcal{E}\mathrm{t}(X) \to \mathcal{P}\mathrm{r}(\mathcal{O}(X))$; hence it is the sheafification functor. The construction is based on the extraction of the stalks of a presheaf F. For any point x, let $U(x)$ be the filter of neighborhoods of x; the stalk of F at x is $F(x) = \mathrm{colim}_{V \in U(x)} F(V)$. The functor $F \mapsto F(x)$ is left-exact because $U(x)$ is a filter and filtered colimits preserve finite limits. Let V be an open subset of X. Any point x in V defines a map $F(V) \to F(x)$, which sends a local section $\mathbf{S}$ of F to its *germ* $s(x)$ at x. Then, the underlying set of Y is $\coprod_{x \in X} F(x)$ and a basis for the topology is given by the sets $\{s(x) | x \in U\}$ for any s in $F(U)$. This geometric construction produces a functor $\mathcal{P}\mathrm{r}(\mathcal{O}(X)) \to \mathcal{E}\mathrm{t}(X)$, which is left-exact because the construction of the stalks is:

$$\begin{tikzcd} \mathcal{P}\mathrm{r}(\mathcal{O}(X)) \arrow[rr, "\text{sheafification}", "\text{(left-exact)}"'] & & \mathcal{E}\mathrm{t}(X) \arrow[dll, "\text{sheaf of sections}", "\text{(equivalence)}"'] \\ \mathcal{S}\mathrm{h}(X) \arrow[u, hook] & & \end{tikzcd}$$

3.1.2 Presheaf logoi and Alexandrov topoi The *Alexandrov logos* of a small category C is defined to be the category of set-valued C-diagrams $[C, \mathsf{Set}] = \mathcal{P}\mathrm{r}(C^{op})$. The *Alexandrov topos* of C is defined to be the dual topos, and we shall denote it by $\mathbf{B}C$. This defines a contravariant 2-functor $[-, \mathsf{Set}] : \mathsf{Cat}^{\mathsf{op}} \to \mathsf{Logos}$ and a covariant 2-functor $\mathbb{B} : \mathsf{Cat} \to \mathsf{Topos}$, where Cat denotes the category of small categories. These 2-functors are not conservative since they take Morita-equivalent categories to equivalent Alexandrov logos/topos. Alexandrov topoi are analogues of Alexandrov locales (see Section 2.2.3). Many important examples of logoi/topoi are of this type.

Examples of Alexandrov topoi

(i) When $C = \varnothing$, we get that the category 1 is a logos. It is the terminal object of Logos. Hence, the corresponding topos, denoted $\varnothing$, is the initial object of Topos and is called the *empty topos*. In the analogy logoi/commutative rings, this is the analogue of the zero ring.

(ii) When $C = 1$, the category Set is a logos. It is the initial object of Logos. In the analogy logoi/commutative rings, this is the analogue of the ring $\mathbb{Z}$. The corresponding topos, denoted $\mathbf{1}$, is the terminal object of Topos and will play the role of the *point*.

(iii) Let C be a small category; the presheaf category $\mathcal{P}\mathrm{r}(C) := [C^{op}, \mathsf{Set}]$ is a particular case of an Alexandrov logoi. The corresponding Alexandrov topos is $\mathbf{B}(C^{op})$. In particular, for a topological space X, the category $\mathcal{P}\mathrm{r}(\mathcal{O}(X))$ is a logos and the sheafification $\mathcal{P}\mathrm{r}(\mathcal{O}(X)) \to \mathcal{S}\mathrm{h}(X)$ is a morphism of logoi. Recall the locale $\widehat{X}$ dual to the frame $\left[\mathcal{O}(X)^{op}, \underline{2}\right]$. Then we have in fact $\mathcal{P}\mathrm{r}(\mathcal{O}(X)) = \mathcal{S}\mathrm{h}\left(\widehat{X}\right)$. For this reason, we shall denote by $\widehat{X}$ the topos dual to $\mathcal{P}\mathrm{r}(\mathcal{O}(X))$. We already saw the existence of an embedding $X \to \widehat{X}$, which is a kind of compactification of X. This will stay true in Topos.

(iv) The category of simplicial sets is a logos since it is defined as $\mathcal{P}\mathrm{r}(\Delta)$, where Δ is the simplicial category, that is, the category of nonempty finite ordinals.

(v) When C is a set E, that is, a discrete category, then $\mathcal{P}r(E) = \mathsf{Set}^E$ is a logos. The corresponding Alexandrov topos $\mathbf{B}E$ is called *discrete*. In the analogy logoi/commutative rings, Set^E is analogue to $\prod_E \mathbb{Z}$.

(vi) Another example is the logos $[\mathsf{Fin}, \mathsf{Set}]$, where Fin is the category of finite sets. This logos is arguably the central piece of the whole theory, and we are going to denote it by $\mathsf{Set}[X]$. The notation is chosen to recall the free ring $\mathbb{Z}[x]$. The logos $\mathsf{Set}[X]$ is in fact the free logos on one generator: for any logos $\mathcal{E}$, a logos morphism $\mathsf{Set}[X] \to \mathcal{E}$ is the same thing as an object of $\mathcal{E}$. The "generic object" X in $\mathsf{Set}[X]$ corresponds to the canonical inclusion $\mathsf{Fin} \to \mathsf{Set}$. It is also the functor represented by the object 1 in Fin. The topos corresponding to $\mathsf{Set}[X]$ will be denoted $\mathbf{A}$ and called the *topos of sets* or the *topos classifying objects*. It will play a role analogous to the affine line $\mathbb{A}^1$ in algebraic geometry. Table 10 details some aspects of the structural analogy between $\mathbb{Z}[x]$ and $\mathsf{Set}[X]$.

(vii) Let $\mathsf{Fin}^\bullet$ be the category of pointed finite sets. The logos $\mathsf{Set}[X^\bullet] := [\mathsf{Fin}^\bullet, \mathsf{Set}]$ is an important companion of $\mathsf{Set}[X]$. A logos morphism $\mathsf{Set}[X^\bullet] \to \mathcal{E}$ is the same thing as a *pointed object* in $\mathcal{E}$, that is, an object E with the choice of a global section $1 \to E$. The "generic pointed object" $X^\bullet$ in $\mathsf{Set}[X^\bullet]$ corresponds to the functor $\mathsf{Fin}^\bullet \to \mathsf{Set}$, forgetting the base point. It is also the functor representable by the object $1 \to 1 \coprod 1$ in $\mathsf{Fin}^\bullet$. The topos corresponding to $\mathsf{Set}[X^\bullet]$ will be denoted $\mathbf{A}^\bullet$ and called the *topos of pointed sets*, or the *topos classifying pointed objects*. There is a distinguished topos morphism $\mathbf{A}^\bullet \to \mathbf{A}$ corresponding to the unique logos morphism $\mathsf{Set}[X] \to \mathsf{Set}[X^\bullet]$ sending X to $X^\bullet$.

(viii) Let $\mathsf{Fin}^\circ \subset \mathsf{Fin}$ be the category of nonempty finite sets. The logos $[\mathsf{Fin}^\circ, \mathsf{Set}]$ is denoted by $\mathsf{Set}[X^\circ]$. The canonical object X° corresponds the inclusion $\mathsf{Fin}^\circ \subset \mathsf{Set}$. The corresponding logos is denoted $\mathbf{A}^\circ$. The inclusion $\mathsf{Fin}^\circ \subset \mathsf{Fin}$ produces a morphism of logoi $\mathsf{Set}[X] \to \mathsf{Set}[X^\circ]$ sending X to X° and a morphism of topoi $\mathbf{A}^\circ \to \mathbf{A}$. The factorization $\mathsf{Fin}^\bullet \to \mathsf{Fin}^\circ \subset \mathsf{Fin}$ produces a factorization $\mathbf{A}^\bullet \to \mathbf{A}^\circ \to \mathbf{A}$. We shall see later that $\mathbf{A}^\circ$ classifies nonempty sets and that the factorization $\mathbf{A}^\bullet \to \mathbf{A}^\circ \to \mathbf{A}$ is the image factorization of $\mathbf{A}^\bullet \to \mathbf{A}$.

(ix) The logos of sheaves on the Sierpiński space is $\mathcal{S}\mathrm{h}(\mathbf{S}) = [\underline{2}, \mathsf{Set}] = \mathsf{Set}^\to$, the arrow category of Set. The corresponding logos/topos are called the *Sierpiński logos/topos*. We shall see later that it plays the role of the Sierpiński space in classifying open domains of topoi, that is, that a morphism of topoi $\mathbf{X} \to \mathbf{S}$ is equivalent to the data of an open subtopos of $\mathbf{X}$.

(x) Let $[n]$ be the poset $\{0 < 1 < \dots < n\}$. The category $\mathsf{Set}^{[n]}$ is a logos. Morphisms of topoi $X \to \mathbf{B}[n]$ can be proved to be equivalent to the data of a stratification of depth n, that is, a sequence $U_n \subset U_{n-1} \subset \ldots \subset U_0 = \mathbf{X}$ of open subtopoi of $\mathbf{X}$. More generally, if P is a poset, morphisms $X \to \mathbf{B}P$ can be proved to be stratifications on X, whose strata are indexed by P.

(xi) Let G be a group; then the category Set^G of sets with a G-action is a logos since it can be described as the presheaf category $\mathcal{P}r(G)$, where G is viewed as a category with one object. The corresponding topos $\mathbf{B}G$ will play the role of a classifying space for G. A topos morphism $\mathbf{X} \to \mathbf{B}G$ can be proved to be the same thing as a G-torsor in the category $\mathcal{S}\mathrm{h}(\mathbf{X})$ [26, VIII.2].

(xii) Let $\mathsf{Ring}_{\mathsf{fp}}$ be the category of commutative rings of finite presentations. The opposite category $\mathsf{Ring}_{\mathsf{fp}}^{\mathsf{op}}$ is the category $\mathsf{Aff}_{\mathsf{fp}}$ of affine schemes of finite presentations. The Alexandrov logos $\left[\mathsf{Ring}_{\mathsf{fp}}, \mathsf{Set}\right] = \mathcal{P}r(\mathsf{Aff}_{\mathsf{fp}})$ and the dual topos $\mathbf{B}\left(\mathsf{Ring}_{\mathsf{fp}}\right)$ are *classifying rings*. A logos morphism $\mathcal{P}r(\mathsf{Aff}_{\mathsf{fp}}) \to \mathcal{E}$ is the same thing as a left-exact functor $\mathsf{Aff}_{\mathsf{fp}} \to \mathcal{E}$, which can be unraveled to be the same thing as a commutative ring object in $\mathcal{E}$, that is, a sheaf of rings.

(xiii) Let $\mathbb{T}$ be a category with cartesian products, that is, a (multisorted) algebraic theory (aka a Lawvere theory). We denote by $\mathsf{Mod}(\mathbb{T})$ the category of models and by $\mathsf{Mod}(\mathbb{T})_{\mathsf{fp}}$ the subcategory of models of finite presentation. The Alexandrov logos $\mathsf{Set}\langle\mathbb{T}\rangle := [\mathsf{Mod}(\mathbb{T})_{\mathsf{fp}}, \mathsf{Set}]$ has the property that a logos morphism $\mathsf{Set}\langle\mathbb{T}\rangle \to \mathcal{E}$ is the same thing as a model of $\mathbb{T}$ in the logos $\mathcal{E}$. For this reason, the dual Alexandrov topos $\mathbf{B}(\mathsf{Mod}(\mathbb{T})_{\mathsf{fp}})$ is called the *classifying topos of the algebraic theory* $\mathbb{T}$ and denoted $\mathbf{B}\langle\mathbb{T}\rangle$.

When $\mathbb{T}$ is the full subcategory of $\mathsf{Aff}_{\mathsf{fp}}$ spanned by affine spaces of finite dimension, $\mathsf{Mod}(\mathbb{T})_{\mathsf{fp}} = \mathsf{Ring}_{\mathsf{fp}}$, and we get back the previous example.

Table 10: Polynomial analogies

	Commutative ring	*Logos*
Initial object	$\mathbb{Z}$	Set
Free on one generator	$\mathbb{Z}[x] = \mathbb{Z}^{(\mathbb{N})}$	$\mathsf{Set}[X] = [\mathsf{Fin}, \mathsf{Set}]$
Monomials	x^n, for n in $\mathbb{N}$	X^N, for N in Fin (representable functors $\begin{aligned} X^N : \mathsf{Fin} &\to \mathsf{Set} \\ E &\mapsto E^N \end{aligned}$)
Polynomial	$P(x) = \sum_n p_n x^n$	$F(X) = \int^N F(N) \times X^N$ (coend over Fin)
Polynomial function	for any ring A $\begin{aligned} P : A &\to A \\ a &\mapsto \sum_n p_n a^n \end{aligned}$	for any logos $\mathcal{E}$ $\begin{aligned} F : \mathcal{E} &\to \mathcal{E} \\ E &\mapsto \int^N F(N) \times E^N \end{aligned}$ (coend over Fin in $\mathcal{E}$)
Dual geometric object with an algebra structure	$\mathbb{A}^1$ is a ring object in Schemes	$\mathbf{A}$ is a logos object in Topos
Additive operation	$+ : \mathbb{A}^2 \to \mathbb{A}^1$ dual to $\begin{aligned} \mathbb{Z}[x] &\to \mathbb{Z}[x,y] \\ x &\mapsto x + y \end{aligned}$	$\operatorname{colim} : \mathbf{A}^C \to \mathbf{A}$ dual to $\begin{aligned} \mathsf{Set}[X] &\to \mathsf{Set}[C] \\ X &\mapsto \operatorname{colim} c \end{aligned}$
Multiplicative operation	$\times : \mathbb{A}^2 \to \mathbb{A}^1$ dual to $\begin{aligned} \mathbb{Z}[x] &\to \mathbb{Z}[x,y] \\ x &\mapsto xy \end{aligned}$	$\lim : \mathbf{A}^C \to \mathbf{A}$ (C finite) dual to $\begin{aligned} \mathsf{Set}[X] &\to \mathsf{Set}[C] \\ X &\mapsto \lim c \end{aligned}$

Let $\mathbb{T}$ be the theory of groups; then $\mathbf{B}\langle\mathbb{T}\rangle$ is the topos classifying groups: one can prove that a topos morphism $\mathbf{X} \to \mathbf{B}\langle\mathbb{T}\rangle$ is the same thing as a group object in $\mathcal{S}h(\mathbf{X})$, that is, a sheaf of groups on $\mathbf{X}$.

3.1.3 Other examples

(i) If $\mathcal{E}$ is a logos and E is an object of $\mathcal{E}$, then the category $\mathcal{E}_{/E}$ is again a logos. This is easy to see in the case $\mathcal{E} = \mathsf{Set}$ since $\mathsf{Set}_{/E} = \mathsf{Set}^E = \mathcal{P}\mathrm{r}(E)$. This is also easy to see in the case $\mathcal{E} = \mathcal{P}\mathrm{r}(C)$ since $\mathcal{P}\mathrm{r}(C)_{/E} = \mathcal{P}\mathrm{r}(C_{/E})$, where $C_{/E}$ is the category of elements of the functor $E : C^{op} \to \mathsf{Set}$. The base change along the map $e : E \to 1$ induces a functor $e^* : \mathcal{E} \to \mathcal{E}_{/E}$, which is a logos morphism. We shall see that such morphisms are étale maps.

(ii) Every logos $\mathcal{E}$ is a left-exact localisation of a presheaf logos $\mathcal{P}\mathrm{r}(C)$. The localisation functor $\mathcal{P}\mathrm{r}(C) \to \mathcal{E}$ is a surjective morphism of logoi. We shall see that the left-exact localisations of $\mathcal{P}\mathrm{r}(C)$ are the "quotients" of $\mathcal{P}\mathrm{r}(C)$ in the category of logoi.

(iii) Let G be a discrete group acting on a topological space X, and let $\mathcal{S}h(X, G)$ be the category of equivariant sheaves on X. Then $\mathcal{S}h(X, G)$ is a logos, and the corresponding topos $X /\!\!/ G$ is the quotient of X by the action of G in the 2-category of topoi. The functor $q^* : \mathcal{S}h(X, G) \to \mathcal{S}h(X)$, forgetting the action, corresponds to the quotient map $q : X \to X /\!\!/ G$.

(iv) Let G be a topological group, and let $\mathsf{Set}^{(G)}$ be the category of sets equipped with a continuous action of G. Then, $\mathsf{Set}^{(G)}$ is a logos. If G is a connected group, then any continuous action of G on a set is trivial, and $\mathsf{Set}^{(G)} = \mathsf{Set}$. In fact, the logos $\mathsf{Set}^{(G)}$ does depends only on the totally disconnected space of connected components of G, which is also a group. In particular, if G is locally connected, the connected components form a discrete group $\pi_0(G)$, and we have $\mathsf{Set}^{(G)} = \mathsf{Set}^{\pi_0(G)}$.

(v) Let K be a profinite group (e.g., the Galois group of some field). Recall that K can be faithfully represented as a totally disconnected topological group. Then, by the previous example, the category $\mathsf{Set}^{(K)}$ of continuous action of K is a logos.

3.2 Elements of topos geometry

As for locales, the fact that $\mathsf{Topos} = \mathsf{Logos}^{\mathsf{op}}$ is indeed a category of geometric objects is proved by the possibility to define there all the classical topological notions. The strategy to generalize topological notions to topoi is the same as before: first, find a formulation in terms of sheaves, then generalize the notion to any logos.

3.2.1 Free logoi and affine topoi As with locales, the fact that topoi are defined as dual to some algebraic structure singularizes the class of topoi corresponding to the free algebras. Let C be a small category and C^{lex} the free completion of C for finite limits.[32] Then $\mathsf{Set}[C] := \mathcal{P}\mathrm{r}(C^{\mathrm{lex}}) = [(C^{\mathrm{lex}})^{op}, \mathsf{Set}]$ is a logos called the *free logos* on C. The logos $\mathsf{Set}[C]$ has the following fundamental property, which justifies its name: if $\mathcal{E}$ is a logos, then co-continuous and left-exact functors $\mathsf{Set}[C] \to \mathcal{E}$ are equivalent to functors $C \to \mathcal{E}$.[33] Inspired by algebraic geometry, the topos corresponding to $\mathsf{Set}[C]$ will be denoted $\mathbf{A}^C$ and called an *affine topos*.

Examples of free logoi/affine topoi

(i) When $C = \varnothing$, we have $\varnothing^{\mathrm{lex}} = 1$ and $\mathsf{Set}[\varnothing] = \mathsf{Set}$ is the initial logos, corresponding to the terminal topos $\mathbf{A}^0 = \mathbf{1}$.

[32] This means that, if $\mathcal{E}$ is a category with finite limits, the data of a functor preserving finite limits $C^{\mathrm{lex}} \to \mathcal{E}$ is equivalent to the data of a functor $C \to \mathcal{E}$.

[33] From a functor $C \to \mathcal{E}$, we get a functor $C^{\mathrm{lex}} \to \mathcal{E}$ by right Kan extension and a function $\mathcal{P}\mathrm{r}(C^{\mathrm{lex}}) \to \mathcal{E}$ by left Kan extension. The fact that this last functor is co-continuous and left-exact is characteristic of logoi [11]. It would not be true if $\mathcal{E}$ were an arbitrary category with colimits and finite limits.

(ii) When $C = 1$, we have $1^{\text{lex}} = \mathsf{Fin}^{\mathsf{op}}$, and $\mathsf{Set}[1]$ is the logos $\mathsf{Set}[X] = [\mathsf{Fin}, \mathsf{Set}]$ introduced before. The corresponding topos is $\mathbf{A}^1 = \mathbf{A}$. If $\mathcal{E}$ is a logos, a logos morphism $\mathsf{Set}[X] \to \mathcal{E}$ is equivalent to the data of an object of $\mathcal{E}$. Geometrically, this gives the fundamental remark that the logos $\mathcal{S}\mathrm{h}(\mathbf{X})$ of sheaves on a topos $\mathbf{X}$ can be described as topos morphisms into $\mathbf{A}$:

$$\mathcal{S}\mathrm{h}(\mathbf{X}) = \mathrm{Hom}_{\mathsf{Topos}}(\mathbf{X}, \mathbf{A}). \qquad \text{(Sheaves as functions)}$$

This formula is analogous to $\mathcal{O}(X) = C^0(X, \mathbf{S})$ for locales. The morphism $\mathbf{X} \to \mathbf{A}$ corresponding to some F in $\mathcal{S}\mathrm{h}(\mathbf{X})$ will be denoted χ_F and called the *classifying morphism* or *characteristic morphism* of F.

(iii) When $C = \{0 \to 1\}$, the category with one arrow, we have $C^{\text{lex}} = (\mathsf{Fin}^{\to})^{op}$ where $\mathsf{Fin}^{\to}$ is the arrow category of Fin, and $\mathsf{Set}[\{0 \to 1\}] = [\mathsf{Fin}^{\to}, \mathsf{Set}]$. The corresponding topos is denoted $\mathbf{A}^{\to}$. A topos morphism $\mathbf{X} \to \mathbf{A}^{\to}$ is the same thing as a map $A \to B$ in $\mathcal{S}\mathrm{h}(\mathbf{X})$. For this reason, $\mathbf{A}^{\to}$ is called the *topos classifying maps*.

(iv) When $C = \{0 \simeq 1\}$, the category with one isomorphism, the affine topos $\mathbf{A}^{\{0 \simeq 1\}}$ is denoted $\mathbf{A}^{\simeq}$. A topos morphism $\mathbf{X} \to \mathbf{A}^{\simeq}$ is the same thing as an isomorphism $A \simeq B$ in $\mathcal{S}\mathrm{h}(\mathbf{X})$, and $\mathbf{A}^{\simeq}$ is called the *topos classifying isomorphisms*. The canonical functor $\{0 \to 1\} \to \{0 \simeq 1\}$ induces a map $\mathbf{A}^{\simeq} \to \mathbf{A}^{\to}$ of affine topoi. Intuitively, $\mathbf{A}^{\simeq}$ is the subtopos of $\mathbf{A}^{\to}$ classifying those maps that are isomorphisms.

Since $\{0 \simeq 1\}$ is equivalent to the punctual category 1, we have in fact $\mathbf{A}^{\simeq} = \mathbf{A}$. Intuitively, this says that the data of an isomorphism between two objects is equivalent to the data of a single object.

Table 11 summarizes some of the classifying properties of affine and Alexandrov topoi (some of these features will be explained later in the text).

Table 11: Classifying properties of affine and Alexandrov topoi

	Topos morphism	*Logos morphism*	*Interpretation*
C small category	$\mathbf{X} \to \mathbf{A}^C$	$\mathsf{Set}[C] \to \mathcal{S}\mathrm{h}(\mathbf{X})$	diagram $C \to \mathcal{S}\mathrm{h}(\mathbf{X})$
E set	$\mathbf{X} \to \mathbf{A}^E$	$\mathsf{Set}[E] \to \mathcal{S}\mathrm{h}(\mathbf{X})$	family of sheaves $\mathbf{X}$ indexed by E
C small category	$\mathbf{X} \to \mathbf{B}C$	$\mathsf{Set}^C \to \mathcal{S}\mathrm{h}(\mathbf{X})$	flat C-diagram $C^{op} \to \mathcal{S}\mathrm{h}(\mathbf{X})$
D small category with finite colimits	$\mathbf{X} \to \mathbf{B}D$	$\mathsf{Set}^D \to \mathcal{S}\mathrm{h}(\mathbf{X})$	lex functor $D^{op} \to \mathcal{S}\mathrm{h}(\mathbf{X})$
E set	$\mathbf{X} \to \mathbf{B}E$	$\mathsf{Set}^E \to \mathcal{S}\mathrm{h}(\mathbf{X})$	partition of $\mathbf{X}$ indexed by E
P poset	$\mathbf{X} \to \mathbf{B}P$	$\mathsf{Set}^P \to \mathcal{S}\mathrm{h}(\mathbf{X})$	stratification of $\mathbf{X}$ indexed by P
G group	$\mathbf{X} \to \mathbf{B}G$	$\mathsf{Set}^G \to \mathcal{S}\mathrm{h}(\mathbf{X})$	G-torsor in $\mathcal{E}$

3.2.2 The category of points As mentioned in the introduction to this chapter, one of the differences between topological spaces and topoi is that the latter have a category of points instead of a mere set. The category of topoi has a terminal object $\mathbf{1}$ that corresponds to the logos Set. A *point* of a topos $\mathbf{X}$ is defined as a morphism of topoi $x : \mathbf{1} \to \mathbf{X}$. Equivalently, a point is a morphism of logoi $x^* : \mathcal{S}\mathrm{h}\,(\mathbf{X}) \to \mathsf{Set}$. The category of points of $\mathbf{X}$ is

$$\mathcal{P}\mathrm{t}(\mathbf{X}) \coloneqq \mathrm{Hom}_{\mathsf{Topos}}(\mathbf{1}, \mathbf{X}) = \mathrm{Hom}_{\mathsf{Logos}}(\mathcal{S}\mathrm{h}\,(\mathbf{X}), \mathsf{Set}) = [\mathcal{S}\mathrm{h}\,(\mathbf{X}), \mathsf{Set}]^{\mathrm{lex}}_{\mathrm{cc}}\,,$$

which is the full subcategory of $[\mathcal{S}\mathrm{h}\,(\mathbf{X}), \mathsf{Set}]$ spanned by functors preserving colimits and finite limits. Geometrically, a point x of $\mathbf{X}$ sends a sheaf F on $\mathbf{X}$ to its stalk $F(x) \coloneqq x^* F$ at x.

Examples of categories of points

(i) When X is a locale, the category of points of $\mathcal{S}\mathrm{h}\,(X)$ coincides with the poset $\mathcal{P}\mathrm{t}(X)$ of points of X defined in Section 2.2.4.

(ii) By the universal property of free logoi, the category of points of $\mathbf{A}$ is the category Set. If E is a set, the logos morphism $\mathsf{Set}\,[X] \to \mathsf{Set}$ corresponding to E sends $X : \mathsf{Fin} \to \mathsf{Set}$ to E. More generally a functor $F : \mathsf{Fin} \to \mathsf{Set}$ is sent to the coend $\int^{N \in \mathsf{Fin}} F(N) \times E^N$.

(iii) More generally, the category of points of $\mathbf{A}^C$ is the category $[C, \mathsf{Set}] = \mathcal{P}\mathrm{r}\,(C^{op})$.

(iv) The classifying map $\chi_F : \mathbf{X} \to \mathbf{A}$ of some sheaf F on $\mathbf{X}$ induces a functor $\mathcal{P}\mathrm{t}(\mathbf{X}) \to \mathcal{P}\mathrm{t}(\mathbf{A}) = \mathsf{Set}$ that sends a point x to the stalk $F(x)$. In other words, the topos theory formalizes in a precise way the intuition that a sheaf is a continuous function with values in sets. In a sense, this fact is the whole point of topos theory.

(v) The category of points of an Alexandrov topos $\mathbf{B}C$ is the category $\mathcal{I}\mathrm{nd}(C)$, the free completion of C for filtered colimits.

(vi) In particular, for a topological space X, the points of the topos $\widehat{X}$, dual to the logos $\mathcal{P}\mathrm{r}\,(\mathcal{O}\,(X))$, form the category $\mathcal{I}\mathrm{nd}(\mathcal{O}\,(X))$. This category is equivalent to the poset of filters in $\mathcal{O}\,(X)$. We already mentioned that the inclusion $X \to \widehat{X}$ sends a point of X to the filter of its open neighborhoods.

(vii) When $C = \mathsf{fInj}$ the category of finite sets and injections, the category of points of $\mathbf{B}(\mathsf{fInj})$ is the category of all sets and injections.

(viii) Let $\mathbb{T}$ be an algebraic theory, that is, a category with cartesian products. The points of the topos $\mathbf{B}\langle\mathbb{T}\rangle$ do form the category $\mathcal{P}\mathrm{t}(\mathbf{B}\langle\mathbb{T}\rangle) = [\mathbb{T}, \mathsf{Set}]^{\times}$ of functors preserving cartesian products. Such functors are also called the *models* of the theory $\mathbb{T}$. If $\mathbb{T}$ is the category opposite to the category of free groups on finite sets, then $\mathcal{P}\mathrm{t}(\mathbf{B}\langle\mathbb{T}\rangle)$ is the category of all groups. If $\mathbb{T}$ is the category of affine spaces of finite dimension and algebraic maps, then then $\mathcal{P}\mathrm{t}(\mathbf{B}\langle\mathbb{T}\rangle)$ is the category of all commutative rings.

(ix) For a group G in Set, the category of points of $\mathbf{B}G$ is G itself viewed as a category with one object. This is a way to say that $\mathbf{B}G$ has essentially one point, but this point has G as its group of symmetries. The unique point of $\mathbf{B}G$ is given by the functor $U : \mathsf{Set}^G \to \mathsf{Set}$ sending a G-set to its underlying set.

(x) If G is a group acting on a space X, the category of points of the quotient topos $X /\!\!/ G$ is the groupoid associated to the action of G on the points of X. In comparison, the points of the classical topological quotient X/G are only the isomorphism classes of objects of this groupoid. The difference is that the groupoid keeps the information about the stabilizers of each point.

In the case of the quotient $\mathbb{R}/\!\!/\mathbb{Q}$, the category of points is the set of orbits of $\mathbb{Q}$ in $\mathbb{R}$. In the case of $\mathbb{R}/\!\!/\mathbb{R}_{dis}$ (where $\mathbb{R}_{dis}$ is $\mathbb{R}$ viewed as a discrete space), the category of point is a single point. Nonetheless, $\mathbb{R}/\!\!/\mathbb{R}_{dis}$ is not a point, and there exist many topos morphisms $\mathbf{X} \to \mathbb{R}/\!\!/\mathbb{R}_{dis}$. For example, when X is a manifold, the set of closed differential forms embeds into the set of morphisms $X \to \mathbb{R}/\!\!/\mathbb{R}_{dis}$.

(xi) The category of points of $\mathbf{A}^\bullet$ is the category $\mathsf{Set}^\bullet$ of pointed sets. The functor $\mathcal{P}\mathrm{t}(\mathbf{A}^\bullet) \to \mathcal{P}\mathrm{t}(\mathbf{A})$ induced by the topos morphism $\mathbf{A}^\bullet \to \mathbf{A}$ mentioned earlier is the forgetful functor $\mathsf{Set}^\bullet \to \mathsf{Set}$.

(xii) At the level of points, the embedding $\mathbf{A}^\circ \subset \mathbf{A}$ corresponds to the inclusion of nonempty sets into sets.

(xiii) The category of points of $\mathbf{A}^\to$ is the arrow category $\mathsf{Set}^\to$.

(xiv) We define an *interval* to be a totally ordered set with a minimal and a maximal element that are distinct. For example, the real interval $[0,1]$ is an interval. A morphism of intervals is an increasing map preserving the minimal and maximal elements. It can be proved that the category of points of the topos $\mathcal{P}\mathrm{r}\,(\Delta)$ of simplicial sets is the category of intervals.

Recall that a simplicial set has a geometric realization that is a topological space. The functor $x^* : \mathcal{P}\mathrm{r}\,(\Delta) \to \mathsf{Set}$ corresponding to the interval $[0,1]$ sends a simplicial set to (the underlying set of) its geometric realization.

3.2.3 Quotient logoi and embeddings of topoi Let $u : Y \subset X$ be an embedding of topological spaces. We saw that $\mathcal{O}\,(X) \to \mathcal{O}\,(Y)$ was a surjective map of frames. The situation is the same for the corresponding map of logoi $u^* : \mathcal{S}\mathrm{h}\,(\mathbf{X}) \to \mathcal{S}\mathrm{h}\,(\mathbf{Y})$, which is essentially surjective. In fact, more is true since u^* can be proved to have a fully faithful right adjoint u_*, that is, it is a left-exact localization. If Y is closed and F is a sheaf on Y, the sheaf u_*F is intuitively the extension of F to X obtained by declaring the fibers of u_*F outside of Y to be a single point.[34]

A morphism of logoi $\mathcal{E} \to \mathcal{F}$ shall be called a *quotient* if it is a left-exact localization. The corresponding morphism of topoi shall be called an *embedding*. If $\mathbf{Y} \hookrightarrow \mathbf{X}$ is an embedding, we shall also say that $\mathbf{Y}$ is a *subtopos* of $\mathbf{X}$. At the level of points, the functor $\mathcal{P}\mathrm{t}(\mathbf{Y}) \to \mathcal{P}\mathrm{t}(\mathbf{X})$ induced by an embedding is fully faithful. Classically, the data of a quotient $\mathcal{E} \to \mathcal{F}$ is encoded by the data of a *Lawvere–Tierney topology* on $\mathcal{E}$. In the case where $\mathcal{E} = \mathcal{P}\mathrm{r}\,(C)$ is a presheaf logos, this is also equivalent to the data of a *Grothendieck topology* on the category C. We shall come back to the notion of quotient of logoi in Section 3.4.2.

Examples of embeddings

(i) From our definition of logoi, it is clear that every logos is a quotient of a presheaf logos, that is, that every topos $\mathbf{X}$ is a subtopos of an Alexandrov topos $\mathbf{X} \hookrightarrow \mathbf{B}C$. In fact, it can be proved that every logos is also a quotient of a free logos, that is, that every topos is a subtopos of an affine topos. This situation is similar to that of affine schemes.

(ii) If $Y \hookrightarrow X$ is an embedding of topological spaces or of locales, the corresponding map of topos is also an embedding. Moreover, any subtopos of a localic topos is localic.

(iii) For X a topological space or a locale, the logos morphism $\mathcal{P}\mathrm{r}\,(\mathcal{O}\,(X)) \to \mathcal{S}\mathrm{h}\,(X)$ is a quotient and the corresponding topos morphism $X \to \widehat{X}$ is an embedding of localic topoi. Recall that the points of $\widehat{X}$ are filters in $\mathcal{O}\,(X)$ and that the embedding $X \hookrightarrow \widehat{X}$ sends a point of X to the filters of its open neighborhoods.

(iv) Any fully faithful functor $C \hookrightarrow D$ between small categories induces a quotient $[D, \mathsf{Set}] \to [C, \mathsf{Set}]$ and an embedding $\mathbf{B}C \hookrightarrow \mathbf{B}D$. At the level of points, this embedding corresponds to the fully faithful functor $\mathcal{I}\mathrm{nd}(C) \hookrightarrow \mathcal{I}\mathrm{nd}(D)$.

(v) In particular, the embedding $\underline{2} = \{\varnothing, \{\star\}\} \subset \mathsf{Fin}$ induces a quotient $\mathsf{Set}\,[X] = [\mathsf{Fin}, \mathsf{Set}] \to [\underline{2}, \mathsf{Set}] = \mathsf{Set}^{\to}$. Recall that $[\underline{2}, \mathsf{Set}] = \mathcal{S}\mathrm{h}\,(\mathbf{S})$. We deduce that the Sierpiński space, when viewed as a topos, is a subtopos of the topos of sets: $\mathbf{S} \hookrightarrow \mathbf{A}$. At the level of points, this embedding corresponds to the inclusion $\{\varnothing, \{\star\}\} \subset \mathsf{Set}$. In other words, the Sierpiński topos can be said to classify sets with at most one element.

(vi) Another example is given by $\mathsf{Fin}^\circ \hookrightarrow \mathsf{Fin}$. This describes the topos $\mathbf{A}^\circ$ as a subtopos of $\mathbf{A}$. We already saw that at the level of points, this corresponds to the inclusion of nonempty sets in sets.

(vii) Yet another example is given by $C \hookrightarrow C_{\mathrm{rex}}$, where C_{rex} is the free completion of C for finite colimits. This builds a quotient of logoi $\mathsf{Set}\,[C^{op}] = [C_{\mathrm{rex}}, \mathsf{Set}] \to [C, \mathsf{Set}]$ and a dual embedding of topoi $\mathbf{B}C \to \mathbf{A}^{C^{op}}$. At the level of points, this embedding corresponds to the fully faithful functor $\mathcal{I}\mathrm{nd}(C) \hookrightarrow \mathcal{P}\mathrm{r}\,(C)$. With the first example, this proves that any topos $\mathbf{X}$ can be embedded in some affine topos $\mathbf{X} \hookrightarrow \mathbf{B}C \hookrightarrow \mathbf{A}^{C^{op}}$.

[34] When Y is not closed, the values of u_*F at the boundary of F are more involved.

(viii) The fully faithful inclusion $\mathsf{Fin}^{\simeq} \hookrightarrow \mathsf{Fin}^{\to}$ of isomorphisms into morphisms builds an embedding $\mathbf{A}^{\simeq} \hookrightarrow \mathbf{A}^{\to}$.

3.2.4 Products of topoi In analogy with locales/frames and commutative rings/schemes, the cartesian products of topoi correspond dually to a tensor product of logoi. If we forgot the existence of finite limits in a logos, the resulting category is a *presentable category*, that is, a localization of a presheaf category. We shall say a few words about presentable categories in Section 3.3.3. The tensor product of logoi is defined at the level of their underlying presentable categories. A morphism of presentable categories is defined as a functor preserving all colimits. For three such categories $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, a functor $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is called *bilinear* if it preserves colimits in each variable. Then, the data of a bilinear functor $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is equivalent to that of a morphism of presentable categories $\mathcal{A} \otimes \mathcal{B} \to \mathcal{C}$ for a certain presentable category $\mathcal{A} \otimes \mathcal{B}$. This category can be described as $\mathcal{A} \otimes \mathcal{B} = [\mathcal{A}^{op}, \mathcal{B}]^{c}$ (where $[\mathcal{A}^{op}, \mathcal{B}]^{c}$ is the category of functors preserving limits). This formula shows in particular that Set is the unit of this product. A comparison between this tensor product and that of abelian groups is sketched in Table 15.

Examples of products

(i) The punctual topos $\mathbf{1}$ is the unit for the product. The equation $\mathbf{1} \times \mathbf{X} = \mathbf{X}$ for topoi is equivalent to $\mathsf{Set} \otimes \mathcal{E} = \mathcal{E}$ for logoi.

(ii) The tensor product of presentable categories is such that $\mathcal{P}\mathrm{r}\,(C) \otimes \mathcal{P}\mathrm{r}\,(D) = \mathcal{P}\mathrm{r}\,(C \times D)$. We deduce that $\mathbf{B}C \times \mathbf{B}D = \mathbf{B}(C \times D)$.

(iii) The free nature of $\mathsf{Set}\,[C]$ and the universal property of sums implies that $\mathsf{Set}\,[C] \otimes \mathsf{Set}\,[D] = \mathsf{Set}\,[C \coprod D]$, that is, $\mathbf{A}^{C} \times \mathbf{A}^{D} = \mathbf{A}^{C \coprod D}$.

(iv) Given two topoi $\mathbf{X}$ and $\mathbf{Y}$, the logos corresponding to $\mathbf{X} \times \mathbf{Y}$ can be described as the category of sheaves on $\mathbf{X}$ with values in $\mathcal{S}\mathrm{h}\,(Y)$ (or reciprocally):

$$\mathcal{S}\mathrm{h}\,(\mathbf{X}) \otimes \mathcal{S}\mathrm{h}\,(\mathbf{Y}) = \left[\mathcal{S}\mathrm{h}\,(\mathbf{X})^{op}, \mathcal{S}\mathrm{h}\,(\mathbf{Y})\right]^{c} = \left[\mathcal{S}\mathrm{h}\,(\mathbf{Y})^{op}, \mathcal{S}\mathrm{h}\,(\mathbf{X})\right]^{c}.$$

3.2.5 Fiber products of topoi An important difference between topoi and topological space is the way fiber products are computed. The fact that topoi live in a 2-category requires the use of the so-called *pseudo fiber products*. We are only going to explain intuitively the situation. Let us consider a cartesian square

$$\begin{tikzcd}
\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbf{X} \arrow[d, "f"] \\
\mathbf{Y} \arrow[r, "g"] & \mathbf{Z}
\end{tikzcd}$$

If $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ were topological spaces or locales, $\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}$ would be the subspace of $\mathbf{X} \times \mathbf{Y}$ spanned by pairs (x, y) such that $f(x) = g(y)$ in $\mathbf{Z}$. The computation of fiber product of topoi is similar, but since the points of topoi leave in categories, the previous equality has to be replaced by an isomorphism. The choice of an isomorphism $f(x) \simeq g(y)$ being a structure and not a property, the map $\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y} \to \mathbf{X} \times \mathbf{Y}$ will no longer be an embedding.[35] In the simplest case of the fiber product

$$\begin{tikzcd}
\mathbf{1} \times_{\mathbf{B}G} \mathbf{1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbf{1} \arrow[d, "b"] \\
\mathbf{1} \arrow[r, "b"] & \mathbf{B}G
\end{tikzcd}$$

we have $\mathbf{1} \times_{\mathbf{B}G} \mathbf{1} = G$, since the choice of an isomorphism $b \simeq b$ is the choice of an element of G.

More generally, let X be a space and G a discrete group acting on X. Recall from the examples that the quotient $X /\!\!/ G$ of X by G computed in the category of topoi is dual to the logos $\mathcal{S}\mathrm{h}\,(X, G)$ of equivariant

[35] The fiber of this maps at a pair (x, y) being the choices of isomorphisms $f(x) \simeq g(y)$.

sheaves on X. It can be proved that the fibers of the quotient map $q : X \to X /\!\!/ G$ are isomorphic to G. Let x be a point of X and $\overline{x}$ be the corresponding point in $X /\!\!/ G$; then we have a cartesian square in the 2-category of topoi

$$\begin{tikzcd} G = \mathbf{1} \times_{X/\!\!/G} X \arrow[r, "\mathrm{orbit}(x)"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "q"] \\ \mathbf{1} \arrow[r, "\overline{x}"] & X /\!\!/ G \end{tikzcd}$$

where the top map sends G to the orbit of x. We mentioned that the category of points of $X /\!\!/ G$ is the groupoid associated to the action of G on the points of X. So an isomorphism $y \simeq x$ in this groupoid is equivalent to the choice of y in the orbit of x and of an element of g such that $g.x = y$. But this data is equivalent to the choice of g only. This is why the fiber is G. In fact, the morphism $X \to X /\!\!/ G$ can even be proved to be a principal G-cover. This is one of the nice features of quotients of discrete group actions in Topos–the quotient map is always a principal cover.

A variation on the same theme is the computation of fibers of the diagonal map $\mathbf{X} \to \mathbf{X} \times \mathbf{X}$ of a topos. Let $(x, y) : \mathbf{1} \to \mathbf{X} \times \mathbf{X}$ be a pair of points of $\mathbf{X}$. By a classical trick of category theory, the fiber product $\mathbf{1} \times_{\mathbf{X}\times\mathbf{X}} \mathbf{X}$ is equivalent to $\Omega_{x,y}\mathbf{X} := \mathbf{1} \times_{\mathbf{X}} \mathbf{1}$, that is, to "path space" between x and y in $\mathbf{X}$. If $\mathbf{X}$ is a topological space or even a locale, this intersection is empty if $x \neq y$ and a single point if $x = y$. But within a topos, points can have isomorphisms, and the topos $\mathbf{1} \times_{\mathbf{X}} \mathbf{1}$ is precisely the topos classifying the isomorphisms between x and y. It is empty if x and y are not isomorphic, but its category of points is the set $\mathrm{Iso}_{\mathcal{P}\mathrm{t}(\mathbf{X})}(x, y)$ if they are. It is possible to prove that $\Omega_{x,y}\mathbf{X}$ is always a localic topos. It follows from these observations that the diagonal map $\mathbf{X} \to \mathbf{X} \times \mathbf{X}$ of a topos is not necessarily an embedding!

Another important example of fiber product is the computation the fiber of the map $\mathbf{A}^\bullet \to \mathbf{A}$. Recall that this map sends a pointed space to its underlying set. Intuitively, the fiber over a set E should be the choice of a base point in E. One can prove that this is indeed the case: recall that $\mathbf{B}E$ is the discrete topos associated to a set E; then there exists a cartesian square

$$\begin{tikzcd} \mathbf{B}E \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbf{A}^\bullet \arrow[d] \\ \mathbf{1} \arrow[r, "\chi_E"] & \mathbf{A}. \end{tikzcd}$$

For this reason, $\mathbf{A}^\bullet \to \mathbf{A}$ is called the *universal family of sets*.

3.2.6 Étale domains We now turn to a central notion of topos theory. We explained in the introduction that, in the same way locales are based on the notion of open domain, the theory of topoi is based on the notion of étale morphism (see Table 4). Recall that an open embedding $U \to X$ was defined as an open quotient of frames $U \cap - : \mathcal{O}(X) \to \mathcal{O}(X)_{/U}$ for some U in $\mathcal{O}(X)$. The corresponding notion for logoi will correspond to étale maps. Let $\mathcal{E}$ be a logos and F an object of $\mathcal{E}$. The base change along the map $F \to 1$ provides a morphism of logoi $\epsilon_F^* : \mathcal{E} \to \mathcal{E}_{/F}$ called an *étale extensions*. If $\mathcal{E} = \mathcal{S}\mathrm{h}(\mathbf{X})$, the corresponding morphisms of topoi will be denoted $\epsilon_F : \mathbf{X}_F \to \mathbf{X}$ and called an *étale morphism* or a *local homeomorphism*. Intuitively, an étale morphism is a morphism whose fibers are discrete. We are going to see that this is indeed the case. We are also going to explain the universal property of $\mathcal{E} \to \mathcal{E}_{/F}$.

Examples of étale morphisms

(i) The identity morphism of a topos $\mathbf{X}$ is étale.

(ii) The morphism $\varnothing \to \mathbf{X}$ from the empty topos is étale.

(iii) The morphism $\mathbf{A}^\bullet \to \mathbf{A}$ is étale. Recall that the object X in $\mathsf{Set}[X] = [\mathsf{Fin}, \mathsf{Set}]$ is represented by the object 1 in Fin. Then the result is a consequence of the formula $[\mathsf{Fin}, \mathsf{Set}]_{/X} = \left[\mathsf{Fin}_{1/}, \mathsf{Set}\right] = [\mathsf{Fin}^\bullet, \mathsf{Set}]$. We shall see that it is the universal étale morphism.

(iv) The proof is the same to show that the morphism $\mathbf{A}^\bullet \to \mathbf{A}^\circ$ is étale. We shall see that it is also surjective.

(v) The morphism $b : \mathbf{1} \to \mathbf{B}G$ is étale. Recall that it corresponds dually to the forgetful functor $U : \mathsf{Set}^G \to \mathsf{Set}$. Let G_λ be the action of G on itself by left translation. Then we have $\mathsf{Set} = \left(\mathsf{Set}^G\right)_{/G_\lambda}$.[36]

The morphism $b : \mathbf{1} \to \mathbf{B}G$ is moreover étale, it can be proved to be a principal covering with structure group G. It is in fact the universal cover of $\mathbf{B}G$.

The étale extension $\epsilon_F^* : \mathcal{E} \to \mathcal{E}_{/F}$ has an important universal property. The object $\epsilon_F^*(F)$ in $\mathcal{E}_{/F}$ corresponds to the map $p_1 : F^2 \to F$, which admits a canonical section given by the diagonal $\Delta : F \to F^2$. Then pair (ϵ_F^*, Δ) is universal for creating a global section of F. More precisely, if $u^* : \mathcal{E} \to \mathcal{F}$ is a logos morphism and $\delta : 1 \to u^* F$ a global section of F in $\mathcal{F}$, there exists a unique factorization of u^* via $\mathcal{E}_{/F}$ such that $v^*(\Delta) = \delta$:

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "u^*"] \arrow[dr, "\epsilon_F^*"'] & & \mathcal{F} \\ & \mathcal{E}_{/F} \arrow[ur, "v^*"'] & \end{tikzcd}$$

This property is to be compared with the splitting of a polynomial in commutative algebra, as shown in Table 12.

Table 12: Étale analogies

Algebraic geometry	*Topos theory*
ring A	logos $\mathcal{E}$
separable polynomial $P(x)$ in $A[x]$	object F of $\mathcal{E}$
separable (or étale) extension $A \to A[x]/P(x)$	étale extension $\mathcal{E} \to \mathcal{E}_{/F}$
root of P in A $=$ retraction of $A \to A[x]/P(x)$	global section $1 \to F$ $=$ retraction of $\mathcal{E} \to \mathcal{E}_{/F}$

This property has also an important geometric interpretation. Suppose that $\mathcal{E} = \mathcal{S}\mathrm{h}(\mathbf{X})$ and $\mathcal{F} = \mathcal{S}\mathrm{h}(\mathbf{Y})$. Recall from the examples that the data of a pointed object $\delta : 1 \to F$ in $\mathcal{F}$ is equivalent to a logos morphism $\mathsf{Set}[X^\bullet] \to \mathcal{F}$. Then, the data of (u^*, δ) above is equivalent to a commutative square of logoi

$$\begin{tikzcd} \mathsf{Set}[X] \arrow[r, "X \,\mapsto\, u^* F"] \arrow[d] & \mathcal{E} \arrow[d, "u^*"] \\ \mathsf{Set}[X^\bullet] \arrow[r, "1\to X^\bullet \,\mapsto\, 1 \xrightarrow{\delta} u^* F"'] & \mathcal{F}. \end{tikzcd}$$

Geometrically, this corresponds to a square of topoi

$$\begin{tikzcd} \mathbf{Y} \arrow[r, "\chi_\delta"] \arrow[d, "u"'] & \mathbf{A}^\bullet \arrow[d] \\ \mathbf{X} \arrow[r, "\chi_F"] & \mathbf{A} \end{tikzcd}$$

[36] For a G-set F, the data of an equivariant morphism $\varphi : F \to G_\lambda$ is equivalent to a trivialization of the action of G on F. Let $E \subset F$ be the elements of F sent to the unit of G by φ; then we have $G \times E \simeq F$ as G-sets.

Therefore, the universal property of $\mathbf{X}_F$ says exactly that it is the fiber product of $\mathbf{X} \to \mathbf{A} \leftarrow \mathbf{A}^\bullet$:

$$\begin{tikzcd}
\mathbf{Y} \arrow[dr, dashed, "v"] \arrow[drr, bend left, "\chi_\delta"] \arrow[ddr, bend right, "u"'] & & \\
& \mathbf{X}_F \arrow[r] \arrow[d, "\epsilon_F"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbf{A}^\bullet \arrow[d] \\
& \mathbf{X} \arrow[r, "\chi_F"] & \mathbf{A}
\end{tikzcd}$$

The fact that any étale morphism is a pull back of the universal family of sets $\mathbf{A}^\bullet \to \mathbf{A}$ says that it is also the *universal étale morphism*. The previous computation of the fibers of $\mathbf{A}^\bullet \to \mathbf{A}$ gives a proof that the fiber of ϵ_F at a point x of $\mathbf{X}$ is the stalk $F(x)$ of F. If X is a topological space and F is a sheaf on X, one can prove that $X_F \to X$ is the *espace étalé* corresponding to the sheaf [12, II.1.2]. The construction $F \mapsto \mathbf{X}_F$ of the "topos étalé" of a sheaf builds a functor

$$\mathcal{S}\mathrm{h}(\mathbf{X}) \hookrightarrow \mathsf{Topos}_{/\mathbf{X}} \qquad \text{(Sheaves as étale maps)}$$

whose image is spanned by étale morphisms over $\mathbf{X}$, or *étale domains of* $\mathbf{X}$. This functor is fully faithful and preserves colimits and finite limits. In other words, sheaves and their operations are faithfully represented as étale maps. Together with (Sheaves as functions), this completes the algebraic/geometric interpretation of sheaves mentioned in Table 4.

3.2.7 Open domains In accordance with what is true for topological spaces, we define an *open embedding* of a topos $\mathbf{X}$ to be an étale morphism $\mathbf{Y} \to \mathbf{X}$ that is also an embedding. The corresponding morphisms of logoi will be called *open quotients*. For an object U in a logos $\mathcal{S}\mathrm{h}(\mathbf{X})$, the functor $\epsilon_U^* : \mathcal{S}\mathrm{h}(\mathbf{X}) \to \mathcal{S}\mathrm{h}(\mathbf{X})_{/U}$ is a quotient if and only if the canonical morphism $U \to 1$ is a monomorphism. This characterizes open domains as the étale domains $\mathbf{X}_U \to \mathbf{X}$ where U is a subterminal object. The étale domains of a topos $\mathbf{X}$ form a full subcategory $\mathcal{O}(\mathbf{X}) \subset \mathcal{S}\mathrm{h}(\mathbf{X})$ that coincides with the poset $Sub(1)$ of subobjects of 1 in $\mathcal{S}\mathrm{h}(\mathbf{X})$.

Intuitively, an étale morphism is an embedding if its fibers are either empty or a point. Recall the embedding $\mathbf{S} \subset \mathbf{A}$ of Sierpiński space into the topos of sets. It can be proved that an étale domain is open if and only if the classifying map $\mathbf{X} \to \mathbf{A}$ factors through $\mathbf{S} \subset \mathbf{A}$. This says that the Sierpiński space, when viewed as a topos, keeps the nice property of classifying open domains:

$$\begin{tikzcd}
\mathbf{X}_U \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbf{1} \arrow[r] \arrow[d, "\text{univ. open map}"] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbf{A}^\bullet \arrow[d, "\text{univ. étale map}"] \\
\mathbf{X} \arrow[r] \arrow[rr, bend right, "\chi_U"'] & \mathbf{S} \arrow[r, hook, "\chi_{\{1\}}"] & \mathbf{A}
\end{tikzcd}$$

Examples of open embeddings

(i) The open embeddings of a localic topos coincides with the open domains of the corresponding locale.

(ii) Let $C \subset D$ be a full subcategory that is a *cosieve* (stable by post-composition). Then the localization $[D, \mathsf{Set}] \to [C, \mathsf{Set}]$ is open and the embedding $\mathbf{B}C \to \mathbf{B}D$ is open. In fact, the poset of open quotients of $[D, \mathsf{Set}]$ can be proved to be exactly the poset of cosieves of D.

(iii) For any topos $\mathbf{X}$, the identity of $\mathbf{X}$ and the canonical morphism $\varnothing \to \mathbf{X}$ are always open embeddings.

(iv) The subtopos $\mathbf{A}^\circ \subset \mathbf{A}$ is open. This is the only non trivial open subtopos of $\mathbf{A}$. The classifying morphism $\mathbf{A} \to \mathbf{S}$ of this open domain is a retraction of the embedding $\mathbf{S} \hookrightarrow \mathbf{A}$.

A topos $\mathbf{X}$ is said to have *enough open domains* if all sheaves on $\mathbf{X}$ can be written as pastings of open domains, that is, if the subcategory $\mathcal{O}(\mathbf{X}) \subset \mathcal{S}\mathrm{h}(\mathbf{X})$ generates by colimits. A topos has enough open domains if and only if it is localic, that is, in the image of the functor $\mathsf{Locale} \to \mathsf{Topos}$. Not every topos has enough open domain and this is a very important fact of the theory. The topos $\mathbf{B}G$ does not have enough open domains. The computation shows that the only open domains of $\mathbf{B}G$ are the identity and $\varnothing \to \mathbf{B}G$, that is, $\mathbf{B}G$ has the same open domains as the point.

The intuitive explanation of what is going on is simple enough. Any morphism $\mathbf{B}G \to \mathbf{S}$ induces a functor $G = \mathcal{P}\mathrm{t}(\mathbf{B}G) \to \mathcal{P}\mathrm{t}(\mathbf{S}) = \{0 < 1\}$. Since the only isomorphisms in the poset $\{0 < 1\}$ are the identities, any functor from G has to be constant. This is why there are so few open domains. In other words, the Sierpiński space does not have "enough room" to reflect that some spaces have many morphisms between points. This is actually the source of the insufficiency of the notion topological space. In its essence, the theory of topoi proposes to enlarge the "gauge" poset $\{0 < 1\}$ by the "gauge" category Set. Doing so creates "enough room" to capture faithfully many spaces with a category of points.

3.2.8 Closed embedding Let $\mathbf{X}_U \hookrightarrow \mathbf{X}$ be an open domain corresponding to an object U in $\mathcal{S}\mathrm{h}(\mathbf{X})$. It is possible to define a *closed complement* for $\mathbf{X}_U$, but we shall not detail this.

Examples of closed embeddings

(i) The closed embeddings of locales gives closed embeddings of topoi.

(ii) We saw that cosieves $C \subset D$ correspond to open embeddings $\mathbf{B}C \to \mathbf{B}D$. Reciprocally, *sieves* (subcategories stable by pre-composition) corresponds to closed embeddings. If $C \subset D$ is a cosieve, the full subcategory C' of D spanned by the objects not in C is a sieve. Then $\mathbf{B}C \hookrightarrow \mathbf{B}D$ and $\mathbf{B}C' \hookrightarrow \mathbf{B}D$ are complementary open and closed embeddings.

(iii) The closed complement of the open embedding $\mathbf{A}^\circ \subset \mathbf{A}$ is the morphism $\chi_\varnothing : \mathbf{1} \hookrightarrow \mathbf{A}$ classifying the empty set.

3.2.9 Socle and hyperconnected topoi For any topos $\mathbf{X}$, the poset $\mathcal{O}(\mathbf{X})$ of its open domains is a frame and defines a locale $\mathcal{S}\mathrm{ocle}(\mathbf{X})$. This provides a functor $\mathcal{S}\mathrm{ocle} : \mathsf{Topos} \to \mathsf{Locale}$, which is the left adjoint to the inclusion $\mathsf{Locale} \to \mathsf{Topos}$. The unit of this adjunction provides a canonical projection $\mathbf{X} \to \mathcal{S}\mathrm{ocle}(\mathbf{X})$. Intuitively, the socle of $\mathbf{X}$ is the best approximation of $\mathbf{X}$ that can be built out of open domains only.[37] A topos is called *hyperconnected* if its socle is a point. In other words, the hyperconnected topoi are exactly the kind of spatial object invisible from the usual point of view on topology (see [19] for more properties).

Examples of socles and hyperconnected topoi

(i) The inclusion of categories $\mathsf{Poset} \to \mathsf{Cat}$ has a left adjoint τ. The poset $\tau(C)$ has the same objects as C and $x \le y$ if there exists an arrow $x \to y$ in C. The socle of $\mathbf{B}C$ is the Alexandrov locale associated to $\tau(C)$. Its frame of open domains is $[C, \underline{2}]$.

(ii) A category C is called hyperconnected if any two objects have arrows going both ways between them. This is equivalent to $\tau(C) = 1$. Then, the corresponding Alexandrov topos $\mathbf{B}C$ is hyperconnected.

(iii) In particular, the topoi $\mathbf{A}^\bullet$, $\mathbf{A}^\circ$, $\mathbf{B}G$ are all hyperconnected, but not $\mathbf{A}$ (because of the strictness of $\varnothing$).

(iv) Examples of hyperconnected topoi are also given by the so-called "bad quotients" in topology. Let $\mathbb{Q}$, viewed as discrete group, act on $\mathbb{R}$ by translation. Every orbit is dense, and the topological quotient is an uncountable set with the discrete topology. The topos quotient $\mathbb{R}/\!\!/\mathbb{Q}$ is the topos corresponding to the logos of $\mathbb{Q}$-equivariant sheaves on $\mathbb{R}$. It stays true in the category of topoi that open domains of the quotient $\mathbb{R}/\!\!/\mathbb{Q}$ are equivalent to saturated open domains of $\mathbb{R}$, and this proves that $\mathbb{R}/\!\!/\mathbb{Q}$ is a hyperconnected topos. One can compute that its category of points, is exactly the set of orbits of the action. So the topos $\mathbb{R}/\!\!/\mathbb{Q}$ has the same points and open domains as the topological quotient, but it

[37] The corresponding logos morphism $\mathcal{S}\mathrm{h}(\mathcal{S}\mathrm{ocle}(\mathbf{X})) \to \mathcal{S}\mathrm{h}(\mathbf{X})$ is full and faithful. Its image is the smallest full category containing $\mathcal{O}(\mathbf{X})$ and stable by colimits and finite limits. In other words, it is the subcategory of sheaves that can be generated by open domains.

has more sheaves! This topos enjoys many nice properties missing for the topological quotient. For example, it can be proved that its fundamental group is $\mathbb{Q}$. This is a good example of how defining a spatial object by its category of étale domains and not only its open domains leads to more regular objects.

3.2.10 Surjections The notion of surjection of topoi is more subtle than the one of locales. The definition is based on the following property of surjection of spaces. Let $u : Y \to X$ be a continuous map and $f : F \to G$ a morphism of sheaves on X. Intuitively, f an isomorphism if and only if all the maps $f(x) : F(x) \to G(x)$ between the stalks are bijections. If f is an isomorphism, then so is $u^*f : u^*F \to u^*G$ in $\mathcal{S}\mathrm{h}(Y)$. If u is not surjective, the condition "u^*f is an isomorphism" is weaker than $F \simeq G$ because it does not say anything about the stalks that are not in the image of u. But if u is surjective, the condition "u^*f is an isomorphism" becomes equivalent to "f is an isomorphism".

A functor $f : C \to D$ is called *conservative* if it is true that "u is an isomorphism" $\Leftrightarrow$ "$f(u)$ is an isomorphism". A morphism of topoi $f : \mathbf{Y} \to \mathbf{X}$ is called a *surjection* if the corresponding morphism of logoi $f^* : \mathcal{S}\mathrm{h}(\mathbf{X}) \to \mathcal{S}\mathrm{h}(\mathbf{Y})$ is conservative.

Examples of surjections

(i) The morphism $\mathbf{1} \to \mathbf{B}G$ is a surjection. This is because the forgetful functor $\mathsf{Set}^G \to \mathsf{Set}$ is conservative.

(ii) The functor $[\mathsf{Fin}^\circ, \mathsf{Set}] \to [\mathsf{Fin}^\bullet, \mathsf{Set}]$ is conservative. Thus the morphism $\mathbf{A}^\bullet \to \mathbf{A}^\circ$ is surjective.

(iii) Let $\mathbf{X}$ be a topos and E be a set of points of $\mathbf{X}$. Then there exists a logos morphism $\mathcal{S}\mathrm{h}(\mathbf{X}) \to [E, \mathsf{Set}]$ sending a sheaf F to the family of its stalks corresponding to the points in E. Dually, this corresponds to a topos morphism $\mathbf{B}E \to \mathbf{X}$ where $\mathbf{B}E$ is the discrete topos associated to the set E. A topos is said to have *enough points* if there exists a set E such that the topos morphism $\mathbf{B}E \to \mathbf{X}$ is surjective. Intuitively, this means that a morphism $F \to G$ between sheaves on $\mathbf{X}$ is a isomorphism if and only if the morphism $F(x) \to G(x)$ is a bijection for all x in E.

Recall from Section 2.2.13 that topological spaces can be faithfully described as locales equipped with a surjective map from a discrete locale. The corresponding notion for topoi, which would be a categorification of topological spaces, is a topos equipped with a surjective morphism from a discrete topos. Such a notion has been studied in [10].

3.2.11 Image factorization With the notions of embedding and surjection, it is possible to define the image of a morphism of topoi $u : \mathbf{Y} \to \mathbf{X}$. From the corresponding morphism of logoi $f^* : \mathcal{S}\mathrm{h}(\mathbf{X}) \to \mathcal{S}\mathrm{h}(\mathbf{Y})$, we extract the class W of maps inverted by u^* and construct the left-exact localization of $\mathcal{S}\mathrm{h}(\mathbf{X}) /\!\!/ W$ generated by W.[38] We deduce a factorization

$$\begin{tikzcd}
\mathcal{S}\mathrm{h}(\mathbf{X}) \arrow[rr, "u^*"] \arrow[dr, "e^*", "\text{lex localization}"'] & & \mathcal{S}\mathrm{h}(\mathbf{Y}) \\
& \mathcal{S}\mathrm{h}(\mathbf{X}) /\!\!/ W \arrow[ur, "s^*", "\text{conservative}"'] &
\end{tikzcd}$$

where e^* is a quotient and s^* is conservative by design. In the corresponding geometric factorization

$$\begin{tikzcd}
\mathbf{Y} \arrow[rr, "u"] \arrow[dr, "s", "\text{surjection}"'] & & \mathbf{X} \\
& \mathrm{Im}(u) \arrow[ur, "e", "\text{embedding}"'] &
\end{tikzcd}$$

the subtopos $\mathrm{Im}(u) \hookrightarrow \mathbf{X}$ is called the *image* of u.

[38] Technically, there is a size issue, and we need to prove that W can be generated by a single map $f : A \to B$ in $\mathcal{S}\mathrm{h}(\mathbf{X})$. This is possible because f is an accessible functor between accessible categories.

Examples of image factorization

(i) Given a functor $C \to D$ between small categories, the image factorization of $\mathbf{B}C \to \mathbf{B}D$ is $\mathbf{B}C \to \mathbf{B}C' \to \mathbf{B}D$, where $C \to C' \to D$ is the essentially surjective/fully faithful factorization of $C \to D$.

(ii) In particular, the image of the morphism $\mathbf{A}^{\bullet} \to \mathbf{A}$ is the topos $\mathbf{A}^{\circ}$.

(iii) In the case of an object $x : 1 \to D$, the image $\mathbf{1} \to \mathbf{B}D$ is $\mathbf{B}(\mathrm{End}\,(x))$ (dual to the logos of action of the monoid $\mathrm{End}\,(x)$ on sets). The category of points of this topos consists in all the retracts of x in D.

3.2.12 Étale covers The image factorization in the category Topos echoes with another image factorization that exists *within* a given logos $\mathcal{E}$. Recall that for any map $f : A \to B$, the diagonal of f is the map $A \to A \times_B A$. The object $A \times_B A$ is a subobject of $A \times A$ that intuitively corresponds to the relation "having the same image by f". The coequalizer of $A \times_B A \rightrightarrows A$ is the quotient of A by this relation. The map f is called a *cover* if this coequalizer is B. This is a way to say that f is surjective. The map f is called a *monomorphism* if its diagonal $A \to A \times_B A$ is an isomorphism. This is a way to say that f is injective. We shall denote by $A \twoheadrightarrow B$ the covers and by $A \rightarrowtail B$ the monomorphisms. In the logos Set, the covers and monomorphisms are exactly the surjections and injections. In the logos $\mathcal{S}\mathrm{h}\,(X)$ of sheaves on a topological space X, covers and monomorphisms are the maps that are surjective and injective stalk-wise.

Any map f in a logos can be factored uniquely in a cover followed by a monomorphism:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, two heads, "c", "\text{cover}"'] & & B \\ & \mathrm{Im}\,(f) \arrow[ur, tail, "m"', "\text{monomorphism}"] & \end{tikzcd}$$

where the object $\mathrm{Im}\,(f)$, called the *image* of f, is the defined as the coequalizer of $A \times_B A \rightrightarrows A$.

If $\mathcal{E} = \mathcal{S}\mathrm{h}\,(\mathbf{X})$, the correspondence (Sheaves as étale maps) transforms the previous factorization into the surjection–embedding factorization:

$$\begin{tikzcd} \mathbf{X}_A \arrow[rr, "\mathbf{x}_f"] \arrow[dr, "\mathbf{x}_c", "\text{étale + surjection = étale cover}"'] & & \mathbf{X}_B \\ & \mathbf{X}_{\mathrm{Im}(f)} = \mathrm{Im}\,(\mathbf{X}_f) \arrow[ur, "\mathbf{x}_m"', "\text{étale + embedding = open embedding}"] & \end{tikzcd}$$

In other words, the correspondence (Sheaves as étale maps) transforms covers into surjections and monomorphisms into embeddings. We saw that the class of monomorphisms produced this way, that is, monomorphisms that are étale, are the open embeddings. The class of surjections produced this way, that is, surjections that are étale, are called *étale covers*.

Examples of étale covers

(i) Any surjective local homeomorphism between topological spaces defines an étale cover between the associated topoi.

(ii) In particular, if U_i is an open covering of a space X, then $U = \coprod_i U_i \to X$ is an étale cover of the topos corresponding to X.

(iii) The étale covers of a topos $\mathbf{X}$ are equivalent to objects U in $\mathcal{S}\mathrm{h}\,(\mathbf{X})$ such that the map $U \to 1$ is a cover. Such objects are also called *inhabited* since they correspond intuitively to sheaves whose stalks are never empty. When viewed as a function, a sheaf $\mathbf{X} \to \mathbf{A}$ is inhabited if and only if it takes its values in the subtopos $\mathbf{A}^{\circ} \subset \mathbf{A}$. Finally, an étale cover of $\mathbf{X}$ is equivalent to a morphism $\mathbf{X} \to \mathbf{A}^{\circ}$.

(iv) The map $\mathbf{1} \to \mathbf{B}G$ is an étale cover since it is étale and surjective.

(v) More generally, if a discrete group G acts on a space X, the quotient map $q : X \to X /\!\!/ G$ is also an étale cover. In particular, the map $\mathbb{R} \to \mathbb{R} /\!\!/ \mathbb{Q}$ is étale.

(vi) The maps $\mathbf{A}^\bullet \to \mathbf{A}^\circ$ is an étale cover since we saw that it was étale and surjective. Recall that it is given by $\mathsf{Set}[X^\circ] \to \mathsf{Set}[X^\circ]_{/X^\circ}$. The fact that X° is an inhabited object is the universal property of the logos $\mathsf{Set}[X^\circ]$. Any non empty object E in a logos $\mathcal{E}$ defines a unique logos morphism $\mathsf{Set}[X^\circ] \to \mathcal{E}$ sending X° to E.

(vii) The factorization $\mathbf{A}^\bullet \to \mathbf{A}^\circ \to \mathbf{A}$ corresponds to the image factorization $X \to X^\circ \to 1$ of the map $X \to 1$ in $\mathsf{Set}[X]$. It is in fact the universal such factorization. Let F be a sheaf on $\mathbf{X}$ and let $F \to \mathrm{Im}(F) \to 1$ be the cover-monomorphism factorization of the canonical mal $F \to 1$. Then the image factorization of $\mathbf{X}_F \to \mathbf{X}$ can be defined by the pullbacks

$$\begin{tikzcd}
\mathbf{X}_F \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] \arrow[dd, bend right=60, "\text{étale}"'] & \mathbf{A}^\bullet \arrow[d, "\text{étale cover}"] \\
\mathbf{X}_{\mathrm{Im}(F)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbf{A}^\circ \arrow[d, "\text{open embedding}"] \\
\mathbf{X} \arrow[r, "\chi_F"] & \mathbf{A}.
\end{tikzcd}$$

3.2.13 Constant sheaves Since Set is the initial logos, every logos $\mathcal{E}$ comes with a canonical morphism $e^* : \mathsf{Set} \to \mathcal{E}$. This functor is left adjoint to the *global section functor* $\Gamma = e_* : \mathcal{E} \to \mathsf{Set}$, which sends a sheaf F to $\Gamma(F) = \mathrm{Hom}_{\mathcal{E}}(1, F)$. The sheaves in the image of e^* are called *constant sheaves*. Geometrically, $e^* : \mathsf{Set} \to \mathcal{S}\mathrm{h}(\mathbf{X})$ corresponds to the unique morphism $\mathbf{X} \to \mathbf{1}$. The interpretation of constant sheaves is that they are the pullback of sheaves on the point. In other words, they are the sheaves with a constant classifying morphism $\mathbf{X} \to \mathbf{1} \to \mathbf{A}$.

3.2.14 Connected topos The previous functor $e^* : \mathsf{Set} \to \mathcal{E}$ is not fully faithful in general. The only case where it is not faithful is when $\mathcal{E} = 1$ is the terminal logos (empty topos). But, when e^* is faithful, there might still be more morphisms between constant sheaves than between the corresponding sets. This is in fact characteristic of spaces with several connected components. For this reason, the logos $\mathcal{E}$ and the corresponding topos are called *connected* whenever e^* is fully faithful. More generally, a morphism of topos $u : \mathbf{Y} \to \mathbf{X}$ is called *connected* if the corresponding morphism of logoi $u^* : \mathcal{S}\mathrm{h}(\mathbf{X}) \to \mathcal{S}\mathrm{h}(\mathbf{Y})$ is fully faithful. The geometric intuition is that u has connected fibers. These definitions coincides with the existing notions for topological spaces.

Examples of connected topoi

(i) If X is a connected topological space or locale, then the corresponding topos is also.

(ii) An Alexandrov topos $\mathbf{B}C$ is connected if and only if the category C is connected (all objects can be linked by a zig-zag of morphisms).

(iii) In particular, the topoi $\mathbf{1}$, $\mathbf{A}$, $\mathbf{A}^C$, $\mathbf{A}^\bullet$, $\mathbf{A}^\circ$, and $\mathbf{B}G$ are all connected.

(iv) Any hyperconnected topos is connected.

3.2.15 connected–disconnected factorization Given a morphism of topoi $u : \mathbf{Y} \to \mathbf{X}$, there exists a factorization related to connected morphisms. We define the *image* of $u^* : \mathcal{S}\mathrm{h}(\mathbf{X}) \to \mathcal{S}\mathrm{h}(\mathbf{Y})$ to be the smallest full subcategory $\mathcal{E}$ of $\mathcal{S}\mathrm{h}(\mathbf{Y})$ containing the image of $\mathcal{S}\mathrm{h}(\mathbf{X})$ and stable by colimits and finite limits.[39] It happens that $\mathcal{E}$ is a logos and that the functors $\mathcal{E} \to \mathcal{S}\mathrm{h}(\mathbf{Y})$ and $\mathcal{S}\mathrm{h}(\mathbf{X}) \to \mathcal{E}$ are logos morphisms. Let $\mathbf{Z}$ be the topos corresponding to $\mathcal{E}$. By design, the morphism $\mathcal{S}\mathrm{h}(\mathbf{Z}) \to \mathcal{S}\mathrm{h}(\mathbf{Y})$ is fully faithful, hence the corresponding topos morphism $\mathbf{Y} \to \mathbf{Z}$ has connected fibers. We shall call *dense* a morphism of logoi $\mathcal{S}\mathrm{h}(\mathbf{Z}) \to \mathcal{S}\mathrm{h}(\mathbf{Y})$ whose image is the whole of $\mathcal{S}\mathrm{h}(\mathbf{Y})$ and *disconnected* the corresponding morphisms of topoi:

[39] The construction is akin to that of the subring image of a ring morphism.

$$
\begin{tikzcd}
\mathcal{S}\mathrm{h}\left(\mathbf{X}\right) \arrow[rr, "u^*"] \arrow[dr, "d^*", "\text{dense}"'] & & \mathcal{S}\mathrm{h}\left(\mathbf{Y}\right) \\
& \mathcal{E} \arrow[ur, "c^*", "\text{fully faithful}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Y} \arrow[rr, "u"] \arrow[dr, "c", "\text{connected}"'] & & \mathbf{X} \\
& \mathbf{Z} \arrow[ur, "d", "\text{disconnected}"'] &
\end{tikzcd}
$$

A topos $\mathbf{X}$ is called *disconnected* if the morphism $\mathbf{X} \to \mathbf{1}$ is. A disconnected topos $\mathbf{X}$ is such that the constant sheaves generate the whole of $\mathcal{S}\mathrm{h}\left(\mathbf{X}\right)$ by means of colimits and finite limits. Intuitively, it is easy to understand how this cannot be the case over a connected space like $\mathbb{R}$ of S^1: there is no way to build the open domains from constant sheaves since all morphisms between them are also constant. Therefore, the connected components of a disconnected topos must have "constant" trivial open domains and be points. In fact, it can be proved that disconnected topoi are totally disconnected spaces. Finally, the geometric intuition behind the connected–disconnected factorization $\mathbf{X} \to \mathbf{Z} \to \mathbf{1}$ is that $\mathbf{Z}$ is the disconnected space of connected components of the fiber. The intuition for the factorization of a morphism is the same fiber-wise.

Examples of disconnected morphisms

(i) Any discrete topos $\mathbf{B}E$ is disconnected over $\mathbf{1}$.

(ii) Any étale morphism, in particular, any open embedding, is disconnected. This is indeed the intuition of étale morphism, since we saw that the fibers are discrete topoi $\mathbf{B}E$.

(iii) Any limit of disconnected topoi is a disconnected topos. In fact, it can be proved that any disconnected morphism is, in a certain sense, a limit of étale maps.

(iv) Any embedding of topoi can be proved to be disconnected.

(v) Let K be the Cantor set; then the topos morphism $K \to \mathbf{1}$ dual to the canonical functor $\mathsf{Set} \to \mathcal{S}\mathrm{h}\left(K\right)$ is disconnected. This is true essentially because K can be written as a limit of discrete spaces. Recall that the Cantor set is a profinite set. Let $\mathsf{Pro(Fin)}$ be the category of profinite sets. The functor $\mathsf{Fin} \to \mathsf{Topos}$ sending a finite set F to the discrete topos $\mathbf{B}F$ can be extended (by commutation to filtered limits) into a functor $\mathsf{Pro(Fin)} \to \mathsf{Topos}$ that is fully faithful. The image of this functor is inside disconnected topoi.

(vi) Let $\mathbb{Q}$ be the set of rational numbers with the topology induced by $\mathbb{R}$: then the logos morphism $\mathsf{Set} \to \mathcal{S}\mathrm{h}\left(\mathbb{Q}\right)$ is dense. (It is sufficient to reconstruct from constant sheaves a basis of the topology of $\mathbb{Q}$. The open subsets (a, b) with a and b irrational numbers are a basis. Any such open subset can be written as the kernel of some maps $1 \rightrightarrows 2$.)

(vii) The diagonal map $\mathbf{X} \to \mathbf{X} \times \mathbf{X}$ of a topos $\mathbf{X}$ can be proved to be a disconnected map. Recall that we saw that the fiber of this map at a pair of points (x, y) is a (localic) topos $\Omega_{x,y}\mathbf{X}$ whose points are the isomorphisms between x and y. The disconnection of the diagonal implies that $\Omega_{x,y}\mathbf{X}$ is a disconnected topos.[40]

(viii) Let G be a topological group and $\mathsf{Set}^{(G)}$ be the logos of continuous action of G on sets. Let $\mathbf{X}$ be the corresponding topos. Then $\mathbf{X}$ is a connected topos, and the fibers of its diagonal map are torsors over the totally disconnected space of connected components of G.

3.2.16 Locally connected maps and π_0 theory The simple definition of the connected–disconnected factorization in terms of sheaves shows that the theory of topoi is particularly suited to deal with connected

[40]This result is actually a source of a limitation of the theory of topoi. Once the notion of a space with a category of points makes sense, it is reasonable to assume that the automorphisms of a given point do form a topological group. The answer is positive, but the disconnection of the diagonal of a topos says that the topology of these automorphism groups is at best disconnected. In particular, it is impossible to obtain S^1 or other connected topological groups as such groups. Indeed, because S^1 is connected, any action on a set is constant, i.e., $\mathsf{Set}^{S^1} = \mathsf{Set}$. Hence, from the point of view of topoi and sheaves of sets, the classifying space of S^1 is indistinguishable from a point. This is an example of a space without enough étale domains, i.e., beyond the world of topoi. The theory of topological stacks is better suited for dealing with these objects.

components. This factorization can also be defined for topological spaces, but the definition of disconnected spaces and disconnected maps in terms of open domains only is more complex.

It is an important feature of topological spaces that not all spaces have a nice *set* of connected components (the easiest counter-examples being the Cantor set or $\mathbb{Q}$). This says that the functor $(-)_{dis} : \mathsf{Set} \to \mathsf{Top}$ sending a set E to the corresponding discrete space E_{dis} does not have a globally defined left adjoint. The situation is a fortiori the same for topoi, and not every topos has a set of connected components. Somehow, the disconnected topoi enlarge the class of discrete topoi just by what is needed so that every space always has a disconnected topos of connected components.

Classically, the spaces whose connected components form a set are the locally connected spaces. Recall that a space X is locally connected if any open subset is a union of connected open subsets. In fact, more is true, and any étale domains $Y \to X$ is also a union of connected open domains. Let $\pi_0(Y)$ be the set of connected components of such a Y. This produces a functor $\pi_0 : \mathcal{S}\mathrm{h}(X) \to \mathsf{Set}$ that is left adjoint to the canonical logos morphism $\mathsf{Set} \to \mathcal{S}\mathrm{h}(X)$. The existence of this left adjoint is essentially the definition of a locally connected topos.[41] More generally, a morphism of topos $u : \mathbf{Y} \to \mathbf{X}$ is *locally connected* if the functor $u^* : \mathcal{S}\mathrm{h}(\mathbf{X}) \to \mathcal{S}\mathrm{h}(\mathbf{Y})$ has a (local) left adjoint $u_!$.[42] Intuitively, this means that its fibers are locally connected topoi. When $u : \mathbf{Y} \to \mathbf{X}$ is locally connected, the disconnected part $\mathbf{Z} \to \mathbf{X}$ of its connected–disconnected factorization $u : \mathbf{Y} \to \mathbf{Z} \to \mathbf{X}$ is an étale morphism.[43]

Examples of locally connected topoi

(i) Any locally connected space is a locally connected topos.

(ii) Any Alexandrov topos $\mathbf{B}C$ is locally connected topos.

(iii) In particular, the topoi $\mathbf{1}$, $\mathbf{A}$, $\mathbf{A}^C$, $\mathbf{A}^\bullet$, $\mathbf{A}^\circ$, $\mathbf{B}G$ are all locally connected.

(iv) The topoi corresponding to the Cantor set and $\mathbb{Q}$ are not locally connected.

3.2.17 Locally constant sheaves and π_1 theory Fundamental groupoids are related to locally constant sheaves, and the theory of topoi is also well suited to work with them. However, the resulting theory has a formulation that is more sophisticated than the π_0 theory [9]. The main difficulty is in fact the definition of locally constant sheaves and particularly of locally constant morphisms between them.[44] Another aspect is that the analogue of the connected–disconnected factorization system is difficult to define in terms of sheaves of sets only. If sheaves of sets are enhanced into sheaves of groupoids (i.e., 1-stacks), then the theory of fundamental groupoids can be nicely formulated in a way analogous to the theory of connected components. We shall see later how the notion of ∞-topos helps to have a nice theory for the whole homotopy type of topoi.

Examples of fundamental groupoids

(i) The fundamental groupoids of a locally simply connected space and of its corresponding topos are the same.

(ii) When $\mathbb{Q}$ is viewed as a discrete group, the quotient $\mathbb{R}/\!\!/\mathbb{Q}$ is a connected and locally simply connected topos, and its fundamental group is $\mathbb{Q}$. More amusing, if $\mathbb{R}_{dis}$ is $\mathbb{R}$ viewed as a discrete space, the quotient $\mathbb{R}/\!\!/\mathbb{R}_{dis}$ is connected and locally simply connected, with a single point but with $\mathbb{R}_{dis}$ as its fundamental group.

(iii) The fundamental groupoid of an Alexandrov topos $\mathbf{B}C$ is the groupoid G obtained from C by inverting all arrows.

[41] In fact, a stronger condition is required: the adjoint π_0 must be *local*, i.e., satisfy the technical assumption that, for any set E and any sheaf F, we have $\pi_0(E \times F) \simeq E \times \pi_0(F)$.

[42] Here again, $u_!$ must satisfy a locality condition: for any sheaf E in $\mathcal{S}\mathrm{h}(\mathbf{X})$ and any sheaf F in $\mathcal{S}\mathrm{h}(\mathbf{Y})$, we need to have $u_!(u^*E \times F) \simeq E \times u_!(F)$.

[43] In this case, we have $\mathcal{S}\mathrm{h}(\mathbf{Z}) = \mathcal{S}\mathrm{h}(\mathbf{X})_{/u_!1}$.

[44] When a space X (or a topos) is not locally 1-connected, the category of locally constant sheaves is not a full subcategory of $\mathcal{S}\mathrm{h}(X)$.

(iv) In particular, the fundamental groupoid of $\mathbf{B}G$ is the group G viewed as a groupoid with one object. The map $\mathbf{1} \to \mathbf{B}G$ is an étale map from a connected space; it is then a universal cover of $\mathbf{B}G$. This is compatible with the earlier computation that the fibers of this map are copies of G.

(v) We deduce also that $\mathbf{1}$, $\mathbf{A}$, $\mathbf{A}^\circ$, and $\mathbf{A}^\bullet$ have trivial fundamental groupoids. (They are in fact examples of topoi with trivial homotopy type.)

3.2.18 Compact topoi We mention briefly how to define a condition of compactness on topoi. Recall that a locale X is called *compact* if the functor $\mathrm{Hom}_{\mathcal{O}(X)}(1,-) : \mathcal{O}(X) \to \underline{2}$ preserves directed unions. The corresponding property for a topos is to ask that the global section functor $\Gamma : \mathcal{S}\mathrm{h}(\mathbf{X}) \to \mathsf{Set}$ to preserve filtered colimits. A topos is called *tidy* if this is the case. As it happens, the condition to be tidy on a topological space or on a locale is a bit stronger than the compactness condition. More details can be found in [19].

Examples of tidy topoi

(i) Any compact Hausdorff space is tidy as a topos.

(ii) When G is a group of finite generation, the topos $\mathbf{B}G$ is tidy.

(iii) All $\mathbf{A}^C$ are tidy. The global section $\Gamma : \left[C^{\mathrm{lex}}, \mathsf{Set}\right] \to \mathsf{Set}$ is simply the evaluation at the terminal object 1 in C^{lex}. In particular, this is a co-continuous functor.

(iv) An Alexandrov topos $\mathbf{B}C$ is tidy if C is a cofiltered category. This is true as soon as C has a terminal object.

(v) In particular, $\mathbf{A}^\circ$ and $\mathbf{A}^\bullet$ are tidy.

(vi) For any locale, we saw that the topos $\widehat{X}$, dual to the presheaf logos $\mathcal{P}\mathrm{r}(\mathcal{O}(X))$, is a localic and compact as a locale. It is in fact tidy as a topos. The coherent envelope $X_{\mathrm{coh}} \hookrightarrow \widehat{X}$ is also tidy as a topos.

3.2.19 Cohomology It should not be a surprise that the setting of topoi is convenient for sheaf cohomology. This includes cohomology with constant coefficients or locally constant coefficients. This has actually been a motivation for the theory. We shall not develop this and refer to the literature for details [5]. However, as for the theory of fundamental groupoids and higher homotopy invariants, the notion of topos turns out to be less suited than that of ∞-topos for the purposes of cohomology theory (see Section 4.2.8).

3.2.20 Topoi as groupoids Topoi turned out to have a close relationship with stacks on the category of locales. A *localic groupoid* G is a groupoid $G_1 \rightrightarrows G_0$ where G_0 and G_1 are locales. The category of such groupoids is denoted $\mathsf{GpdLocale}$. To any such groupoid, we can associate a logos $\mathcal{S}\mathrm{h}(G)$ of equivariant sheaves on G_0. This produces a functor $\mathsf{GpdLocale} \to \mathsf{Topos}$ between 2-categories. The main theorem of [21] proves that this functor is essentially surjective. However, this functor is not fully faithful (see [28] for a study of its fully faithfulness).

3.3 Descent and other definitions of logoi/topoi

The previous section explained how a number of topological features could be extended to topoi. We focus now more on the algebraic side of topos theory, that is, logos theory. The basic idea we have laid out is that a logos is a category $\mathcal{E}$ with finite limits, (small) colimits, and a compatibility relation between them akin to distributivity. There exist several ways to formulate this relation, and this is essentially the difference between the several definitions of topoi. We are going to present a unified view on the structure of logoi based in the geometric theory of descent, that is, the art of gluing. Such a path will also make it clear what is gained with the notion of ∞-logos/topos.

We start by some recollections on descent. Then, we formulate descent in a way that makes it closer to a distributivity condition. This will help us to explain Giraud and Lawvere axioms. Finally, we will sketch the deep analogy of structure between logoi, frames, and commutative rings.

3.3.1 Descent for sheaves We first recall some facts about the gluing of sheaves. Let $U_i \to X$ be an open covering of a space X, and let $U_{ij} = U_i \cap U_j$, and $U_{ijk} = U_i \cap U_j \cap U_k$. Let F be a sheaf on X. We define F_i, F_{ij} and F_{ijk} to be the pullbacks of F along $U_i \to X$, $U_{ij} \to X$, and $U_{ijk} \to X$. All this data organizes into a diagram[45]

$$\begin{tikzcd} \coprod_{ijk} F_{ijk} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \coprod_{ij} F_{ij} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \coprod_i F_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & F \arrow[d] \\ \coprod_{ijk} U_{ijk} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \coprod_{ij} U_{ij} \arrow[r, shift left] \arrow[r, shift right] & \coprod_i U_i \arrow[r] & X \end{tikzcd}$$

where the vertical maps are the étale maps corresponding to the sheaves. By construction of this diagram by pullback, all the squares of the diagram are cartesian. The cartesian nature of this diagram is a clever way to encode the data of the *cocycle* gluing the F_i together to get back F. The cartesianness of the middle square says that the two pullbacks of F_i and F_j along $U_{ij} \to U_i$ and $U_{ij} \to U_j$ are isomorphic and gives $\phi_{ij} : F_{i|ij} \simeq F_{j|ij}$. The cartesianness of the left square says that these isomorphisms satisfy a coherence condition on U_{ijk}: $\phi_{ki}\phi_{jk}\phi_{ij} = id$.[46]

We define a *descent data* relative to the covering $\{U_i\}$ as the data of a cartesian diagram of sheaves

$$\begin{tikzcd} \coprod_{ijk} F_{ijk} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \coprod_{ij} F_{ij} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \coprod_i F_i \arrow[d] \\ \coprod_{ijk} U_{ijk} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \coprod_{ij} U_{ij} \arrow[r, shift left] \arrow[r, shift right] & \coprod_i U_i . \end{tikzcd} \tag{Descent data}$$

Morphisms of descent data are defined as morphisms of diagrams. The category of descent data is denoted $\mathrm{Desc}(\{U_i\})$ and called the *descent category* of the covering U_i.

This category has a conceptual definition. The vertical maps of (Descent data) define objects in the categories $\prod_i \mathcal{S}\mathrm{h}(U_i)$, $\prod_{ij} \mathcal{S}\mathrm{h}(U_{ij})$, and $\prod_{ijk} \mathcal{S}\mathrm{h}(U_{ijk})$. These categories are related by pullback functors:

$$\prod_i \mathcal{S}\mathrm{h}(U_i) \rightrightarrows \prod_{ij} \mathcal{S}\mathrm{h}(U_{ij}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \prod_{ijk} \mathcal{S}\mathrm{h}(U_{ijk}).$$

Then, a descent data is the same thing as an object in the limit of this diagram of categories.[47] In other terms, we can define the descent category as

$$\mathrm{Desc}(\{U_i\}) = \lim\Big(\prod_i \mathcal{S}\mathrm{h}(U_i) \rightrightarrows \prod_{ij} \mathcal{S}\mathrm{h}(U_{ij}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \prod_{ijk} \mathcal{S}\mathrm{h}(U_{ijk}) \Big). \tag{Descent category}$$

The construction of the beginning builds a *restriction functor*:

$$\begin{aligned} \mathrm{rest}_{\{U_i\}} : \mathcal{S}\mathrm{h}(X) &\longrightarrow \mathrm{Desc}(\{U_i\}) \\ F &\longmapsto (F_i, F_{ij}, F_{ijk}). \end{aligned}$$

It is a classical result about sheaves that, reciprocally, it is possible to define a sheaf F on X by gluing a descent data (F_i, F_{ij}, F_{ijk}) relative to a covering U_i. In terms of category theory, this gluing is nothing but the colimit of the diagram

$$\coprod_{ijk} F_{ijk} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \coprod_{ij} F_{ij} \rightrightarrows \coprod_i F_i.$$

This constructs a functor

$$\mathrm{glue}_{\{U_i\}} : \mathrm{Desc}(\{U_i\}) \longrightarrow \mathcal{S}\mathrm{h}(X)$$

that is left adjoint to the restriction functor.

[45] This diagram is technically a truncated simplicial diagram. We have not drawn the degeneracy arrows to facilitate reading, but they are part of the diagram.

[46] The degeneracy maps not drawn in the diagram also gives conditions on the ϕ_{ij}. In the middle square, we get the condition $\phi_{ii} = id$. In the left square, we get the conditions $\phi_{ij}\phi ji = id = \phi_{ji}\phi ij$.

[47] More precisely, it is a *pseudo-limit* computed in the 2-category of categories.

We shall say that descent data along the covering $\{U_i\}$ are *faithful* if the functor $\mathrm{rest}_{\{U_i\}}$ is fully faithful, and *effective* if the functor $\mathrm{colim}_{\{U_i\}}$ is fully faithful. Intuitively, the faithfulness of descent data means that, given a sheaf F, its decomposition into (F_i, F_{ij}, F_{ijk}) followed by the gluing of the (F_i, F_{ij}, F_{ijk}) reconstructs F. The effectivity of descent data says that the gluing of (F_i, F_{ij}, F_{ijk}) into some F followed by the decomposition of F reconstructs the diagram (F_i, F_{ij}, F_{ijk}). We shall say that the *descent property holds* along the covering $\{U_i\}$ if descent data are effective and faithful, that is, if the adjunction $\mathrm{colim}_{\{U_i\}} \dashv \mathrm{rest}_{\{U_i\}}$ is an equivalence of categories,[48]

$$\mathcal{S}\mathrm{h}\,(X) \simeq \mathrm{Desc}(\{U_i\}).$$

These considerations can be extended to a topos $\mathbf{X}$ in a straightforward way. The only difference is that the open embeddings $U_i \to X$ can be enhanced into étale maps $\mathcal{U}_i \to \mathbf{X}$. Then, the $\mathcal{U}_{ij}$ are defined by the fiber products $\mathcal{U}_i \times_{\mathbf{X}} \mathcal{U}_j$, and so on. Let U_i be the object of $\mathcal{S}\mathrm{h}\,(\mathbf{X})$ corresponding to the étale morphisms $\mathcal{U}_i \to \mathbf{X}$ by the correspondence (Sheaves as étale maps). Recall that this correspondence preserves finite limits. This says that the fiber products $\mathcal{U}_i \times_{\mathbf{X}} \mathcal{U}_j$ can be dealt with by means of the corresponding object $U_{ij} = U_i \times U_j$ in $\mathcal{S}\mathrm{h}\,(\mathbf{X})$. The category $\mathrm{Desc}(\{\mathcal{U}_i\})$ is defined by the same diagrams (Descent category), the restriction and gluing functors $\mathrm{rest}_{\{\mathcal{U}_i\}}$ and $\mathrm{colim}_{\{\mathcal{U}_i\}}$ are defined similarly, and the same vocabulary makes sense.

Examples of descent data

(i) Recall the étale cover $\mathbf{1} \to \mathbf{B}G$. Using the computation of $G = \mathbf{1} \times_{\mathbf{B}G} \mathbf{1}$ made earlier, a descent data with respect to this map is the data of an object in the limit of the diagram

$$\begin{tikzcd} \mathcal{S}\mathrm{h}\,(\mathbf{1}) \arrow[r, shift left] \arrow[r, shift right] & \mathcal{S}\mathrm{h}\,(\mathbf{1} \times_{\mathbf{B}G} \mathbf{1}) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathcal{S}\mathrm{h}\,(\mathbf{1} \times_{\mathbf{B}G} \mathbf{1} \times_{\mathbf{B}G} \mathbf{1}) \end{tikzcd}$$

$$\begin{tikzcd} = \ \mathsf{Set} \arrow[r, shift left] \arrow[r, shift right] & \mathsf{Set}_{/G} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathsf{Set}_{/G\times G}, \end{tikzcd}$$

that is, a diagram of sets of the type

$$\begin{tikzcd} G \times G \times E \arrow[r, shift left=2, "p_1 \times a"] \arrow[r, "m \times id" description] \arrow[r, shift right=2, "p_{23}"'] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near start] & G \times E \arrow[r, shift left, "a"] \arrow[r, shift right, "p_2"'] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near start] & E \arrow[d] \\ G \times G \arrow[r, shift left=2, "p_1"] \arrow[r, "m" description] \arrow[r, shift right=2, "p_2"'] & G \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & 1. \end{tikzcd}$$

Such a data is the same thing as an action of the group G over a set E. The action is given by the map $a : G \times E \to E$, and the diagram relations ensure that it is unital and associative.

(ii) More generally, if a discrete group G acts on a space X, the quotient map $q : X \to X /\!\!/ G$ is also an étale cover of topoi. A descent data with respect to this cover is the same thing as a sheaf on X with an equivariant action of G.

3.3.2 Descent and distributivity We abstract from the previous section the structure of descent. This will lead us to conditions with a flavor of distributivity, summarized in Table 14.

The distributivity relation $c(a + b) = ca + cb$ has an obvious analogue in terms of colimits and limits, which is the property of *universality of colimits*. Let A_i be a diagram $I \to \mathcal{E}$, $u : C \to B$ be a map in $\mathcal{E}$, and $\mathrm{colim}_i A_i \to B$ be another map. Then, the universality of colimits is the condition that the base change along u preserves the colimit of A_i:

$$C \times_B (\operatorname*{colim}_i A_i) = \operatorname*{colim}_i (C \times_B A_i).$$

[48]Given an adjoint pair of functors $L \dashv R$, recall that L is fully faithful if and only if the unit $1 \to RL$ is an isomorphism, and R is fully faithful if and only if the co-unit $LR \to 1$ is an isomorphism. Then, L and R are inverse equivalences of categories if and only if they are both fully faithful.

The analogy with the distribution of products over sums should be clear.

There exist a number of equivalent formulations for this condition. For example, this is equivalent to saying that the pullback, or base change, functor

$$u^* : \mathcal{E}_{/B} \longrightarrow \mathcal{E}_{/C}$$

preserves colimits. Geometrically, this says that the pullback of sheaves along étale maps preserves the colimits. By symmetry of the fiber product, this says also that, for any B in $\mathcal{E}$, the fiber product $- \times_B -$ preserves colimits in both variables. This is somehow analogue to the bilinearity of the product $m : R^2 \to R$ of a commutative ring R.

The universality of colimits will be one of the conditions to hold in a logos, but to formulate the other conditions, we need to reformulate it. Let us assume that $B = A$ is the colimit of the A_i and let $C_i = A_i \times_A C$; then we have two co-cones $A_i \to A$ and $C_i \to C$ and a morphism between them (represented vertically):

$$\begin{tikzcd}
C_i \arrow[r] \arrow[d] \arrow[drr] \arrow[dr, phantom, "\lrcorner", very near start] & C_j \arrow[d, dashed] \arrow[dr] & \\
A_i \arrow[r] \arrow[drr] & A_j \arrow[dr] & C \arrow[d, "u"] \\
 & & A
\end{tikzcd}$$

By construction, all the square faces of this diagram are cartesian. Then, the universality of colimits is the condition for C to be the colimit of the diagram C_i.

The other condition we are looking for is a kind of reciprocal statement. We are going to need a few prior steps to be able to formulate it properly. Let us assume that we have a natural transformation of diagrams $C_i \to A_i$ such that, for any map $u : i \to j$ in the indexing category I, the corresponding square is cartesian:

$$\begin{tikzcd}
C_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C_j \arrow[d] \\
A_i \arrow[r] & A_j.
\end{tikzcd} \qquad \text{(Generalized descent data)}$$

An example of such a cartesian natural transformations is given by descent data along a covering (see (Descent data) and the following examples). In this case, the role of the diagram A_i is played by the so-called *nerve* of the covering family $U_i \to X$, which is the truncated simplicial diagram[49]

$$\textstyle\coprod_{ijk} U_i \times_X U_j \times_X U_k \Rrightarrow \coprod_{ij} U_i \times_X U_j \rightrightarrows \coprod_i U_i. \qquad \text{(Nerve of a covering)}$$

Intuitively, the cartesian transformations between diagrams corresponds also to descent data, but relative to an arbitrary diagram A_i instead of the nerve of a covering family.

From there, the situation is very similar to what we did with descent. For a diagram $A_\bullet : I \to \mathcal{E}$, let $\mathrm{Desc}(A_\bullet)$ be the category of cartesian natural transformations $C_i \to A_i$, as above. For each map $i \to j$ in I, we have a map $A_i \to A_j$ and a base change functor $\mathcal{E}_{/A_j} \to \mathcal{E}_{/A_i}$. Then, the category $\mathrm{Desc}(A_\bullet)$ can be described as the limit this diagram of $\mathcal{E}_{/A_i}$,[50]

$$\mathrm{Desc}(A_\bullet) = \lim_i \ \mathcal{E}_{/A_i}. \qquad \text{(Descent category 2)}$$

[49] Precisely, the indexing category is $(\Delta_{\leq 2})^{op}$, where $\Delta_{\leq 2}$ is the full subcategory of the simplex category Δ spanned by simplices of dimensions 0, 1, and 2 only.

[50] This limit is a pseudo-limit in the 2-category of categories. It can be computed as the category of cartesian sections of a certain fibered category over the indexing category I.

Let A be the colimit of A_i; then we have a natural "restriction" functor (pull back along the maps $A_i \to A$) and a "gluing" functor (colimit of the diagram)

$$\mathcal{E}_{/A} \underset{\mathrm{rest}_{A_\bullet}}{\overset{\mathrm{glue}_{A_\bullet}}{\rightleftarrows}} \mathrm{Desc}(A_\bullet) = \lim_i \ \mathcal{E}_{/A_i}. \qquad \text{(Descent adjunction)}$$

We shall say that the colimits of A_i are *faithful* if the functor $\mathrm{rest}_{A_\bullet}$ is fully faithful, and that they are *effective* if the functor $\mathrm{glue}_{A_\bullet}$ is fully faithful. The faithfulness condition says that, given $C \to A$, C can be decomposed into the pieces $C_i = A_i \times_A C$ and recomposed as the colimit of this diagram. The effectivity condition says that, given a cartesian morphism $C_i \to A_i$, we can compose the diagram C_i into its colimit $C = \operatorname{colim} C_i$ and then decompose the resulting object C into its original pieces by $C_i = A_i \times_A C$. In other words, the effectivity of the colimit of A_i is equivalent to the following squares being cartesian for all i:

$$\begin{tikzcd} C_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & C = \operatorname{colim} C_i \arrow[d] \\ A_i \arrow[r] & A = \operatorname{colim} A_i. \end{tikzcd}$$

The descent property along the diagram A_i is then formulated by the equivalence of categories

$$\mathcal{E}_{/\operatorname{colim} A_i} \simeq \lim_i \ \mathcal{E}_{/A_i}. \qquad \text{(Generalized descent property)}$$

We have finally arrived at the end of the formulation of the descent property. The slice categories $\mathcal{E}_{/A}$ and the base change functors define a functor, called the *universe*, with values in the 2-category of categories:

$$\begin{array}{rcl} \mathbb{U} : \mathcal{E}^{op} & \longrightarrow & \mathsf{Cat} \\ A & \longmapsto & \mathcal{E}_{/A} \\ f : A \to B & \longmapsto & f^* : \mathcal{E}_{/B} \ \to \ \mathcal{E}_{/A} \end{array} \qquad \text{(Universe)}$$

By the formula (Generalized descent property), the diagrams for which the descent property holds are precisely those for which their colimit is sent to a limit by the functor $\mathbb{U}$.

For example, let G be a sheaf of groups acting on a sheaf F over some space X. The group action defined a simplicial diagram in $\mathcal{S}\mathrm{h}(\mathbf{X})$:

$$\ldots G \times G \times F \ \substack{\xrightarrow{p_1 \times a} \\ \xrightarrow{m \times id} \\ \xrightarrow[p_{23}]{}} \ G \times F \ \substack{\xrightarrow{a} \\ \xrightarrow[p_2]{}} \ F.$$

The quotient of the action $F/\!\!/G$ is the colimit of this diagram in $\mathcal{S}\mathrm{h}(\mathbf{X})$. A descent data associated to this diagram is equivalent to the data of a sheaf E with an action of G and an equivariant map of sheaves $E \to F$. Then, the descent property says that a sheaf over the quotient $F/\!\!/G$ is equivalent to an equivariant sheaf over F. This equivalence does not hold for a general group action, but it holds when the action is free. The general descent condition can be understood intuitively in the same way: a diagram has the descent property if working over its colimit is equivalent to working "equivariantly" over the diagram.[51]

Table 13 summarizes all the descent conditions, and Table 14 sets up the comparison with the distributivity relation in a commutative ring.[52] The descent conditions make sense in any category $\mathcal{E}$ with colimits and finite limits, but they do not hold in general. Whether they hold or not is going to define logoi. As it happens, every diagram in a logos is going to be of faithful descent, but not every diagram is going to be

[51] We shall see that in sheaves of ∞-groupoids, within an ∞-logos, all diagrams have the descent property. In particular, any group action will be qualified for working equivariantly. This property is one of the motivations to define ∞-logoi/topoi.

[52] The conditions of Table 14 do have a flavor of distributivity, but a better formulation would be to have a general relation of commutation of finite limits and colimits, like $\lim_i \operatorname{colim}_j X_{ij} = \operatorname{colim}_k \lim_i X_{i,k(i)}$. However, we do not know any such formulation.

of effective descent.[53] There are two natural ways to restrict the effectivity condition: either we ask that a specific class of diagrams be of effective descent, or we can ask that all diagrams be of effective descent but for a restricted class of descent data. The first condition will lead us to Giraud axioms, the second to Lawvere–Tierney axioms.

Table 13: Descent conditions for a diagram $A_\bullet : I \to \mathcal{E}$

Descent category	
$\mathrm{Desc}(A_\bullet) = \lim_i \mathcal{E}_{/A_i}$	
Descent property	
$\mathcal{E}_{/\operatorname{colim}_i A_i} \simeq \lim_i \mathcal{E}_{/A_i}$	
Faithfulness	*Effectivity*
$\mathrm{rest}_{A_\bullet} : \mathcal{E}_{/\operatorname{colim}_i A_i} \longrightarrow \lim_i \mathcal{E}_{/A_i}$ is fully faithful	$\mathrm{glue}_{A_\bullet} : \lim_i \mathcal{E}_{/A_i} \longrightarrow \mathcal{E}_{/\operatorname{colim}_i A_i}$ is fully faithful
$C = \operatorname{colim}_i(C \times_{\operatorname{colim}_i A_i} A_i)$ decomposition-then-composition identity	$C_i = (\operatorname{colim}_i C_i) \times_{\operatorname{colim}_i A_i} A_i$ composition-then-decomposition identity
Case of a group action $F/\!\!/G$	
Faithfulness	*Effectivity*
a sheaf on $F/\!\!/G$ can be described faithfully by an equivariant sheaf on F	any equivariant sheaf of F describes faithfully a sheaf on $F/\!\!/G$

3.3.3 Presentable categories The last ingredient before we are able to state the definitions of a logos is the notion of presentable category, which, in the analogy between logoi and commutative rings, plays the role of abelian groups. The structural analogy between presentable categories and abelian groups is presented in Table 15.

The notion of presentable category is one of the most crucial notions of category theory. They are a particularly nice class of categories with all colimits (or cocomplete categories) for which a technical problem of size is tamed. Let $\mathcal{C}$ be a cocomplete category and R be a class of arrows in $\mathcal{C}$. We denote by $\mathcal{C}/\!\!/R$ the localization of $\mathcal{C}$ forcing all the arrows in R to become isomorphisms.[54] We called it the *quotient* of $\mathcal{C}$ by R.[55] A category $\mathcal{C}$ is called *presentable* if it is equivalent to some quotient $\mathcal{P}\mathrm{r}(C)/\!\!/R$, were C is a small category and R a *set* (rather than a class). The intuitive idea is that, even though presentable categories are not small, they still are controlled by the small data (C, R).

Here follows a list of some properties for which presentable categories are so nice. Let $\mathcal{C}$ be a presentable

[53] For a counter-example, see [32]. The condition for every diagram to be of effective descent is going to be the definition of an ∞-logos.

[54] This localization is taken in the category of cocomplete categories and functors preserving colimits. This forces $\mathcal{C}/\!\!/R$ to have all colimits and the canonical functor $\mathcal{C} \to \mathcal{C}/\!\!/R$ to preserves them.

[55] The vocabulary is a bit awkward here–the classical name of the operation $\mathcal{C} \to \mathcal{C}/\!\!/R$ is *localization* because the operation is thought from the point of view of the arrows of $\mathcal{C}$, but, from the point of view of the objects of $\mathcal{C}$, this operation is in fact a *quotient* of $\mathcal{C}$ identifying the domain and codomain of the maps $A \to B$ in R. This second point of view is better for our purposes. The notation $\mathcal{C}/\!\!/R$ is intended to be more evocative of this fact than the classical notation $\mathcal{C}[R^{-1}]$.

Table 14: Descent and distributivity

Logos		*Commutative ring*
Faithfulness (decomposition-then-composition condition)	$C = \operatorname{colim}_i \left(C \times_{\operatorname{colim}_i A_i} A_i\right)$	distributivity relation $c\sum_i a_i = \sum_i c a_i$
Effectivity (composition-then-decomposition condition)	given $\begin{array}{ccc} C_i & \longrightarrow & C_j \\ \downarrow & \ulcorner & \downarrow \\ A_i & \longrightarrow & A_j \end{array}$ $C_i = (\operatorname{colim}_j C_j) \times_{\operatorname{colim}_j A_j} A_i$ (not a consequence of faithfulness)	given elements a_i and c_i such that $c_i a_j = a_i c_j$ $c_i \sum_j a_j = a_i \sum_j c_j$ (consequence of distributivity)

category, then

(a) $\mathcal{C}$ has (small) limits in addition to (small) colimits;

(b) (special adjoint functor theorem) if $\mathcal{D}$ is a cocomplete category, a functor $\mathcal{C} \to \mathcal{D}$ has a right adjoint if and only if it preserves (small) colimits;

(c) (representability theorem) in particular, a functor $\mathcal{C}^{op} \to \mathsf{Set}$ is representable by an object X in $\mathcal{C}$ if and only if it sends colimits to limits;

(d) (quotients as full subcategories) if R is a set of maps in $\mathcal{C}$, the quotient $\mathcal{C}/\!\!/R$ is again presentable, and the right adjoint to the quotient functor $\mathcal{C} \to \mathcal{C}/\!\!/R$ is fully faithful.

The last property is the one we need now. The existence of a fully faithful right adjoint $q_* : \mathcal{C}/\!\!/R \to \mathcal{C}$ to the quotient functor $q^* : \mathcal{C} \to \mathcal{C}/\!\!/R$ means that any quotient of $\mathcal{C}$ can be identified canonically to a full subcategory of $\mathcal{C}$ (however, this embedding does not preserves colimits). An object X of $\mathcal{C}$ is called *orthogonal* to R if, for any $f : A \to B$ in R, the map $\operatorname{Hom}(B, X) \to \operatorname{Hom}(A, X)$ is a bijection. This relation is denoted $R \perp X$. Intuitively, this says that, from the point of view of X, the maps in R are isomorphisms. Then, the image of $q_* : \mathcal{C}/\!\!/R \to \mathcal{C}$ is the full subcategory $R^\perp$ spanned by the objects orthogonal to all maps in R.[56]

Examples of presentable categories

(i) The categories Set, $\mathcal{P}\mathrm{r}(C)$, $\mathsf{Set}[C]$ are presentable. Set^{op} is not a presentable category.

(ii) An important example of quotient is the construction of categories of sheaves. Let C be a small category with finite limits, and for each object X in C, let $J(X)$ be a set of covering families $U_i \to X$. A presheaf F in $\mathcal{P}\mathrm{r}(C)$ is a *sheaf* if and only if, for each covering family, we have

$$F(X) = \lim\Big(\prod_i F(U_i) \Longrightarrow \prod_{ij} F(U_i \times_X U_j) \Big).$$

Let $U = \operatorname{colim}\left(\coprod_{ij} U_i \times_X U_j \rightrightarrows \coprod U_i\right)$ computed in $\mathcal{P}\mathrm{r}(C)$. The canonical map $U \to X$ is a monomorphism in $\mathcal{P}\mathrm{r}(C)$, called the *covering sieve* associated to the covering family $U_i \to X$. Let J be the set

[56] This is how quotients are dealt with in practice: they are defined as categories $R^\perp$ (see the example of sheaves below). The quotient functor $\mathcal{C} \to R^\perp$ is then constructed by a small object argument from the set R.

of all the covering sieves. Then, the previous condition can be reformulated as: F is a sheaf if and only if $J \perp F$. In other words, $\mathcal{S}\mathrm{h}(C,J) = J^{\perp} \subset \mathcal{P}\mathrm{r}(C)$. The property that $J^{\perp} = \mathcal{P}\mathrm{r}(C)/\!\!/J$, says that the category of sheaves can be thought as the quotient of $\mathcal{P}\mathrm{r}(C)$ by the relations given by the topology J. This is actually the proper way to think about it.

Table 15: Presentable categories v. abelian groups

	Presentable categories	*Aelian groups*
Operations	colimits $\mathcal{A}^I \to \mathcal{A}$	sums $A^n \to A$
Morphisms	functors $\mathcal{A} \to \mathcal{B}$ preserving colimits (cc functors)	linear maps $A \to B$
Initial object	$0 = \{\star\}$	0
Free object on one gen.	Set	$\mathbb{Z}$
Free objects	$\mathcal{P}\mathrm{r}(C)$	$\mathbb{Z}.E := \oplus_E \mathbb{Z}$
Quotients	$\mathcal{P}\mathrm{r}(C)/\!\!/(A_i \to B_i \text{ iso})$	$\mathbb{Z}.E/(a_i - b_i = 0)$
Additivity	$\mathcal{A} \oplus \mathcal{B} = \mathcal{A} \times \mathcal{B}$	$A \oplus B = A \times B$
Self-enrichment	the category of cc functors $[\mathcal{A},\mathcal{B}]_{\mathrm{cc}}$ is presentable	the set of group maps $\mathrm{Hom}(A,B)$ is an abelian group
Tensor product	functor preserving colimits in each variable $\mathcal{A} \times \mathcal{B} \to \mathcal{C} =$ functor preserving colimits $\mathcal{A} \otimes \mathcal{B} \to \mathcal{C}$ $\mathcal{A} \otimes \mathcal{B} = [\mathcal{A}^{op},\mathcal{B}]^{\mathrm{c}}$ $\mathcal{P}\mathrm{r}(C) \otimes \mathcal{P}\mathrm{r}(D) = \mathcal{P}\mathrm{r}(C \times D)$	bilinear map $A \times B \to C =$ linear map $A \otimes B \to C$ $\mathbb{Z}.E \otimes \mathbb{Z}.F = \mathbb{Z}.(E \times F)$
Closure of the tensor product	$[\mathcal{A} \otimes \mathcal{B},\mathcal{C}]_{\mathrm{cc}} = [\mathcal{A},[\mathcal{B},\mathcal{C}]_{\mathrm{cc}}]_{\mathrm{cc}}$	$\mathrm{Hom}(A \otimes B, C) = \mathrm{Hom}(A,\mathrm{Hom}(B,C))$
Dual objects	$\mathcal{A}^\star = [\mathcal{A},\mathsf{Set}]_{\mathrm{cc}}$ $\mathcal{P}\mathrm{r}(C)^\star = \mathcal{P}\mathrm{r}(C^{op})$	$A^\star = \mathrm{Hom}(A,\mathbb{Z})$ $(\mathbb{Z}.E)^\star = \mathbb{Z}.E$
Dualizable objects	retracts of $\mathcal{P}\mathrm{r}(C)$	retracts of $\mathbb{Z}.E$

3.3.4 Definitions of a logos/topos We are now ready to present several definitions of logoi. We are going to explain in detail the ones of Giraud and Lawvere. The comparison between these definitions is summarized in Table 17.

A presentable category $\mathcal{E}$ is a *logoi* if

Def. 1. (Our first definition) it is a left-exact localization of some presheaf category $\mathcal{P}\mathrm{r}(C)$;

Def. 2. (Original definition in [5, IV]) it is a category of sheaves on a site;

Def. 3. (Giraud) it has universal colimits, disjoint sums and effective equivalence relations;

Def. 4. (Lawvere) it is locally cartesian closed and has a subobject classifier Ω.[57]

Universality of colimits & local cartesian closeness We defined the universality of colimits as the condition that, for any map $u : C \to B$ in $\mathcal{E}$, the base change functor

$$u^* : \mathcal{E}_{/B} \longrightarrow \mathcal{E}_{/C}$$

preserve colimits. When the category $\mathcal{E}$ is assumed presentable, this condition is also equivalent to the existence of a right adjoint for this functor,

$$u_* = \prod_u : \mathcal{E}_{/C} \longrightarrow \mathcal{E}_{/B}.$$

This functor is called the *relative limit*, the *multiplicative direct image*, or the *depend product*, along u. A category $\mathcal{E}$ such that, for every map u in $\mathcal{E}$, the adjoint pair $u^* \dashv u_*$ exists is called *locally cartesian closed.* These conditions are also equivalent to the condition that every diagram be of faithful descent. Hence, although they are stated differently, Giraud's and Lawvere's definitions both assume this half of the descent property.

Giraud definition The first condition of Giraud axioms is that all diagrams be of faithful descent. The idea behind the other axioms is to ask for the effectivity of descent for some diagrams only. Intuitively, these diagrams are going to be the nerves of covering families (Nerve). But such a characterization of these diagrams will be true only if the Giraud axiom holds. So we need to define them without the fact that they correspond to nerves of covering families. There are going to be two cases. The first case is that of unions. The second case is that of the quotient of an object by an equivalence relation.

Let A_i be a set of objects; the descent property for the sum of the A_i is the condition:

$$\mathcal{E}_{/\coprod_i A_i} \simeq \prod_i \mathcal{E}_{/A_i}.$$

This is sometimes called *extensivity of sums.* As it happens, this whole condition boils down to a single simpler condition called the *disjointness of sums.* Sums are said to be disjoint if, for any $i \neq j$, the following square is cartesian:

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & A_i \arrow[d] \\ A_j \arrow[r] & \coprod_i A_i. \end{tikzcd}$$

The second condition concerns equivalence relations within the category $\mathcal{E}$ that we now define. Let A_0 be an object in $\mathcal{E}$. An *equivalence relation* on A_0 is the data of a relation $A_1 \rightarrowtail A_0 \times A_0$ (a monomorphism) satisfying the following:

(i) (reflexivity) the diagonal of $A_0 \rightarrowtail A_0 \times A_0$ factors through A_1 ($A_0 \subset A_1$ as subobjects of $A_0 \times A_0$)

(ii) (transitivity) for $A_2 = A_1 \times_{p_2, A_0, p_1} A_1$, we have $A_2 \subset A_1$ as subobjects of $A_0 \times A_0$

(iii) (symmetry) $A_1 \rightarrowtail A_0 \times A_0 \overset{\sigma}{\simeq} A_0 \times A_0$ is A_1.

[57] Lawvere's original definition does not in fact require the category $\mathcal{E}$ to be presentable. Without this hypothesis, we get the notion of an *elementary topos* (but we shall say *elementary logos*). This notion is not equivalent to the other definitions. By comparison, the other notion is called a *Grothendieck topos* (but we shall say *Grothendieck logos*). To view topoi as spatial objects, as is the purpose of this chapter, we need to use Grothendieck's definition, not Lawvere's. This is why we have chosen not to present Lawvere's definition in full generality, but to restrict it to the case of a presentable category only.

Such a data provides a truncated simplicial diagram[58]

$$A_2 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; A_1 \;\substack{\longrightarrow\\ \longrightarrow}\; A_0.$$

The equivalence relation $A_1 \rightrightarrows A_0$ is said to be of effective descent if the previous diagram is. As with sums, this condition boils down to a single simpler condition, called the *effectivity of equivalence relations*. The quotient A of the equivalence relation is defined to be the colimit of the previous diagram.[59] Then, the equivalence relation is of effective descent if and only if the following square is cartesian:

$$\begin{tikzcd}
A_1 \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\ulcorner", very near start] & A_0 \arrow[d] \\
A_0 \arrow[r] & A.
\end{tikzcd}$$

Table 16 summarizes the Giraud axioms and the descent conditions they correspond to.

We have already said that the descent condition is not true for all diagrams within a logos. This raises the question to characterizes the diagrams for which it holds. Giraud axioms give a family of diagrams (sums and equivalence relations) sufficient to define the structure of logos, but more diagrams have the descent property. They are the π_1-*acyclic* diagrams, that is, the diagrams A_i for which the ∞-colimit, computed in sheaves of ∞-groupoids, have trivial fundamental group.

Table 16: Giraud axioms

<table>
<tr><td colspan="3">Under assumption of universality of colimits</td></tr>
<tr><td>descent for sums
$\mathcal{E}_{/\coprod_i A_i} \simeq \prod_i \mathcal{E}_{/A_i}$</td><td>$\iff$</td><td>disjointness of sums
$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & A_i \arrow[d] \\ A_j \arrow[r] & \coprod_i A_i \end{tikzcd}$$</td></tr>
<tr><td>descent for equivalence relations
$\mathcal{E}_{/A} = \lim\left(\mathcal{E}_{/A_0} \rightrightarrows \mathcal{E}_{/A_1} \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathcal{E}_{/A_2}\right)$</td><td>$\iff$</td><td>effectivity of equivalence relations
$$\begin{tikzcd} A_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & A_0 \arrow[d] \\ A_0 \arrow[r] & \operatorname{colim}(A_1 \rightrightarrows A_0) \end{tikzcd}$$</td></tr>
</table>

Lawvere definition We already explain the local cartesian closure property of Lawvere definition. The definition of Lawvere of a logos emphasizes the so-called subobject classifier Ω. For an object A in $\mathcal{E}$, a subobject of A is a monomorphism $B \rightarrowtail A$.[60] The subobjects of A span a full subcategory $\mathrm{Sub}(A) \subset \mathcal{E}_{/A}$ which is equivalent to a poset. We denote by $\mathrm{sub}(A)$ the set of objects of this poset. Since monomorphisms

[58] The indexing category is $(\Delta_{\leq 2})^{op}$. Again we are drawing only the face maps.
[59] Or equivalently, the coequalizer of $A_1 \rightrightarrows A_0$.
[60] Recall that a monomorphism is a morphism $f : B \rightarrowtail A$ such that the diagonal $\Delta f : B \rightarrowtail B \times_A B$ is an isomorphism.

are preserved by base change, the family of all $\mathrm{sub}(A)$ defines a functor[61]

$$\begin{aligned} \mathrm{sub} : \mathcal{E}^{op} &\longrightarrow \mathsf{Set} \\ A &\longmapsto \mathrm{sub}(A). \end{aligned}$$

Since we have assumed $\mathcal{E}$ to be a presentable category, the property (c) of such categories says that this functor is representable by an object Ω, that is, $\mathrm{sub}(A) = \mathrm{Hom}(A, \Omega)$, if and only if it sends colimits in $\mathcal{E}$ to limits in Set. But this condition is exactly a descent condition,[62] but for the class of diagrams (Generalized descent data) where the vertical maps are monomorphisms only:

$$\begin{tikzcd} C_i \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\ulcorner", very near start] & C_j \arrow[d, tail] \\ A_i \arrow[r] & A_j \end{tikzcd}$$

In other words, Lawvere's axiom of existence of Ω is a way to impose a general descent property but for a restricted class of descent data.

Table 17: Definitions of logoi/topoi

<table>
<tr><th></th><th colspan="2">Giraud</th><th>Lawvere–Tierney</th></tr>
<tr><td>decomposition-then-composition condition</td><td colspan="3">universality of colimits ($\Leftrightarrow$ all diagrams are of faithful descent)</td></tr>
<tr><td rowspan="2">composition-then-decomposition condition</td><td colspan="2">only π_1-acyclic diagrams are of effective descent</td><td>all diagrams are of effective descent, but for subobjects only</td></tr>
<tr><td>sums are disjoint
$\varnothing = X_i \times_{\coprod_k X_k} X_j$</td><td>equivalence relations are effective
$X_1 \simeq X_0 \times_{X_{-1}} X_0$</td><td>the functor $\mathrm{Sub} : \mathcal{E}^{op} \to \mathsf{Set}$ of subobjects is representable by an object Ω</td></tr>
</table>

3.4 Elements of logos algebra

3.4.1 Structural analogies In this section, we sketch the structural analogy between the theories of logoi, frames, and commutative rings. We already saw the analogy between presentable categories and abelian groups in Table 15. We are going to continue along the same spirit.

The theory of commutative rings is related in a fundamental way to that of abelian groups and that of commutative monoids. Between these structures, there exists forgetful functors and their left adjoints, or

[61] The family of all $\mathrm{Sub}(A)$ defines also a functor Sub with values in Poset, which is a subfunctor of the universe $\mathbb{U}$, but we shall not need this functor.

[62] Strictly speaking, the descent condition would be for the functor Sub defined in the previous footnote. We are smoothing things out a bit here.

free constructions.

$$\begin{tikzcd}
\text{Commutative rings} \arrow[r, shift left] \arrow[d, shift left] & \text{Abelian groups} \arrow[l, shift left, "\mathrm{Sym}"'] \arrow[d, shift left] \\
\text{Commutative monoids} \arrow[u, shift left, "\mathbb{Z}."] \arrow[r, shift left] & \text{Sets} \arrow[l, shift left, "M(-)"'] \arrow[u, shift left, "\mathbb{Z}."] \arrow[ul, shift left, "\mathbb{Z}[-]"']
\end{tikzcd}$$

The functor $\mathbb{Z}.$ constructs the free abelian group on a set. The functor M constructs the free commutative monoid. The functor Sym constructs the symmetric tensor algebra. The functor $\mathbb{Z}[-]$ constructs the free commutative ring on a set. The commutativity of the square says that this last construction can be obtained either by taking first the free abelian group and then the symmetric algebra, or first the free monoid and then linear combinations of the resulting set.

The analogues of these structures for locales and topos are summarized in Table 18 (we have included also ∞-topoi for future reference). The notion of sup-lattice is a poset with arbitrary suprema. The notion of meet-lattice is a poset with finite infima. The notion of lex category is a category with finite limits. And we already saw the notion of presentable category. These structures are also related by a number of forgetful and free functors, presented in Figure 1.[63] In the diagram for frames, the functor $\underline{2}[-]$ is the free frame functor, mentioned earlier. The functor $\bigvee$ is the free sup-lattice functor; if P is a small poset, $\bigvee P = [P^{op}, \underline{2}]$. The functor $(-)^{\wedge}$ is the free meet semi-lattice functor. For a poset P $(P^{\wedge})^{op}$ is the subposet of $[P, \underline{2}]$ generated by finite unions of elements of P. The functor Sym is an analogue of the symmetric algebra functor. In the diagram for logoi, the functor $\mathcal{P}$ is the free cocompletion functor. It is defined only for small categories C, where it is given by the presheaves $\mathcal{P}\mathrm{r}(C) = [C^{op}, \mathsf{Set}]$. The functor $(-)^{\mathrm{lex}}$ is the free finite limit completion functor. The functor Sym is an analogue of the symmetric algebra functor; we refer to [7] for details. The functor $\mathsf{Set}[-]$ is the free logos functor. It is defined only for small categories C by the formula that we have seen already

$$\mathsf{Set}[C] = \mathcal{P}\mathrm{r}\left(C^{\mathrm{lex}}\right) = \left[(C^{\mathrm{lex}})^{op}, \mathsf{Set}\right].$$

Figure 1: Free constructions

$$\begin{tikzcd}
\text{Frames} \arrow[r, shift left] \arrow[d, shift left] & \text{Sup-lattice} \arrow[l, shift left, "\mathrm{Sym}"'] \arrow[d, shift left] \\
\text{Meet semi-lattice} \arrow[u, shift left, "\bigvee"] \arrow[r, shift left] & \text{Posets} \arrow[l, shift left, "(-)^{\wedge}"'] \arrow[u, shift left, "\bigvee"] \arrow[ul, shift left, "\underline{2}[-]"']
\end{tikzcd}$$

$$\begin{tikzcd}
\text{Logoi} \arrow[r, shift left] \arrow[d, shift left] & \text{Presentable categories} \arrow[l, shift left, "\mathrm{Sym}"'] \arrow[d, shift left] \\
\text{Lex categories} \arrow[u, shift left, dashed, "\mathcal{P}"] \arrow[r, shift left] & \text{Categories} \arrow[l, shift left, "(-)^{\mathrm{lex}}"'] \arrow[u, shift left, dashed, "\mathcal{P}"] \arrow[ul, shift left, dashed, "\mathsf{Set}[-]"']
\end{tikzcd}$$

3.4.2 Presentation of logoi by generators and relations The previous paragraph essentially detailed the construction of the free logos. As it is true for any kind of algebraic structure, any logos is a quotient of a free logos. This leads to the possibility to define logoi by generators and relations. This is a key feature in the connection of logoi with classifying problems and logic.

[63] In the right diagram of Fig. 1, the left adjoint functors going up do not, strictly speaking, exist for problems of size. This is why we put them in dashed arrows. They are only defined for small categories and small lex categories.

Table 18: Analogies of structure

Algebraic geometry	*Locale theory*	*Topos theory*	*∞-Topos theory*
Set	Poset	Category	∞-Category
Abelian group addition $(+, 0)$ $\mathbb{Z}$	Sup-lattice suprema $(\bigvee, \bot)$ $\underline{2} = \{0 < 1\}$	Presentable category colimits, initial object Set	Presentable ∞-category colimits, initial object $\mathcal{S}$
Commutative monoid multiplication $(\times, 1)$ $x^{\mathbb{N}}$	Meet semi-lattice finite infima $(\wedge, \top)$ $\underline{2}^{op}$	Lex category finite limits, terminal object $\mathsf{Fin}^{\mathsf{op}}$	Lex ∞-category finite limits, terminal object $\mathcal{S}_{\text{fin}}^{op}$
Commutative ring $\mathbb{Z}[x] = \mathbb{Z}.x^{\mathbb{N}}$	Frame $\underline{2}[x] = [\underline{2}, \underline{2}]$	Logos $\mathsf{Set}[X] = [\mathsf{Fin}, \mathsf{Set}]$	∞-Logos $\mathcal{S}[X] = [\mathcal{S}_{\text{fin}}, \mathcal{S}]$
Distributivity relation $c \sum a_i = \sum c a_i$	Distributivity relation $c \wedge \bigvee_i a_i = \bigvee_i c \wedge a_i$	Distributivity relations (see Tables 14 and 17)	Distributivity relation (all colimits have the descent property)
Affine scheme affine line $\mathbb{A}^1$	Locale Sierpiński space $\mathbf{S}$	Topos topos of sets $\mathbf{A}$	∞-Topos ∞-topos $\mathbf{A}_\infty$ of ∞-groupoids

Relations and quotients of logoi The computation of quotients of logoi is one of the most fundamental pieces of technology of the theory. The collection of quotients of a given logos $\mathcal{E}$ is a poset. Given any family R of maps in a logos $\mathcal{E}$, the class of all quotients of $\mathcal{E}$ where all maps in R becomes an invertible map has a minimal element $\mathcal{E} \to \mathcal{E}/\!\!/R$ called the *quotient generated by* R.[64] Any quotient can be generated this way. Geometrically, the situation is clear: in the case of a single map, if $f : A \to B$ is a map of sheaves on a topos $\mathbf{X}$, the subtopos $\mathbf{X}^f$ corresponding to $\mathcal{S}\mathrm{h}(\mathbf{X})/\!\!/f$ is intuitively the subspace of points x where the map $f(x) : A(x) \to B(x)$ between the stalks of A and B is a bijection.[65]

The construction $\mathcal{E}/\!\!/R$ has the following universal property: given a logos morphism $u^* : \mathcal{E} \to \mathcal{F}$ such that, for any f in R, $u^*(f)$ is an isomorphism in $\mathcal{F}$, there exists a unique logos morphism $\mathcal{E}/\!\!/R \to \mathcal{F}$ and a factorization $u^* : \mathcal{E} \to \mathcal{E}/\!\!/R \to \mathcal{F}$. Geometrically, this factorization says that if $u : \mathbf{Y} \to \mathbf{X}$ is such that the pullback of the maps $f : A \to B$ of R on $\mathbf{Y}$ are isomorphisms, then the image of u is within the subtopos of $\mathbf{X}$ where all maps in R are isomorphisms.

Recall from Section 2.2.7 that the quotients of a frame F were encoded by nuclei $j : F \to F$. There exists an analogue notion for quotient of logoi called a *left-exact idempotent monad* (we shall say *lex reflector* for short). Such an object is an (accessible) endofunctor $j : \mathcal{E} \to \mathcal{E}$ with a natural transformation $1 \to j$ such that the induced transformation $j \to j \circ j$ is an isomorphism and j is a left-exact functor. Recall that quotients of logoi $q^* : \mathcal{E} \to \mathcal{F}$ are reflective, that is, have a fully faithful right adjoint $q_* : \mathcal{F} \to \mathcal{E}$. In this situation, the endofunctor j is q_*q^* and projects $\mathcal{E}$ to the full subcategory equivalent to $\mathcal{F}$. Reciprocally, any lex reflector j determines a quotient $\mathcal{E} \to \mathcal{F}$ where $\mathcal{F}$ is the full subcategory of *fixed points* of j (objects F such that the map $F \to j(F)$ is an isomorphism). Table 19 presents a comparison of the theory of quotients of logoi and commutative rings.

Table 19: Quotients of logoi & commutative rings

Comm. ring A	ideal $J \subseteq A$	generators a_i for J	projection $\pi : A \to A$ on a complement of J in A	quotient A/J in bijection with the set of fixed points $a = \pi(a)$
Logos $\mathcal{E}$	the class W of all maps $A \to B$ inverted by the quotient	a generating set R of maps $A_i \to B_i$	left-exact idempotent monad $j : \mathcal{E} \to \mathcal{E}$	quotient $\mathcal{E}/\!\!/W$ equivalent to the category of fixed points $F \simeq j(F)$

Examples of quotients and reflectors

(i) For X a topological space or a locale, the lex reflector associated to the quotient $\mathcal{P}\mathrm{r}(X) \to \mathcal{S}\mathrm{h}(X)$ is the sheafification endo-functor.

(ii) (Open reflector) Let $\mathbf{Y} \to \mathbf{X}$ be the open embedding associated to the subterminal object U in $\mathcal{S}\mathrm{h}(\mathbf{X})$. The associated lex reflector is the functor $\mathcal{S}\mathrm{h}(\mathbf{X}) \to \mathcal{S}\mathrm{h}(\mathbf{X})$ sending F to $U \times F$. Intuitively, this functor replaces the stalks of F outside U by a point, leaving the others unchanged.

(iii) (Closed reflector) Let $\mathbf{Y} \to \mathbf{X}$ be the closed embedding associated to the subterminal object U in $\mathcal{S}\mathrm{h}(\mathbf{X})$. For F in $\mathcal{S}\mathrm{h}(\mathbf{X})$, we define $U \star F$ as the pushout of the diagram $U \leftarrow U \times F \to F$. The

[64] Technically, $\mathcal{E} \to \mathcal{E}/\!\!/R$ is the left-exact localization generated by the family of maps R. The detailed construction is given in the examples. There exists the same problem of vocabulary (localization or quotient) as with presentable categories (see Footnote 55). Again, thinking a logos in terms of its objects and not its arrows, the term quotient is more appropriate.

[65] This construction is what becomes the construction of a subspace $Y \subset X$ as equalizer of two maps $a, b : X \rightrightarrows A$ ($Y = \{x | a(x) = b(x)\}$). When sets of points are replaced by categories of points, the equality of two objects has to be replaced by an isomorphism.

associated lex reflector is the functor sending F to $U \star F$. Intuitively, this functor replaces the stalks of F in U by a point, leaving the others unchanged.

(iv) We detail the general construction of the $\mathcal{E} \to \mathcal{E}/\!\!/R$. Thanks to the reflectivity of localizations, $\mathcal{E}/\!\!/R$ can be described as the full subcategory $\mathcal{E}^R$ of $\mathcal{E}$ of objects X satisfying the following condition. Let G be a small category of generators for $\mathcal{E}$. We define R' to be the smallest class of maps in $\mathcal{E}$ containing R that is (1) stable by diagonals (if $f : A \to B$ is in R', then $\Delta f : A \to A \times_B A$ is in R'), and (2) stable by all base change along maps in G (if $f : A \to B$ is in R', then for any $g : C \to B$ in G, the map $f' : C \times_B A \to C$ is in R'). Then, X is in $\mathcal{E}^R$ if, for any map $u : C \to D$ in R', the canonical map of sets $\mathrm{Hom}(D, X) \to \mathrm{Hom}(C, X)$ is a bijection. With the notation introduced for quotients of presentable categories, we have $\mathcal{E}^R = (R')^\perp$. The corresponding reflector and the localization functor $\mathcal{E} \to \mathcal{E}/\!\!/R$ are then constructed with a small object argument.

(v) If R is made of monomorphisms only, the previous description simplifies. It is enough to defined the class R' to satisfy condition (2) only, that is, that R' be stable by base change (along generators). Then, an object X is in $\mathcal{E}^R$ if $\mathrm{Hom}(D, X) \to \mathrm{Hom}(C, X)$ is a bijection for any map $u : C \to D$ that is a base change of some map in R. The reflector is again constructed with a small object argument.

Presentations We define a *logos presentation* as the data of a pair (G, R), where G is a small category and R a set of maps in $\mathsf{Set}[G]$. The objects of G are called the *generators*, and the maps in R the *relations*. A *presentation of a logoi* $\mathcal{E}$ is a triple (G, R, p), where (G, R) is a presentation and p is a functor $p : G \to \mathcal{E}$ inducing an equivalence $\mathsf{Set}[G]/\!\!/R \simeq \mathcal{E}$. Every logos admits a presentation.

Recall that a logos morphism $\mathsf{Set}[G] \to \mathcal{E}$ is equivalent to a diagram $G \to \mathcal{E}$. Then, a morphism $\mathsf{Set}[G]/\!\!/R \to \mathcal{E}$ corresponds to a diagram $G \to \mathcal{E}$ satisfying extra conditions. It is useful to introduce the vocabulary that $\mathsf{Set}[G]$ is the *logos classifying G-diagrams*, and that $\mathsf{Set}[G]/\!\!/R$ is the logos classifying G-diagrams that are *R-exact*.[66] Any structure that can be described diagrammatically (such as groups; rings, as we saw; and also local rings, as we will see) can be classified in this way by a topos. And since every logos admits a presentation, every logos can be thought as classifying some kind of exact diagram. This fact is important in the relationship of logoi with logical theories (see Section 3.4.2).

Recall from the examples of affine topoi the topos $\mathbf{A}^{\to}$ classifying maps and its subtopos $\mathbf{A} \simeq \mathbf{A}^{\simeq} \subset \mathbf{A}^{\to}$ classifying isomorphisms. Geometrically, the data of a map f in $\mathbf{A}^G$ correspond to a topos morphism $\mathbf{A}^G \to \mathbf{A}^{\to}$. For R a family of maps in $\mathcal{E}$, the topos $\mathbf{X}$ corresponding to $\mathsf{Set}[G]/\!\!/R$ is defined by the fiber product in Topos (or the corresponding pushout in Logos)[67]

$$
\begin{tikzcd}
\mathbf{X} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & (\mathbf{A}^{\simeq})^R \arrow[d, hook] \\
\mathbf{A}^G \arrow[r, "R"] & (\mathbf{A}^{\to})^R
\end{tikzcd}
\qquad
\left(
\begin{tikzcd}
\mathsf{Set}[G]/\!\!/R & \mathsf{Set}[R] \arrow[l] \\
\mathsf{Set}[G] \arrow[u] \arrow[ur, phantom, "\ulcorner", very near end] & \mathsf{Set}[\underline{2} \times R] \arrow[l] \arrow[u]
\end{tikzcd}
\right).
$$

Examples of presentations

(i) (Flat diagrams) Let C be a small category with finite limits. We already mentioned that the logos $\mathcal{P}\mathrm{r}(C)$ classifies diagrams $C \to \mathcal{E}$ that preserve finite limits. Let us compute a presentation of this topos. For a finite diagram c_i in C, let $\lim_i^{(C)} c_i$ be the limit of the diagram in C and let $\lim_i^{(\mathrm{free})} c_i$ be the limit of the same diagram in $\mathsf{Set}[C]$. There is a canonical map $f_c : \lim_i^{(C)} c_i \to \lim_i^{(\mathrm{free})} c_i$ in $\mathsf{Set}[C]$. Let Λ be the collection of all these maps. Then, the logos quotient $\mathsf{Set}[C]/\!\!/\Lambda$ is the logos $\mathcal{P}\mathrm{r}(C)$.

A logos morphism $\mathsf{Set}[C] \to \mathcal{E}$ is the same thing as a diagram $C \to \mathcal{E}$. The logos morphisms $\mathcal{P}\mathrm{r}(C) \to \mathcal{E}$ correspond to those diagrams $C \to \mathcal{E}$ that are *flat*, or *filtering* in the sense of [26, VII.8]. In the case

[66] Recall that any ring can be presented as classifying the solutions to some polynomial equations. Classifying R-exact diagrams is the analogue for logoi.

[67] Notice the analogy with the definition of affine schemes as zeros of a set of polynomials.

where C has finite limits, a diagram $C \to \mathcal{E}$ is flat if and only if it is a left-exact functor.

(ii) (Torsors) In the case where $C = G$ is a group viewed as a category with one object, a diagram $G \to \mathcal{E}$ corresponds to a sheaf with an action of G. Such a diagram is flat if and only if the action is free and transitive, that is, if and only if it is a G-torsor [26, VIII]. Moreover, natural transformations between logos morphisms $\mathsf{Set}^G \to \mathcal{E}$ corresponds to morphisms of G-torsors. This says that $\mathbf{B}G$ is the topos classifying G-torsors.

(iii) Let C be a small category with finite sums; then there exists a topos classifying diagrams $C \to \mathcal{E}$ that preserve sums. For a finite family (c_i) of objects in C, let $\coprod_i^{(C)} c_i$ be the sum of the family in C, and let $\coprod_i^{(\text{free})} c_i$ be the sum of the family in $\mathsf{Set}[C]$. There is a canonical map $\coprod_i^{(\text{free})} c_i \to \coprod_i^{(C)} c_i$ in $\mathsf{Set}[C]$. Let Σ be the collection of all these maps. Then, the logos $\mathsf{Set}[C]/\!\!/\Sigma$ is the logos classifying diagrams $C \to \mathcal{E}$ preserving sums.

More generally, the same construction works for any class of colimits existing on C and leads to a topos classifying diagrams $C \to \mathcal{E}$ preserving any set of colimits.

(iv) (Inhabited sets revisited) The left-exact localizations of the logos $\mathsf{Set}[X]$ classify objects satisfying some conditions. For example, one can ask that the canonical map $X \to 1$ be a cover (see Section 3.2.12). This condition is equivalent to the exactness of the diagram $X \times X \rightrightarrows X \to 1$. One can prove that $\mathsf{Set}[X]/\!\!/(\operatorname{colim}(X \times X \rightrightarrows X) \to 1) = [\mathsf{Fin}^\circ, \mathsf{Set}] = \mathcal{S}\mathrm{h}(\mathbf{A}^\circ)$. That is, the topos classifying inhabited objects is the topos classifying nonempty sets.

(v) (Sierpiński revisited) Another example is to ask that the canonical map $X \to 1$ be a monomorphism, that is, X is subterminal. This condition is equivalent to the diagonal $X \to X \times X$ being an isomorphism. One can prove that $\mathsf{Set}[X]/\!\!/(X \simeq X \times X) = \mathcal{S}\mathrm{h}(\mathbf{S})$, that is, that subterminal objects are classified by the Sierpiński topos. We already saw this, since subterminal objects are equivalent to open domains.

(vi) (Arrow classifier) Let $C = \{Y \to X\} \simeq \underline{2}$ be the category with one arrow. Then $\mathsf{Set}[Y \to X]$ is the logos classifying arrows. It can be proved to be $[\mathsf{Fin}^\to, \mathsf{Set}]$. We can impose the condition that $Y = 1$; this is equivalent to inverting the canonical map $Y \to 1$. The resulting logos is $\mathsf{Set}[Y \to X]/\!\!/(Y \simeq 1) = \mathsf{Set}[X^\bullet]$.

(vii) (Mono classifier) A monomorphism in a logos is defined as a map $A \to B$ such that the diagonal $A \to A \times_B A$ is an isomorphism. Intuitively, a monomorphism of sheaves on a space X is a map $f : A \to B$ that is injective stalk-wise. Let $\mathsf{Fin}^\rightarrowtail$ be the full subcategory of $\mathsf{Fin}^\to$ whose objects are monomorphisms between finite sets. It can be proved that the $[\mathsf{Fin}^\rightarrowtail, \mathsf{Set}]$ is the logos classifying monomorphisms $\mathsf{Set}[Y \rightarrowtail X]$. The corresponding subtopos of $\mathbf{A}^\to$ will be denoted $\mathbf{A}^\rightarrowtail$.

If we further force the map $X \to 1$ to be an isomorphism, we get back the Sierpiński logos.

(viii) (Cover classifier) Let $f : A \to B$ be a map in a logos $\mathcal{E}$. Recall from Section 3.2.12 that the image factorization of f is $A \to \mathrm{im}(f) \to B$, where $\mathrm{Im}(f) = \mathrm{colim}(A \times_B A \rightrightarrows A)$. The map f is a cover if and only if the monomorphim $\mathrm{im}(f) : \mathrm{Im}(f) \to B$ is an isomorphism. Let $\mathsf{Fin}^\twoheadrightarrow$ be the full subcategory of $\mathsf{Fin}^\to$ whose objects are surjections between finite sets. It can be proved that the $[\mathsf{Fin}^\twoheadrightarrow, \mathsf{Set}]$ is the logos classifying surjections $\mathsf{Set}[Y \twoheadrightarrow X]$. The corresponding subtopos of $\mathbf{A}^\to$ will be denoted $\mathbf{A}^\twoheadrightarrow$.

The image factorization of maps gives a topos morphism $\mathbf{A}^\to \to \mathbf{A}^\rightarrowtail$ and a cartesian square

$$\begin{tikzcd}
\mathbf{A}^\twoheadrightarrow \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbf{A}^\simeq \arrow[d, hook] \\
\mathbf{A}^\to \arrow[r, "\text{image}"] & \mathbf{A}^\rightarrowtail.
\end{tikzcd}$$

The fact that a map is an isomorphism if and only if it is a cover and a monomorphism gives a cartesian square

$$\begin{tikzcd}
\mathbf{A}^\simeq \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbf{A}^\rightarrowtail \arrow[d, hook] \\
\mathbf{A}^\twoheadrightarrow \arrow[r, hook] & \mathbf{A}^\to.
\end{tikzcd}$$

Topologies and sites Although presentations may be the most natural way to define logoi by generators and relation, history and practice have imposed another way to do it: the notion of site. In a presentation by means of a site, the free logoi $\mathsf{Set}[G]$ are replaced by presheaf logoi $\mathcal{P}r(C)$, and the relations are replaced by the data of a *topology*. Recall that the quotient of a logos $\mathcal{E}$ generated by a map $f : A \to B$ forces f to become an isomorphism. A variation on this is to force f to become a cover instead. This is the main idea behind the notion of a topology. The comparison between sites and presentations is summarized in Table 21.

Let $A \to \mathrm{Im}(f) \to B$ be the image factorization of f. The image factorizations are built using colimits and finite limits, so they are preserved by any morphism of logoi $\mathcal{E} \to \mathcal{F}$. The map f becomes a cover in $\mathcal{F}$ if and only if the monomorphism $\mathrm{im}(f) : \mathrm{Im}(f) \to B$ becomes an isomorphism in $\mathcal{F}$. Thus, forcing a map to become a cover is equivalent to forcing some monomorphism to become an isomorphism, which is a particular case of a quotient. The data of *topological relations* on a logos $\mathcal{E}$ is defined to be the data of a family J of maps to be forced to become cover. Equivalently, topological relations can be given as the data of a family J of monomorphisms to be inverted.

Let us see how this is related to the so-called sheaf condition. Recall from the examples of quotients the construction of the quotient $\mathcal{E}/\!\!/(\mathrm{im}(f)) \simeq \mathcal{E}^{\mathrm{im}(f)} \hookrightarrow \mathcal{E}$ as a full subcategory of $\mathcal{E}$. A necessary condition for an object F of $\mathcal{E}$ to be in $\mathcal{E}^{\mathrm{im}(f)}$ is that $\mathrm{Hom}(B, F) \simeq \mathrm{Hom}(\mathrm{Im}(f), F)$. Using the fact that $\mathrm{Im}(f) = \mathrm{colim}(A \times_B A \rightrightarrows A)$, this condition becomes the *sheaf condition*:

$$\mathrm{Hom}(B, F) = \lim\Big(\ \mathrm{Hom}(A, F) \rightrightarrows \mathrm{Hom}(A \times_B A, F)\ \Big).$$

Then, one can prove that F is in $\mathcal{E}^{\mathrm{im}(f)}$ if and only if it satisfies the same condition not only for f but for all base changes of f.

A *site* is the data of a small category C and a set J of topological relations on $\mathcal{P}r(C)$ satisfying some extra conditions (stability by base change, composition, etc.) We shall not detail them since most of them are superfluous to characterize the corresponding reflective subcategory. Only the stability by base change is crucial.[68]

As for presentations, the notion of a site can be interpreted geometrically in Topos. Recall the subtopos $\mathbf{A}^{\twoheadrightarrow} \hookrightarrow \mathbf{A}^{\to}$ classifying arrows that are covers. Let $\mathbf{B}(C^{op})$ be the Alexandrov topos dual to $\mathcal{P}r(C)$ and J a topological relation in $\mathcal{P}r(C)$. The subtopos $\mathbf{X}$ of $\mathbf{B}(C^{op})$ defined by J can be defined as the following pullback in Topos:

$$\begin{tikzcd}
\mathbf{X} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & (\mathbf{A}^{\twoheadrightarrow})^J \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & (\mathbf{A}^{\simeq})^J \arrow[d, hook] \\
\mathbf{B}(C^{op}) \arrow[r, "J"] \arrow[rr, bend right, "\text{fam. of monos}"'] & (\mathbf{A}^{\to})^J \arrow[r, "\text{image}"] & (\mathbf{A}^{\rightarrowtail})^J.
\end{tikzcd}$$

It is a very important feature of logoi that the two conditions of forcing some maps to become isomorphisms and forcing some maps to become surjective are in fact equivalent, that is, every quotient can be described in terms of topological relations.[69] Recall that the diagonal of $f : A \to B$ is the map $\Delta f : A \to A \times_B A$, which is always a monomorphism. The map $f : A \to B$ is a monomorphism if and only if Δf is an isomorphism. Then a map f is an isomorphism if and only if it is a cover and a monomorphism if and only if both monomorphisms $\mathrm{im}(f)$ and Δf are isomorphisms. As a consequence, any logos can be presented by means of topological relations. Table 20 recalls how to translate some conditions in terms of topologies, that is, of monomorphisms, and Table 21 summarizes the comparison between sites and presentations.

[68]The situation compares to a more classical one. Recall that any relation R on a set E generates an equivalence relation. But, to compute the quotient E/R, is it not necessary for R to be an actual equivalence relation. Similarly, any set of monomorphism in a logos $\mathcal{E}$ can be completed into a topology, but the characterization of the quotient reflective subcategory can be done directly from the generators.

[69]We shall see that this property fails for ∞-logoi.

Table 20: Quotient and topologies

Forcing condition	*Formulation in terms of monomorphisms*
inverting a map $f : A \to B$	inverting the two monomorphisms $\mathrm{im}\,(f) : \mathrm{Im}\,(f) \to B$ and $\Delta f : A \to A \times_B A$.
forcing a map $c : U \to X$ to become a cover	inverting the monomorphism $\mathrm{im}\,(c) : \mathrm{im}\,(c) \rightarrowtail X$
forcing a family $c_i : U_i \to X$ to become covering	inverting the monomorphism $\left(\bigcup_i \mathrm{im}\,(c_i)\right) \rightarrowtail X$

Table 21: Comparison of sites and presentations

	Site	*Presentation*
Generators	a category C of *representables*	a category G of *generators*
"Free" object	$\mathcal{P}\mathrm{r}\,(C)$ (presheaf logos/ Alexandrov topos)	$\mathsf{Set}\,[G] = \mathcal{P}\mathrm{r}\left(G^{\mathrm{lex}}\right)$ (free logos/affine topos)
Relations	a topology J on C (forcing some maps to become covers)	a set R of maps in $\mathsf{Set}\,[G]$ (forcing some maps to become isomorphisms)
convenient for conditions of the type	colim of representables = representable	colim of lim of generators = lim of colim of generators
Quotient	$\mathcal{P}\mathrm{r}\,(C) /\!\!/ J = \mathcal{S}\mathrm{h}\,(C, J)$	$\mathsf{Set}\,[G] /\!\!/ R$

Examples of topological relations and sites

(i) (Canonical and coherent topologies) Let X be a space. Let J_{can} be the collection of all open covers $U_i \to X$. Then the logos $\mathcal{S}\mathrm{h}(X)$ is the quotient of the logos $\mathcal{P}\mathrm{r}(\mathcal{O}(X))$ forcing the families in J_{can} to be covering families. If we consider instead the class J_{fin} of all finite open covers $U_i \to X$, then the quotient is the logos $\mathcal{S}\mathrm{h}(X_{\text{coh}})$.

(ii) (Stone–Čech) Let E be a set. Recall that the Stone–Čech compactification βE of E is a subtopos of $\widehat{E}$. Let J be the collection of all partitions $E_1 \coprod E_2 \to E$ or E. Then the logos $\mathcal{S}\mathrm{h}(\beta E)$ is the quotient of the logos $\mathcal{S}\mathrm{h}(\widehat{E}) = \mathcal{P}\mathrm{r}(P(E))$ forcing the families in J to be covering families.

(iii) (Zariski spectrum) Let fLoc_A be the poset of finitely generated localisations of a ring A. Every finitely generated localisation of A is of the form $A_f = A[f^{-1}]$ for some element f in A. If f and g are in A, let us write $f \le g$ to mean that g is invertible in A_f. The relation $f \le g$ is a pre-order (it is transitive and reflexive). Let P_A be the associated poset, and let us write $D(f)$ for the image of $f \in A$ in P_A. The poset P_A is an inf-semi-lattice with $D(f) \wedge D(g) = D(fg)$ and $D(1) = 1$. The points of the Alexandrov logos $[P_A, \mathsf{Set}]$ form the poset $\mathsf{Loc}_A = \mathcal{I}\mathrm{nd}(\mathsf{fLoc}_A)$ of all localizations $A \to B$.

If $D(f_i) \le D(f)$ $(1 \le i \le n)$ and $f_1 + \cdots + f_n = f$, let us declare that the family $D(f_i)$ $(1 \le i \le n)$ covers $D(f)$. For example, the pair $(D(f), D(1-f))$ covers $D(1) = 1$ for every $f \in A$. Also, $D(0)$ is covered by the empty family. This defines a topology on the presheaf logos $[P_A, \mathsf{Set}]$. The corresponding topos is the *Zariski spectrum* $Spec_{Zar}(A)$ of A. The topos $Spec_{Zar}(A)$ is localic, and its posets of points is the subposet of Loc_A spanned by localizations $A \to B$, where B is a local ring. This poset is the opposite of the poset of prime ideals of A.

(iv) (Actions of a Galois group) Let fSep_k be the category of finite separable field extensions of a field k. We consider the Alexandrov logos $[\mathsf{fSep}_k, \mathsf{Set}]$. A point of the corresponding topos is a separable field extensions of k. Then, we can construct the localisation forcing all maps in $(\mathsf{fSep}_k)^{\mathsf{op}}$ to become covers in $[\mathsf{fSep}_k, \mathsf{Set}]$. The resulting quotient is the logos $\mathcal{S}\mathrm{h}(\mathsf{fEt}_k, \text{étale})$ of sheaves for the étale topology on fEt_k. The corresponding topos is the so-called *étale spectrum* of k.

Recall that the Galois group $\mathrm{Gal}(k)$ of k is defined as a profinite group. We mentioned that the category $\mathsf{Set}^{(\mathrm{Gal}(k))}$ of sets equipped with a continuous action of $\mathrm{Gal}(k)$ is a logos. The logos $\mathcal{S}\mathrm{h}(\mathsf{Et}_k, \text{étale})$ can be proved to be equivalent to $\mathsf{Set}^{(\mathrm{Gal}(k))}$.

(v) (Schanuel logos) Let fInj be the category of finite sets and injective maps. The category of points of the logos $\mathbf{B}\mathsf{fInj}$ is the category of all sets and injective maps. Then, we can construct the localisation forcing every map in $\mathsf{fInj}^{\mathsf{op}}$ to become cover in $[\mathsf{fInj}, \mathsf{Set}]$. The resulting category of sheaves $\mathcal{S}\mathrm{h}(\mathsf{fInj}^{op})$ is called the *Schanuel logos*. Its category of points is the category of infinite sets and injective maps. Let $G := \mathrm{Aut}(\mathbb{N})$ be the group of automorphisms of $\mathbb{N}$ with the topology induced from the infinite product $\mathbb{N}^{\mathbb{N}}$, and let $\mathsf{Set}^{(G)}$ be the category of continuous G-sets. It can be proved that the logos $\mathcal{E}$ is equivalent to the category $\mathsf{Set}^{(G)}$.

(vi) (Étale spectrum of a commutative ring) Let fSep_A be the category of finite separable extensions of a ring A. The opposite category is the category fEt_A of finite étale extensions of the scheme dual to A. We consider the Alexandrov logos $[\mathsf{fSep}_A, \mathsf{Set}] = \mathcal{P}\mathrm{r}(\mathsf{fEt}_A)$. Its category of point is the category $\mathsf{Sep}_A = \mathcal{I}\mathrm{nd}(\mathsf{fSep}_A)$ of all separable extensions $A \to B$.

The Yoneda embedding $\mathsf{fEt}_A \hookrightarrow \mathcal{P}\mathrm{r}(\mathsf{fEt}_A)$ does not send étale coverings in fEt_A to covering families $\mathcal{P}\mathrm{r}(\mathsf{fEt}_A)$. Forcing this, define the *étale spectrum* $Spec_{Et}(A)$ of A. The category of points of $Spec_{Et}(A)$ is the subcategory of Sep_A spanned by separable extensions $A \to B$ such that B is a strictly Henselian local ring. The isomorphism classes of $\mathcal{P}\mathrm{t}(Spec_{Et}(A))$ are in bijection with prime ideals of A. For an ideal p, the symmetries of the corresponding strict Henselianisation $A \to A_p^h$ are given by the Galois group of the residue field of p. This category is not a poset, and this proves that the topos $Spec_{Et}(A)$ is not localic. However, its localic reflection, that is, the socle of $Spec_{Et}(A)$, is $Spec_{Zar}(A)$. Intuitively, $Spec_{Et}(A)$ is the space $Spec_{Zar}(A)$, but with the extra information of Galois groups at each points.

The construction of étale spectra was the original motivation to develop topos theory. Its most important property is that the functor $Spec_{Et} : \mathsf{Ring}^{\mathsf{op}} \to \mathsf{Topos}$ sends étale maps of schemes to étale maps of topoi. This is what allows us to interpret the algebraic Galoisian or étale descent as an actual topological descent and permits the construction of ℓ-adic cohomology theories.

(vii) (Nisnevich spectrum) In the previous example, if we force only the Nisnevich coverings families to become covering families $\mathcal{P}r(\mathsf{fEt}_A)$, this defines the subtopos *Nisnevich spectrum* $Spec_{Nis}(A)$ of A.

Geometrically, the Nisnevich spectrum is further from the classical intuition of the Zariski spectrum of A than the étale spectrum is. The category of points of $Spec_{Nis}(A)$ is the subcategory of Sep_A spanned by separable extensions $A \to B$ such that B is an Henselian ring. There exists an inclusion $Spec_{Et}(A) \hookrightarrow Spec_{Nis}(A)$, which at the level of points corresponds to that of strict Henselian rings. Since not every Henselian ring is strict, the set of isomorphism classes of $\mathcal{P}t(Spec_{Nis}(A))$ is strictly larger than the set of prime ideals of A. For example, in the case of field k, the Nisnevich topology is trivial, and $Spec_{Nis}(k) = \mathcal{P}r(\mathsf{fEt}_A)$, whose points are all separable extensions of fields $k \to k'$. The poset reflection of this category is the poset of conjucacy classes of intermediate fields between k and some separable closure $\overline{k}$. This proves that the socle of $Spec_{Nis}(A)$ is not $Spec_{Zar}(A)$.

There exists two morphisms of topoi

$$Spec_{Et}(A) \hookrightarrow Spec_{Nis}(A) \twoheadrightarrow Spec_{Zar}(A)$$

where the first one is an embedding and the second a surjection, and the composite is the socle projection of $Spec_{Et}(A)$. Intuitively, the Nisnevich spectrum is a sort of "mapping cone" (in the sense of homotopy theory) interpolating between the étale and Zariski spectra.

(viii) (Zariski sheaves) Let $\mathsf{Ring}_{\mathsf{fp}}$ be the category of commutative rings of finite presentation and $\mathsf{Aff}_{\mathsf{fp}} = \mathsf{Ring}_{\mathsf{fp}}^{\mathsf{op}}$ be the category of affines schemes of finite presentation. We consider the Alexandrov logos $\mathcal{P}r(\mathsf{Aff}_{\mathsf{fp}}) = [\mathsf{Ring}_{\mathsf{fp}}, \mathsf{Set}]$. The Yoneda embedding $\mathsf{Aff}_{\mathsf{fp}} \hookrightarrow \mathcal{P}r(\mathsf{Aff}_{\mathsf{fp}})$ sends $\mathbb{A}^1$ to the forgetful functor

$$\mathbb{A}^1 : \mathrm{Hom}_{\mathsf{Ring}_{\mathsf{fp}}}(\mathbb{Z}[x], -) : \mathsf{Ring}_{\mathsf{fp}} \to \mathsf{Set}.$$

Recall that $\mathbb{A}^1$ is a ring object in the category of affine schemes with addition and multiplication given by maps $+, \times : \mathbb{A}^2 \to \mathbb{A}^1$. The Yoneda embedding preserves products, and $\mathbb{A}^1$ is also a ring object in $[\mathsf{Ring}_{\mathsf{fp}}, \mathsf{Set}]$. If $f^* : [\mathsf{Ring}_{\mathsf{fp}}, \mathsf{Set}] \to \mathcal{E}$ is a morphism of logoi, then $f^*(\mathbb{A}^1)$ is a ring object in $\mathcal{E}$. This defines an equivalence between the category of logos morphisms $[\mathsf{Ring}_{\mathsf{fp}}, \mathsf{Set}] \to \mathcal{E}$ and the category of ring objects in $\mathcal{E}$. Thus, the logos $[\mathsf{Ring}_{\mathsf{fp}}, \mathsf{Set}]$ classifies commutative rings.

Recall that a ring A is *non zero* if $0 \neq 1$ in A. Let $\mathsf{Ring}^{\circ}_{\mathsf{fp}} \subset \mathsf{Ring}_{\mathsf{fp}}$ be the full category of non zero rings. The forgetful functor $\mathbb{A}^1 : \mathsf{Ring}^{\circ}_{\mathsf{fp}} \to \mathsf{Set}$ is a non zero ring object in the logos $[\mathsf{Ring}^{\circ}_{\mathsf{fp}}, \mathsf{Set}]$. The fully faithful inclusion $\mathsf{Ring}^{\circ}_{\mathsf{fp}} \hookrightarrow \mathsf{Ring}_{\mathsf{fp}}$ induces a left-exact localization $[\mathsf{Ring}_{\mathsf{fp}}, \mathsf{Set}] \to [\mathsf{Ring}^{\circ}_{\mathsf{fp}}, \mathsf{Set}]$ that presents $[\mathsf{Ring}^{\circ}_{\mathsf{fp}}, \mathsf{Set}]$ as the logos classifying non zero rings.

Recall that a commutative ring A is a *local ring* if $0 \neq 1$, and for every element a in A, either a or $1-a$ is invertible. An element a in A is the same thing as a map $\mathbb{Z}[x] \to A$. This element is invertible if and only if the classifying map can be factored as $\mathbb{Z}[x] \to \mathbb{Z}[x, x^{-1}] \to A$. The definition of a non zero local ring can be encoded by saying that, in the following diagram, one of the two dashed arrows has to exist:

$$\begin{tikzcd}
\mathbb{Z}[x, x^{-1}] \arrow[dr, dashed, "a \text{ is invertible}"'] & \mathbb{Z}[x] \arrow[l] \arrow[r] \arrow[d, "a"] & \mathbb{Z}[x, (1-x)^{-1}] \arrow[dl, dashed, "1-a \text{ is invertible}"] \\
& A &
\end{tikzcd}$$

Let $A^{\times} = \mathrm{Hom}(\mathbb{Z}[x, x^{-1}], A)$ be the subset of invertible elements in A. The two horizontal maps define two maps $A^{\times} \to A \leftarrow A^{\times}$, and a non zero ring A is local if they are jointly surjective. The two horizontal maps of the diagram above corresponds to two maps in the opposite category $\mathsf{Aff}^{\circ}_{\mathsf{fp}}$:

$$\mathbb{G}_m \xrightarrow{\iota} \mathbb{A} \xleftarrow{1-\iota} \mathbb{G}_m$$

The two maps define a single map $\mathbb{G}_m \coprod \mathbb{G}_m \to \mathbb{A}$ in $\mathcal{P}\mathrm{r}\left(\mathsf{Aff}^{\circ}_{\mathsf{fp}}\right)$. This map is not a cover, but it can be forced to be. And this is exactly the condition that defines local rings. The quotient of $\mathcal{P}\mathrm{r}\left(\mathsf{Aff}^{\circ}_{\mathsf{fp}}\right)$ generated by the condition "$\mathbb{G}_m \coprod \mathbb{G}_m \to \mathbb{A}$ is a cover" is the logos $\mathcal{S}\mathrm{h}\left(\mathsf{Aff}^{\circ}_{\mathsf{fp}}\right)$ that classifies local rings. The image of $\mathbb{A}$ in $\mathcal{S}\mathrm{h}\left(\mathsf{Aff}^{\circ}_{\mathsf{fp}}\right)$ is the generic local ring, and it is often denoted $\mathbb{A}^1$. The category $\mathcal{S}\mathrm{h}\left(\mathsf{Aff}^{\circ}_{\mathsf{fp}}\right)$ can be proved to be the category $\mathcal{S}\mathrm{h}\left(\mathsf{Aff}_{\mathsf{fp}}, \mathrm{Zar}\right)$ of sheaves on $\mathsf{Aff}_{\mathsf{fp}}$ for the Zariski topology.

Similar considerations apply to defining the topoi classifying Henselian rings (with the Nisnevich topology) and strict Henselian rings (with the étale topology). However, these topologies are not nicely generated by a single map, as is the Zariski topology.

Presentations from logical theories We mentioned in the introduction that logoi could be thought as categories of generalized sets and were suited to producing semantics for all sorts of logical theories. A particular aspect of this relationship with logic is that logical theories can be used as generating data for logoi. Roughly presented, a logical theory has sorts (or types), formulas, and axioms. Intuitively, the sorts and formulas generate the objects and morphisms of a category G, and the axioms distinguish a set of maps R in $\mathsf{Set}[G]$ (using the dictionary sketched in Table 7). A model of the logical theory in a logos $\mathcal{E}$ is an interpretation of sorts and formulas such that the axioms are validated. In terms of category, this is a functor $G \to \mathcal{E}$ such that the canonical extension $\mathsf{Set}[G] \to \mathcal{E}$ sends the maps of R to isomorphisms. In other terms, a model in $\mathcal{E}$ is a logos morphism $\mathsf{Set}[G]/\!\!/R \to \mathcal{E}$. For this reason, the logos $\mathsf{Set}[G]/\!\!/R$ is called the *classifying logos* of the theory. Details about this construction can be found in [26, VI, VIII, X]. The previous construction of the logos of Zariski sheaves is an example of this construction. The quotient forcing the map $\mathbb{G}_m \coprod \mathbb{G}_m \to \mathbb{A}^1$ to become a cover corresponds to the axiom that the ring must be local.

However, such a construction is not pertinent for all logical theories. It relies implicitly on the fact that morphisms of logoi preserve the logical constructions, but this is mostly false. Logoi morphisms preserve all colimits, but only finite limits. This means that, in the dictionary of Table 7, they will only be compatible with logical theories involving *finite* conjunctive conditions, that is, only finite conjunctions of propositions and no function type, no universal quantification, no implication, and no subobject classifier. Logical theories compatible with logoi morphisms are called *geometric* (see [19, 26]).

A particular instance of the dictionary of Table 7 is that an existential statement translates into the image of a morphism. This gives an elegant logical interpretation to the presentation of logoi by sites: topological relations correspond to forcing some statements of existence. Again, the previous construction of the logos of Zariski is an example: the axiom forcing a ring to be local is existential.[70]

4 Higher topos–logos duality

4.1 Definitions and examples

4.1.1 Enhancing Set into $\mathcal{S}$ Our presentation should have made it clear that the theory of topoi is essentially what become locale theory when the "basic coefficients" are enhanced from the poset $\{0 < 1\}$ to the category Set. Similary, the theory of ∞-topoi is what becomes topos theory when the category Set is enhanced into the ∞-category $\mathcal{S}$ of ∞-groupoids (e.g., homotopy types of spaces). Intuitively, an ∞-logos is an ∞-category of sheaves with values in ∞-groupoids.[71]

The replacements of $\{0 < 1\}$ by Set and then by $\mathcal{S}$ follow a precise logic. In posets, $\underline{2}$ is the free sup-lattice on one generator. In categories, $\mathsf{Set} = \mathcal{P}\mathrm{r}(1)$ is the free cocomplete category on one generator. And in

[70] $\vdash_a \exists b, (ab = 1) \vee \big((1-a)b = 1\big)$.

[71] Sheaves of ∞-groupoids are also called *stacks* in ∞-groupoids. However, the usage in ∞-topos theory has simplified the vocabulary and kept only the name of sheaves.

∞-categories, $\mathcal{S} = \mathcal{P}r_\infty(1)$ is the free cocomplete ∞-category on one generator. These universal properties are the reason why $\underline{2}$, Set, and $\mathcal{S}$ are so important. This may explains also why, in the setting of ∞-categories, $\mathcal{S}$ is a more fundamental object than Set: the category Set is still cocomplete as an ∞-category, but it is no longer freely generated.[72]

The manipulation of ∞-groupoids is, in practice, remarkably similar to that of sets. The main operations of manipulation of ∞-groupoids are still limits and colimits, but their behavior in the ∞-categorical setting is different. For example, the diagonal $\Delta f : A \to A \times_B A$ of a map $f : A \to B$ need not be a monomorphism any longer. Also, using the embedding $\mathsf{Set} \hookrightarrow \mathcal{S}$ whose image is discrete ∞-groupoids, the colimit of a diagram of sets computed in $\mathcal{S}$ need not be discrete.[73] Otherwise, the theory of ∞-logoi is very similar in its structure to that of logoi (see Table 18). Essentially, it suffices to replace Set by $\mathcal{S}$ everywhere and to interpret all constructions (limits, colimits, adjunctions, commutativity of diagrams, etc.) in the ∞-categorical sense. For example, the free cocompletion of an ∞-category C is now given by the ∞-category of presheaves of ∞-groupoids $\mathcal{P}r_\infty(C) = [C^{op}, \mathcal{S}]$ rather than presheaves with values in Set. An ∞-logos can then be defined as an (accessible) left-exact localization of some $\mathcal{P}r_\infty(C)$. Morphisms of ∞-logoi are defined as functors preserving colimits and finite limits in the ∞-categorical sense. This defines an ∞-category Logos_∞, and the category Topos_∞ is then defined to be $(\mathsf{Logos}_\infty)^{\mathsf{op}}$.[74] We shall denote by $\mathcal{S}\mathrm{h}_\infty(\mathbf{X})$ the ∞-logos dual of an ∞-topos $\mathbf{X}$.

Affine topoi, Alexandrov topoi, points, subtopoi, étale morphisms and so on, are all defined the same way as in topos theory. For this reason, we shall not present the theory of ∞-topoi systematically, as in the case of topoi (see [4, 23]). We will just underline the important new features of the theory. Before we do this, we are going to introduce some examples to play with.

4.1.2 First examples

(i) (Point) The ∞-category $\mathcal{S}$ is the initial ∞-logos. Any ∞-logos $\mathcal{E}$ has a canonical logos morphism $\mathcal{S} \to \mathcal{E}$. The ∞-topos $\mathbf{1}$ dual to $\mathcal{S}$ is terminal. A *point* of an ∞-topos $\mathbf{X}$ is a morphism $\mathbf{1} \to \mathbf{X}$, that is, a logos morphism $\mathcal{S}\mathrm{h}_\infty(\mathbf{X}) \to \mathcal{S}$. The ∞-category of points of a topos $\mathbf{X}$ is $\mathcal{P}\mathrm{t}(\mathbf{X}) \coloneqq \mathrm{Hom}_{\mathsf{Topos}_\infty}(\mathbf{1}, \mathbf{X}) = \mathrm{Hom}_{\mathsf{Logos}_\infty}(\mathcal{S}\mathrm{h}_\infty(\mathbf{X}), \mathcal{S})$.

(ii) (The ∞-topos of a topos) In the same way that any frame $\mathcal{O}(X)$ defines a logos $\mathcal{S}\mathrm{h}(X)$ of sheaves of sets, any logos $\mathcal{S}\mathrm{h}(\mathbf{X})$ defines an ∞-logos $\mathcal{S}\mathrm{h}_\infty(\mathbf{X})$ of sheaves of ∞-groupoids. The ∞-category $\mathcal{S}\mathrm{h}_\infty(\mathbf{X})$ is defined at the full sub-∞-category of $\left[\mathcal{S}\mathrm{h}(\mathbf{X})^{op}, \mathcal{S}\right]$ spanned by functors F satisfying the *higher sheaf condition*: for any covering family $U_i \to U$ in $\mathcal{S}\mathrm{h}(\mathbf{X})$ we must have

$$F(U) \simeq \lim \Big(\textstyle\prod_i F(U_i) \rightrightarrows \prod_{ij} F(U_{ij}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \prod_{ijk} F(U_{ijk}) \dots \Big),$$

where the diagram is now a *full* cosimplicial diagram. This defines a functor $\mathcal{S}\mathrm{h}_\infty : \mathsf{Logos} \to \mathsf{Logos}_\infty$, and dually a functor $\mathsf{Topos} \to \mathsf{Topos}_\infty$, which are both fully faithful. In particular, the ∞-category of points of a topos $\mathbf{X}$ does not change when it is viewed as an ∞-topoi and stays a 1-category.

(iii) (Quasi-discrete ∞-topos) For K an ∞-groupoid, the ∞-category $\mathcal{S}_{/K}$ is a ∞-logos. The dual ∞-topos is denoted $\mathbf{B}_\infty K$ and called *quasi-discrete*. An ∞-topos is called *discrete* if it of the type $\mathbf{B}_\infty E$ for E a set. This construction defines a fully faithful functor $\mathbf{B}_\infty : \mathcal{S} \to \mathsf{Topos}_\infty$, which is analogue to the "discrete topos" functor $\mathsf{Set} \to \mathsf{Topos}$. The ∞-category of points of $\mathbf{B}_\infty K$ is K. In particular, when K is not a 1-groupoid (e.g., the homotopy type $K(\mathbb{Z}, 2)$ of $\mathbb{CP}^\infty$, which is a non trivial 2-groupoid), the quasi-discrete topos $\mathbf{B}_\infty K$ is not in the image of $\mathsf{Topos} \hookrightarrow \mathsf{Topos}_\infty$. This proves that there are more ∞-topoi than topoi.

(iv) (Alexandrov ∞-topos) For C a small ∞-category, the diagram ∞-category $[C, \mathcal{S}] = \mathcal{P}r_\infty(C^{op})$ is an ∞-logoi. The dual *Alexandrov ∞-topos* is denoted $\mathbf{B}_\infty C$. This construction defines a functor $\mathbf{B}_\infty$:

[72] Other motivations to enhance sets into ∞-groupoids are given in [1].

[73] This new colimit is the so-called homotopy colimit. For a description of the notion of homotopy colimit, see [1].

[74] When Logos_∞ is viewed as an $(\infty, 2)$-category, we defined the $(\infty, 2)$-category of ∞-topoi as $\mathsf{Topos}_\infty = (\mathsf{Logos}_\infty)^{\mathsf{1op}}$, i.e., by reversing the direction of 1-arrows only.

$\mathsf{Cat}_\infty \to \mathsf{Topos}_\infty$ that is not fully faithful.[75] The restriction of this functor to ∞-groupoids via $\mathcal{S} \hookrightarrow \mathsf{Cat}_\infty$ gives back the previous example. The ∞-category of points of $\mathbf{B}_\infty C$ is $\mathcal{P}\mathrm{t}(\mathbf{B}_\infty C) = [C^{op}, \mathcal{S}]^{\mathrm{lex}} = \mathcal{I}\mathrm{nd}(C)$.

Quasi-discrete ∞-topoi are examples of Alexandrov ∞-topoi. This is a consequence of the *Galoisian interpretation of homotopy theory* [34, 37] that provides the important equivalence of ∞-categories $\mathcal{S}^K \simeq \mathcal{S}_{/K}$. In the case where $K = BG$ is the classifying space of some group G, this equivalence encodes the statement that a homotopy type with an action of G is the same thing as a homotopy type over BG. In this case, we shall denote simply by $\mathbf{B}_\infty G$ the quasi-discrete ∞-topos $\mathbf{B}_\infty(BG)$.

(v) (Affine ∞-topos) For C a small ∞-category, the *free ∞-logoi* on C is $\mathcal{S}[C] := \mathcal{P}\mathrm{r}_\infty\left(C^{\mathrm{lex}}\right) = \left[(C^{\mathrm{lex}})^{op}, \mathcal{S}\right]$, where the lex completion is taken in the ∞-categorical sense. It satisfies the expected property that an ∞-logos morphism $\mathcal{S}[C] \to \mathcal{E}$ is equivalent to a diagram $C \to \mathcal{E}$. The dual *affine ∞-topos* is denoted $\mathbf{A}^C_\infty$.

(vi) (The ∞-topos of ∞-groupoids) In particular, the free ∞-logos on one generator is $\mathcal{S}[X] = [\mathcal{S}_{\mathrm{fin}}, \mathcal{S}]$, where $\mathcal{S}_{\mathrm{fin}}$ is the ∞-category of finite ∞-groupoids (homotopy types of finite cell complexes). The object X corresponds to the canonical inclusion $\mathcal{S}_{\mathrm{fin}} \to \mathcal{S}$. The corresponding ∞-topos shall be denoted simply by $\mathbf{A}_\infty$. Its ∞-category of points is $\mathcal{P}\mathrm{t}(\mathbf{A}_\infty) = \mathcal{S}$. The universal property of $\mathcal{S}[X]$ translates geometrically into the result that

$$\mathcal{S}\mathrm{h}_\infty(\mathbf{X}) = \mathrm{Hom}_{\mathsf{Topos}_\infty}(\mathbf{X}, \mathbf{A}_\infty).$$

(vii) (∞-Étale morphisms) If $\mathcal{E}$ is an ∞-logos, then so is the slice $\mathcal{E}_{/E}$ for any object E of $\mathcal{E}$. Moreover, the base change along $E \to 1$ in $\mathcal{E}$ provides an ∞-logos morphism $\epsilon_E^* : \mathcal{E} \to \mathcal{E}_{/E}$ called an *∞-étale extension*. Let $\mathbf{X}$ and $\mathbf{X}_E$ be the ∞-topoi dual to $\mathcal{E}$ and $\mathcal{E}_{/E}$. Observe that the diagonal map $\delta_E : E \to E \times E$ is defining a global section of the object $\epsilon_E^*(E) := (E \times E, p_1)$. The pair $(\epsilon_E^*(E), \delta_E)$ is universal in the sense that: for any morphism of ∞-logoi $u^* : \mathcal{E} \to \mathcal{F}$ and any global section $s : 1 \to u^* E$ there exists a morphism of ∞-logoi $v^* : \mathcal{E}_{/E} \to \mathcal{F}$ such that $v^* \circ \epsilon_E^* = u^*$ and $u^*(\delta_E) = s$; moreover, the morphism u^* is essentially unique:

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\epsilon_E^*"] \arrow[dr, "u^*"'] & \mathcal{E}_{/E} \arrow[d, dashed, "v^*"] \\ & \mathcal{F} \end{tikzcd}$$

In other words, the ∞-logos $\mathcal{E}_{/E}$ is obtained from $\mathcal{E}$ by adding freely a global section δ_E to the object E.

The corresponding morphism $\mathbf{X}_E \to \mathbf{X}$ is called *∞-étale* and $\mathbf{X}_E$ is called and *∞-étale domain* of $\mathbf{X}$. Intuitively, in the same way that an étale morphism of topoi has discrete fibers, an ∞-étale morphism of ∞-topoi has quasi-discrete fibers.

(viii) (Pointed objects) Let $\mathcal{S}^\bullet_{\mathrm{fin}}$ be the ∞-category of pointed finite ∞-groupoids (pointed finite cell-complexes). The Alexandrov ∞-topos $\mathbf{A}^\bullet_\infty$ is defined to be the dual of $\mathcal{S}[X^\bullet] := [\mathcal{S}^\bullet_{\mathrm{fin}}, \mathcal{S}]$. It has the classifying property that an ∞-logos morphism $\mathcal{S}[X^\bullet] \to \mathcal{E}$ is equivalent to a pointed object of $\mathcal{E}$, that is, an object E together with a global section $1 \to E$. There exists an equivalence $\mathcal{S}[X^\bullet] = \mathcal{S}[X]_{/X}$ that gives an étale morphism $\mathbf{A}^\bullet_\infty \to \mathbf{A}_\infty$. This map is the universal ∞-étale morphism: for any ∞-topos $\mathbf{X}$ and any object E in $\mathcal{S}\mathrm{h}_\infty(\mathbf{X})$, there exists a unique cartesian square

$$\begin{tikzcd} \mathbf{X}_E \arrow[r] \arrow[d, "\epsilon_E"'] \arrow[dr, phantom, "\ulcorner"] & \mathbf{A}^\bullet_\infty \arrow[d] \\ \mathbf{X} \arrow[r, "\chi_E"] & \mathbf{A}_\infty . \end{tikzcd}$$

The argument is the same as in Section 3.2.6.

[75] Two Morita equivalent ∞-categories define the same Alexandrov ∞-topos.

(ix) (Quotient) Let R be a set of maps in an ∞-logos $\mathcal{E}$. The quotient $\mathcal{E}/\!\!/R$ is defined to be the left-exact localization of $\mathcal{E}$ generated by R. It is equivalent to the sub-∞-category $\mathcal{E}^R$ of $\mathcal{E}$ spanned by objects E satisfying the following condition. Recall that for a map $f : A \to B$, the iterated diagonals of f are defined by $\Delta^0 f := f$ and $\Delta^n f := \Delta(\Delta^{n-1} f)$. Let $C \to D$ be a base change of some $\Delta^n f$ for f in R; then E must satisfy that $\mathrm{Hom}(D, E) \to \mathrm{Hom}(C, E)$ is an invertible map in $\mathcal{S}$.

(x) (Truncated objects) For $-2 \le n \le \infty$, a morphism $f : A \to B$ of $\mathcal{E}$ is said to be *n-truncated* if $\Delta^{n+2} f$ is invertible. A (-1)-truncated morphism is the same thing as a monomorphism. An object E is called n-truncated if the map $E \to 1$ is. In this case, we simply put $\Delta^n E = \Delta^n(E \to 1)$. In the ∞-logos $\mathcal{S}$, the n-truncated objects are the n-groupoids. Intuitively, the n-truncated objects in $\mathcal{E}$ are sheaves with values in n-groupoids. In particular, 0-truncated objects are sheaves with discrete fibers, and (-1)-truncated objects are sheaves with fibers an empty set or a singleton. Given an ∞-logos $\mathcal{E}$, we denote by $\mathcal{E}^{\le n}$ the full sub-∞-category spanned by n-truncated objects. A morphism of ∞-logoi $\mathcal{E} \to \mathcal{F}$ induces a functor $\mathcal{E}^{\le n} \to \mathcal{F}^{\le n}$.

The ∞-logos $\mathcal{S}[X^{\le n}] := \mathcal{S}[X]/\!\!/\big(\Delta^{n+2} X\big)$ is the classifier for n-truncated objects. This means that $\mathrm{Hom}_{\mathsf{Logos}_\infty}(\mathcal{S}[X^{\le n}], \mathcal{E}) = \mathcal{E}^{\le n}$. In particular, the ∞-category of points of $\mathcal{S}[X^{\le n}]$ is the ∞-category $\mathcal{S}^{\le n}$ of n-groupoids. Since any n-truncated object is also $(n+1)$-truncated, we have a tower of quotients of ∞-logoi:

$$\mathcal{S}\big[X^{\le -1}\big] \longleftarrow \mathcal{S}\big[X^{\le 0}\big] \longleftarrow \mathcal{S}\big[X^{\le 1}\big] \longleftarrow \dots \longleftarrow \mathcal{S}[X].$$

We denote by $\mathbf{A}^{\le n}_\infty$ the ∞-topos dual to $\mathcal{S}[X^{\le n}]$. It is a sub-∞-topos of $\mathbf{A}_\infty$. We have $\mathcal{S}\big[X^{\le 0}\big] = \mathcal{S}\mathrm{h}_\infty(\mathbf{A})$ and $\mathcal{S}\big[X^{\le -1}\big] = \mathcal{S}\mathrm{h}_\infty(\mathbf{S})$, hence $\mathbf{A}^{\le 0}_\infty$ and $\mathbf{A}^{\le -1}_\infty$ are respectively the ∞-topos corresponding to the topos of sets and the Sierpiński space though the embeddings $\mathsf{Locale} \hookrightarrow \mathsf{Topos} \hookrightarrow \mathsf{Topos}_\infty$. Altogether, we have an increasing sequence of sub-∞-topoi:

$$\mathbf{S} = \mathbf{A}^{\le -1}_\infty \hookrightarrow \mathbf{A} = \mathbf{A}^{\le 0}_\infty \hookrightarrow \mathbf{A}^{\le 1}_\infty \hookrightarrow \dots \hookrightarrow \mathbf{A}_\infty.$$

4.1.3 Extension and restriction of scalars For $\mathbf{X}$ an ∞-topos, the ∞-category $\mathcal{O}(\mathbf{X}) := \mathcal{S}\mathrm{h}_\infty(\mathbf{X})^{\le -1}$ of (-1)-truncated objects is a frame, called the frame of *open domains* of $\mathbf{X}$. The corresponding locale is denoted $\tau_{-1}(\mathbf{X})$ and called the *socle* of $\mathbf{X}$. The ∞-category $\mathcal{S}\mathrm{h}_\infty(\mathbf{X})^{\le 0}$ of 0-truncated objects is a logos called the *discrete truncation* of $\mathcal{S}\mathrm{h}_\infty(\mathbf{X})$. The corresponding topos is denoted $\tau_0(\mathbf{X})$. The socle of $\tau_0\mathbf{X}$ in the sense of ordinary topoi is the socle of $\mathbf{X}$ in the sense of ∞-topoi.[76] These constructions build left adjoints to the inclusion functors:

$$\begin{tikzcd}
\mathsf{Locale} \arrow[r, shift right, hook] & \mathsf{Topos} \arrow[l, shift right, "\text{Socle}"'] \arrow[r, shift right, hook] & \mathsf{Topos}_\infty . \arrow[l, shift right, "\text{Disc. trunc.}"'] \arrow[ll, bend right, "\text{Socle}"']
\end{tikzcd}$$

At this point, it is perhaps useful to make an analogy with commutative algebra. The embedding $\underline{2} \simeq \big\{\varnothing, \{\star\}\big\} \hookrightarrow \mathsf{Set}$ compares somehow with the inclusion $\{0,1\} \subset \mathbb{Z}$. Schemes over $\mathbb{Z}$ are defined as zeros of polynomial with coefficients in $\mathbb{Z}$. Among them are those that can be defined as zeros of polynomials with coefficients in $\{0,1\}$ (e.g., toric varieties). There are more of the former than the latter. The relation between locales and topoi can be thought the same way: there are more topoi than locales because the latter are allowed to be defined only by equations involving a restricted class of functions. And there are more ∞-topoi than topoi for the same reason. Table 22 details a bit this analogy.

Moreover, the above truncation functors $\mathsf{Topos}_\infty \to \mathsf{Topos} \to \mathsf{Locale}$ can be formalized as actual base change along the coefficient morphisms $\mathcal{S} \xrightarrow{\pi_0} \mathsf{Set} \xrightarrow{\pi_{-1}} \underline{2}$. Presentable ∞-categories have a tensor product, denoted $\otimes_{\mathcal{S}}$, defined similarly to the one of presentable categories (which we rename $\otimes_{\mathsf{Set}}$ here). We shall not expand on it here. We shall only give the computation formula $\mathcal{A} \otimes_{\mathcal{S}} \mathcal{B} = [\mathcal{A}^{op}, \mathcal{B}]^{\mathrm{c}}$ where $[-,-]^{\mathrm{c}}$ refer to the

[76] There exists a notion of n-logos corresponding to the categories $\mathcal{S}\mathrm{h}(\mathbf{X})^{\le n}$ but, once in the paradigm of ∞-categories, the notion of ∞-logos/topos encompasses all the others, and it is also the one with the most regular features. For these reasons we shall not say much about n-logoi/topoi (see [23]).

Table 22: Coefficient analogies

Degree	*Commutative algebra*		*Logos theory*	
	coefficient k	*k-algebra*	*coefficient $\mathcal{K}$*	*$\mathcal{K}$-logos*
-1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$-algebra	$\{0 \to 1\} = \mathcal{S}^{\leq -1}$ (-1-groupoids)	frame = 0-logos
0	$\mathbb{Z}$	$\mathbb{Z}$-algebra	$\mathsf{Set} = \mathcal{S}^{\leq 0}$ (0-groupoids)	logos = 1-logos
1	$\mathbb{Z}[\epsilon] = \mathbb{Z}[x]/\left(x^2\right)$	$\mathbb{Z}[\epsilon]$-algebra	$\mathcal{S}^{\leq 1}$ (1-groupoids)	2-logos
n	$\mathbb{Z}[x]/\left(x^{n+1}\right)$	$\mathbb{Z}[x]/\left(x^{n+1}\right)$-algebra	$\mathcal{S}^{\leq n}$ (n-groupoids)	$(n+1)$-logos
∞	$\mathbb{Z}[\![s]\!]$	$\mathbb{Z}[\![s]\!]$-algebra	$\mathcal{S}$ (∞-groupoids)	∞-logos

∞-category of functors preserving limits. All structural relations of Table 15 make sense also for presentable ∞-categories, provided Set is replaced by $\mathcal{S}$. Using this tensor product, the truncation functor can be written as base change formula

$$\mathcal{S}\mathrm{h}_\infty(\mathbf{X})^{\leq 0} = \mathcal{S}\mathrm{h}_\infty(\mathbf{X}) \otimes_{\mathcal{S}} \mathsf{Set}$$

and

$$\mathcal{S}\mathrm{h}_\infty(\mathbf{X})^{\leq -1} = \mathcal{S}\mathrm{h}_\infty(\mathbf{X})^{\leq 0} \otimes_{\mathsf{Set}} \underline{2} = \mathcal{S}\mathrm{h}_\infty(\mathbf{X}) \otimes_{\mathcal{S}} \underline{2}.$$

4.2 New features

4.2.1 Simplification of descent properties Although the use of ∞-groupoids instead of sets might look like a sophistication, it happens that the characterization of ∞-logoi by their descent properties is actually simpler than the one of logoi. Recall from Section 3.3.4 and Table 17 that not every colimit had the descent property in a logoi and that we had to restrict this condition to characterize logoi. It is a remarkable fact that *all* colimits have the descent property in an ∞-logoi. This leads to a very compact characterization first proposed by Rezk [32]: a presentable ∞-category $\mathcal{E}$ is an ∞-logoi if and only if, for any diagram $X : I \to \mathcal{E}$, we have

$$\mathcal{E}_{/\operatorname{colim}_i X_i} \simeq \lim_i \mathcal{E}_{/X_i}. \qquad \text{(Descent)}$$

In the case of $\mathcal{E} = \mathcal{S}$, this property is equivalent to the Galoisian interpretation of homotopy theory, $\mathcal{S}^K = \mathcal{S}_{/K}$, mentioned in the examples.[77] Definitions à la Giraud or Lawvere can also be given, but we shall not detail them here (see [23, 36, 38]).

This property is equivalent to another one that we will need below. Let $\mathcal{E}^{(\mathrm{core})}_{/E}$ be the *core* of $\mathcal{E}_{/E}$, that is, the sub-∞-groupoid containing all objects and only invertible maps. The core functor $(-)^{(\mathrm{core})} : \mathsf{Cat}_\infty \to \mathcal{S}$ is right adjoint to the inclusion $\mathcal{S} \to \mathsf{Cat}_\infty$. In particular, it preserves limits, and we get from the descent property of the ∞-logos $\mathcal{E}$ that

$$\mathcal{E}^{(\mathrm{core})}_{/\operatorname{colim}_i X_i} \simeq \lim_i \mathcal{E}^{(\mathrm{core})}_{/X_i}. \qquad \text{(Core descent)}$$

Under the assumption that $\mathcal{E}$ has universal colimits, the core descent property, written in terms of ∞-groupoids, turns out to be equivalent to the previous one in terms of ∞-categories.

[77] Essentially, if $\mathcal{S}^K = \mathcal{S}_{/K}$, we deduce $\mathcal{E}_{/\operatorname{colim}_i X_i} = \mathcal{E}^{\operatorname{colim}_i X_i} = \lim_i \mathcal{E}^{X_i} = \lim_i \mathcal{E}_{/X_i}$. Reciprocally, we use $K = \operatorname{colim}_K 1$ to get $\mathcal{E}_{/K} = \mathcal{E}_{/\operatorname{colim}_K 1} = \lim_K \mathcal{E} = \mathcal{E}^K$.

4.2.2 The universe One of the reasons to deal with ∞-groupoids instead of sets is the failure of sets to classify themselves. Letting aside size issues for now, the problem is that sets do not so much form a set as a category, or a groupoid, if we are only interested in classifying them up to isomorphism only. Only ∞-groupoids have a self-classification property: there exists naturally an ∞-groupoid of ∞-groupoids.[78]

The only sets that are classified by an actual set are those without symmetries, that is, the empty set and singletons. This singles out the set $\{\varnothing, 1\}$ as a classifier for these "rigid" sets. In a logos $\mathcal{E}$, thought as a category of generalized sets, the role of $\{\varnothing, 1\}$ is played by the subobject classifier Ω. A map $E \to \Omega$ is intuitively the same thing as a family of empty or singleton sets parameterized by E, that is, a subobject $F \rightarrowtail E$.

To classify more general families, that is, general maps $f : F \to E$, by some characteristic map $\chi_f : E \to U$, the codomain U needs to be able to classify sets of all sizes, that is, sets with symmetries. The symmetries are a well-known obstruction to construct any kind of classifying (or moduli) space with the property that χ_f is *uniquely* determined by f. The solution was found with the idea that the classifying object U need not only classify sets up to symmetries, but sets *and* their symmetries. That is, U needs to have a groupoid of points and not only a set. This is the beginning of stack theory [1, 27].

The formalism of presheaves is actually of great help to formalize classification problems. Let a family of objects of a logos $\mathcal{E}$ parameterized by an object E be a map $F \to E$ in $\mathcal{E}$, that is, an object of $\mathcal{E}_{/E}$ (intuitively, the family is that of the fibers of this map). A morphism of families is a morphism $F \to F'$ compatible with the projections to E, that is, a morphism in $\mathcal{E}_{/E}$. Since we are only interested in classifying objects of $\mathcal{E}$ up to isomorphisms, we are going to consider only the subgroupoid $\mathcal{E}_{/E}^{(\text{core})} \hookrightarrow \mathcal{E}_{/E}$ containing all objects but only isomorphisms. If $E' \to E$ is a map, any family on E can be pulled back on E'. This builds the *functor of families*, called also the *universe* of the logos $\mathcal{E}$:

$$\begin{aligned} \mathbb{U} : \mathcal{E}^{op} &\longrightarrow \mathsf{Gpd} \\ E &\longmapsto \mathbb{U}(E) \coloneqq \mathcal{E}_{/E}^{(\text{core})} \\ f : E' \to E &\longmapsto f^* : \mathcal{E}_{/E}^{(\text{core})} \longrightarrow \mathcal{E}_{/E'}^{(\text{core})}. \end{aligned} \tag{Core universe}$$

There exists a Yoneda embedding $\mathcal{E} \hookrightarrow [\mathcal{E}^{op}, \mathsf{Gpd}]$ sending an object E to the functor $\widehat{E} \coloneqq \mathrm{Hom}(-, E)$ with values in $\mathsf{Set} \hookrightarrow \mathsf{Gpd}$, in particular, the groupoid of natural transformations $\mathrm{Hom}\big(\widehat{E}, \mathbb{U}\big)$ is $\mathbb{U}(E) = \mathcal{E}_{/E}^{(\text{core})}$. This equivalence implies that, in the category of presheaves of groupoids, the object $\mathbb{U}$ has the property that a map $F \to E$ in $\mathcal{E}$ corresponds uniquely to a map $\widehat{E} \to \mathbb{U}$. In other words, the presheaf $\mathbb{U}$ is the formal solution to the classification of families of objects of $\mathcal{E}$.

Now, the classification problem can be formulated properly as the problem of finding an object U in $\mathcal{E}$ such that $\widehat{U} \simeq \mathbb{U}$. There are two obstructions to this:

1. $\mathrm{Hom}(-, U)$ takes values in sets and not groupoids;
2. (size issue) the values of $\mathrm{Hom}(-, U)$ are small, but those of $\mathbb{U}$ are large.

In logos theory, the first obstruction is handled by restricting the functor $\mathbb{U}$. If we limit ourselves to families $F \to E$ which are monomorphisms, then the groupoid of such $F \rightarrowtail E$ is actually a set. This defines a subfunctor $\mathbb{U}^{\leq -1} \hookrightarrow \mathbb{U}$ with values in sets and can be represented by an object of $\mathcal{E}$. This is actually the universal property of subobject classifier: $\mathbb{U}^{\leq -1} = \mathrm{Hom}(-, \Omega)$. But the first obstruction is better dealt with by enhancing sets into ∞-groupoids and logoi into ∞-logoi. When $\mathcal{E}$ in an ∞-logos, both the functor of points $\mathrm{Hom}(-, U)$ of an object U and the core universe $\mathbb{U}$ take values in the ∞-category $\mathcal{S}$ of (large) ∞-groupoids. Moreover, since $\mathcal{E}$ is assumed a presentable ∞-category, a functor $\mathcal{E}^{op} \to \mathcal{S}$ is representable if and only if it sends colimits in $\mathcal{E}$ to limits in $\mathcal{S}$. But this is exactly the descent property of (Core descent) characterizing ∞-logoi. So the object U would exist if it were not for the second obstruction.

This second obstruction is dealt with by considering only partial universes, that is, universes that classified uniquely *some* families. We shall say that an object U of an ∞-logos $\mathcal{E}$ is a *partial universe* if it is equipped with a monomorphism $\widehat{U} \rightarrowtail \mathbb{U}$. This means that, for an object E in $\mathcal{E}$, the ∞-groupoid $\mathrm{Hom}(E, U)$ is a full

[78] Notice that, because n-groupoids form an $(n+1)$-groupoid, we need to go to infinity to have this property.

sub-∞-groupoid of $\mathcal{E}_{/E}^{(\mathrm{core})}$. For example, the subobject classifier Ω classifies only families $F \to E$ that are monomorphisms. Now, a fundamental property of ∞-logoi is that, even though the universe is too big to be an actual object of $\mathcal{E}$, there exists always partial universes. In other words, given any map $f : F \to E$, there exists always a partial universe U such that f is classified by a unique map $\chi_f : E \to U$.[79] Moreover, there are always enough partial universes in the sense that $\mathbb{U}$ is the union of all the partial universes of $\mathcal{E}$. This last property has the practical effect that, for the most part, one can manipulate the universe as if it were an actual object of the ∞-logos.

4.2.3 ∞-Topoi from homology theories Eilenberg–Steenrod axioms for homology theories have a modern formulation in terms of ∞-category theory. Let $\mathcal{S}_{\mathrm{fin}}^{\bullet}$ be the category of pointed finite ∞-groupoids. A functor $H : \mathcal{S}_{\mathrm{fin}}^{\bullet} \to \mathcal{S}$ is a *homology theory* if it satisfies the *excision property*, that is, if it sends pushout squares to pullback squares:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd} \longmapsto \begin{tikzcd} H(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & H(B) \arrow[d] \\ H(C) \arrow[r] & H(D) \end{tikzcd} \qquad \text{(Excision)}$$

A homology theory H is called *reduced* if, moreover, $H(1) = 1$.[80]

Homology theories define a full sub-∞-category $[\mathcal{S}_{\mathrm{fin}}^{\bullet}, \mathcal{S}]^{(1)}$ of $\mathcal{S}[X^\bullet] = [\mathcal{S}_{\mathrm{fin}}^{\bullet}, \mathcal{S}]$. The sub-∞-category of reduced homology theories can be proved to be equivalent to the ∞-category $\mathcal{S}\mathrm{p}$ of *spectra* (in the sense of algebraic topology) and the ∞-category $[\mathcal{S}_{\mathrm{fin}}^{\bullet}, \mathcal{S}]^{(1)} = \mathcal{P}\mathcal{S}\mathrm{p}$ the ∞-category $\mathcal{P}\mathcal{S}\mathrm{p}$ of *parameterized spectra*.[81] Moreover, Goodwillie's calculus of functors proves that $[\mathcal{S}_{\mathrm{fin}}^{\bullet}, \mathcal{S}]^{(1)}$ is in fact a left-exact localization of $\mathcal{S}[X^\bullet]$ (see [2]). Let $\mathbf{A}_\infty^{(1)}$ be the dual ∞-topos; we have an embedding $\mathbf{A}_\infty^{(1)} \hookrightarrow \mathbf{A}_\infty^{\bullet}$.

It is possible to give a presentation of the ∞-logos $[\mathcal{S}_{\mathrm{fin}}^{\bullet}, \mathcal{S}]^{(1)} = \mathcal{P}\mathcal{S}\mathrm{p}$. Let us say that a pointed object $1 \to E$ in a logos is *additive* if sums and products of this object coincide, that is, if the canonical map $E \vee E \to E \times E$ is invertible. An additive pointed object is called *stably additive* if the addivity property extends to all its loop objects, that is, if, for all m, n, $\Omega^m X^\bullet \vee \Omega^n X^\bullet \simeq \Omega^m X^\bullet \times \Omega^n X^\bullet$. The logos classifying stably additive objects is $\mathcal{S}[X^{(1)}] := \mathcal{S}[X^\bullet] /\!\!/ (\Omega^m X^\bullet \vee \Omega^n X^\bullet \to \Omega^m X^\bullet \times \Omega^n X^\bullet,\ m, n \in \mathbb{N})$. In [3], we prove that $[\mathcal{S}_{\mathrm{fin}}^{\bullet}, \mathcal{S}]^{(1)} = \mathcal{S}[X^{(1)}]$. Under the equivalence $[\mathcal{S}_{\mathrm{fin}}^{\bullet}, \mathcal{S}]^{(1)} = \mathcal{P}\mathcal{S}\mathrm{p}$, the universal stably additive object $X^{(1)}$ corresponds to the sphere spectrum $\mathbb{S}$ in $\mathcal{P}\mathcal{S}\mathrm{p}$.

The fact that $\mathcal{P}\mathcal{S}\mathrm{p}$ is an ∞-logos has been a surprise for everybody in the higher category community. The category $\mathcal{S}\mathrm{p}$ is an example of a *stable* ∞-category.[82] Another example is the ∞-category $C(k)$ of chain complexes over a ring k. It is a result of Hoyois that the parameterized version of $C(k)$ (or of any stable ∞-category $\mathcal{C}$) is an ∞-logos [14].[83] Intuitively, if ∞-topoi are ∞-categories of generalized homotopy types,

[79] Partial universe are equivalent to codomains of Voevodsky's notion of *univalent maps*.

[80] For H a reduced homology theory and $B = C = 1$, the excision condition says $H(\Sigma A) = \Omega H(A)$. Passing to the homotopy groups $H_i(A) := \pi_i(H(A))$, we get the more classical form of the excision $H_i(\Sigma A) = H_{i+1}(A)$.

[81] A *spectrum* is a collection of pointed spaces $(E_n)_{n\in\mathbb{N}}$ and of homotopy equivalences $E_n = \Omega E_{n+1}$. Let S^n be the sphere of dimension n viewed as an object of $\mathcal{S}$. A reduced homology theory defines such a sequence by $E_n = H(S^n)$.

A *parameterized spectrum* is the data of an object B of $\mathcal{S}$ (the space of parameters), of a collection of pointed objects $(E_n)_{n\in\mathbb{N}}$ in $\mathcal{S}_{/B}$ and of homotopy equivalences $E_n = \Omega_B E_{n+1}$. Equivalently, spectra parameterized by B can be defined as diagrams $B \to \mathcal{S}\mathrm{p}$. Intuitively, they can be thought as locally constant families of spectra parameterized by B (local systems of spectra). A homology theory defines such a data by putting $B = H(1)$ and $E_n = H(S^n)$.

[82] A presentable ∞-category is called *stable* if its colimits commute with finite limits. In particular, it is an additive category: initial and terminal objects coincide, and so do finite sums and products. Stable categories are the proper higher notion replacing abelian categories. Another example is the ∞-category $C(k)$ obtained by localizing the 1-category of chain complexes over a ring k by quasi-isomorphism.

[83] Parameterized chain complexes are the same thing as local systems of chain complexes.

If $\mathcal{C}$ is an ∞-category, the ∞-category $\mathcal{P}\mathcal{C}$ of parameterized objects of $\mathcal{C}$ is defined in the following way. Its objects are diagrams $x : K \to \mathcal{C}$ where K is an ∞-groupoid. The 1-morphisms $x' \to x$ are pairs (u, v) where $u : K' \to K$ is a map of ∞-groupoids and $v : x' \to x \circ u$ is a natural transformation of diagrams $K' \to \mathcal{C}$. Higher morphisms are defined in the obvious way. There is a canonical embedding $\mathcal{C} \to \mathcal{P}\mathcal{C}$ induced by the choice $K = 1$.

stable ∞-categories are ∞-categories of generalized homology theories (a.k.a generalized stable homotopy types). The two worlds are used to be thought as quite different (stable homotopy types behave very differently than their unstable counter part), but the result of Hoyois shows that they are closer than expected.

4.2.4 ∞-Connected objects The ∞-connected objects are arguably the most important new feature of ∞-topoi. They provide an unexpected bridge between stable and unstable homotopy theories. They are also responsible for the failure of the notion of site to present ∞-logoi by generators and relations.

In the same way that sheaves are continuous families of sets, sheaves of ∞-groupoids are continuous families of ∞-groupoids (their stalks). Therefore, ∞-logoi can be understood as generalized categories of ∞-groupoids, that is, generalized homotopy theories. The operations of manipulation of these generalized homotopy types are the same as for homotopy types, but their behavior is different. The most important difference is arguably the failure of Whitehead theorem to ensure that a homotopy type with trivial homotopy groups is the point. To explain this, we need some definitions.

Given a map $f : A \to B$, the *nerve of* f is the simplicial diagram

$$\dots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; A \times_B A \times_B A \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; A \times_B A \;\substack{\longrightarrow\\\longrightarrow}\; A. \tag{Nerve}$$

The image of f, denoted $\mathrm{Im}\,(f)$, is the colimit of this diagram.[84] The map f is called a *cover*, or a (-1)-*connected* maps, if its image is B. Intuitively, a map is a cover if its fibers are not empty. Recall that a map $f : A \to B$ is a monomorphism if $\Delta f : A \to A \times_B A$ is an invertible map (in $\mathcal{S}$, this corresponds to a fully faithful functor between ∞-groupoids). The construction of the image produces a factorization of any map $f : A \to B$ into a cover followed by a monomorphism $A \to \mathrm{Im}\,(f) \to C$.

More generally, f is called a *n-connected* if all its iterated diagonals $\Delta^k f$ are all covers for $0 \le k \le n+1$. An object E is called a *n-connected* if the map $E \to 1$ is. An object E of $\mathcal{S}$ is n-connected if and only if $\pi_k(E) = 0$ for all $k \le n$. Intuitively, an object in an ∞-logos is n-connected if it is a sheaf with n-connected stalks, and a map between sheaves is n-connected if its fibers are. The definition make sense for $n = \infty$. In $\mathcal{S}$, an ∞-connected object corresponds to an ∞-groupoid with trivial homotopy groups. By Whitehead theorem, only the point satisfies this. However, there exists ∞-logoi with non trivial ∞-connected objects.

Examples of ∞-connected objects

(i) Recall the ∞-logos $\mathcal{PS}\mathrm{p}$ of parameterized spectra and the canonical inclusion $\mathcal{S}\mathrm{p} \hookrightarrow \mathcal{PS}\mathrm{p}$ of reduced homology theories into homology theories. There exists a canonical functor $\mathrm{red} : \mathcal{PS}\mathrm{p} \to \mathcal{S}$, called the *reduction*, sending a parameterized spectrum $B \to \mathcal{S}\mathrm{p}$ to its indexing ∞-groupoid B. This functor is a logos morphism that happens to be the only point of the topos $\mathbf{A}^{(1)}_\infty$. It is possible to prove that an object of $\mathcal{PS}\mathrm{p}$ is ∞-connected if and only if it is in the image of $\mathcal{S}\mathrm{p} \to \mathcal{PS}\mathrm{p}$, that is, a reduced homology theory. More generally, a morphism in $\mathcal{PS}\mathrm{p}$ is ∞-connected if and only if its image under the reduction $\mathrm{red} : \mathcal{PS}\mathrm{p} \to \mathcal{S}$ is an invertible map in $\mathcal{S}$. This proves that there are plenty of ∞-connected morphisms in $\mathcal{PS}\mathrm{p}$.

    It is possible to think the situation intuitively in the following way. The objects of $\mathcal{PS}\mathrm{p}$ are sorts of infinitesimal thickenings of the objects of $\mathcal{S}$. In particular, spectra are infinitesimal thickenings of the point. From this point of view, the morphism $\mathrm{red} : \mathcal{PS}\mathrm{p} \to \mathcal{S}$ is indeed a reduction, forgetting the infinitesimal thickening.[85]

[84] In a 1-category, the beginning of this diagram $A \times_B A \rightrightarrows A$ is sufficient to define covers. It is the graph of the equivalence relation on A "having the same image by f". But in higher categories, in $\mathcal{S}$, for example, "having the same image by f" is no longer a relation but a *structure* on the pairs (a, a') in A: that of the choice of a homotopy $\alpha : f(a) \simeq f(a')$ in B. This is why the higher part of the simplicial diagram is needed. The nerve of f defines a groupoidal relation in $\mathcal{S}$ that encodes the coherent compositions of the homotopies α.

[85] There again, the situation compares to algebraic geometry. Recall that in algebraic geometry, the connected components of a scheme depend only on its reduction. In particular, the spectrum of a local Artinian ring is connected. Similarly, the homotopy invariants of an object E of $\mathcal{PS}\mathrm{p}$ are those if its reduction $B = \mathrm{red}(E)$ in $\mathcal{S}$.

(ii) Another source of examples of ∞-connected objects is the *hypercovers* in the ∞-logos $\mathcal{S}\mathrm{h}_\infty(X)$ associated to a space X, but we shall not detail this here (see [23, 6.5.3]). Because of this example, an ∞-logos such that the only ∞-connected maps are the invertible maps is called *hypercomplete*. This is the case of $\mathcal{S}$ and any diagram category $[C, \mathcal{S}]$. In particular, free ∞-logoi are hypercomplete. The ∞-logos $\mathcal{S}\mathrm{h}_\infty(X)$ of a space of "finite dimension" (like a manifold) is hypercomplete (see [23, 6.5.4]).

An ∞-topos $\mathbf{X}$ is said to have enough points, if a map $A \to B$ in $\mathcal{S}\mathrm{h}_\infty(\mathbf{X})$ is invertible if and only if, for any point x of $\mathbf{X}$, the map $A(x) \to B(x)$ between the stalks is invertible in $\mathcal{S}$. Intuitively, this means that a sheaf is faithfully represented by the diagram of its stalks. If $\mathcal{S}\mathrm{h}_\infty(\mathbf{X})$ has some hyperconnected maps, then it cannot have enough points. This creates the bizarre situation that a topological space X such that $\mathcal{S}\mathrm{h}_\infty(X)$ is non-hypercomplete does not have enough points![86]

(iii) In homotopy theory, the construction of the free group on a pointed homotopy type X is given by $\Omega\Sigma X$, where Σ is the suspension functor. There exists a canonical map $X \to \Omega\Sigma X$ (the inclusion of generator). In $\mathcal{S}$, this maps is invertible if and only if X is the point. But there exists examples of topoi where $X = \Omega\Sigma X$ for some non trivial object. This is the case in $\mathcal{PS}\mathrm{p}$. The embedding $\mathcal{S}\mathrm{p} \hookrightarrow \mathcal{PS}\mathrm{p}$ preserves pushout and fiber product, hence if E is a spectrum, the object $\Omega\Sigma E$ is the same computed in $\mathcal{S}\mathrm{p}$ or in $\mathcal{PS}\mathrm{p}$. But in the first case, we have trivially $E = \Omega\Sigma E$. In other terms, any spectrum viewed in $\mathcal{PS}\mathrm{p}$ provides a pointed object that is its own free group.

The logos classifying these *self-free-groups* is $\mathsf{Set}[X^\bullet]/\!\!/(X^\bullet \simeq \Omega\Sigma X^\bullet)$. Any self-free-group is ∞-connected. This explains why there are not more of them in $\mathcal{S}$.

4.2.5 Insufficiency of topologies We saw that any quotient of a logos $\mathcal{E}$ could be generated by a set of monomorphisms. This property fails drastically for ∞-logoi since there exists quotients of logoi inverting no monomorphims at all. An example is given by the reduction morphism $\mathrm{red} : \mathcal{PS}\mathrm{p} \to \mathcal{S}$. It is a localization because its right adjoint is the canonical ∞-logos morphism $\mathcal{S} \to \mathcal{PS}\mathrm{p}$, which is fully faithful.[87] We saw that a map is inverted by red if and only if it is ∞-connected. So we need to prove that no proper monomorphism can be ∞-connected. This is because an ∞-connected map is in particular a cover and a map that is both a cover and a monomorphism is always invertible.

We now analyze why the trick that worked in logoi no longer works for ∞-logoi. Recall that a map is a monomorphism if and only if its diagonal is invertible. Let $f : A \to B$ be a map in an ∞-logos. We have that "$f : A \to B$ is invertible" if and only if "f is both a cover and a monomorphism" if and only if "f is a cover and Δf is invertible". In the context of logoi, the map Δf is a monomorphism, and the reformulation stops there. But in the context of ∞-logoi, Δf is no longer a monomorphism, so the equivalence of conditions continues into "f is invertible" if and only if "f and Δf are covers and $\Delta^2 f$ is invertible" if and only if "f, Δf, $\Delta^2 f$ are covers and $\Delta^3 f$ is invertible", and so on. At the limit of this process, we get the condition "$\forall n,\ \Delta^n f$ is a cover". But this condition is not equivalent to "f is invertible", it is equivalent to "f is ∞-connected". This explains the failure of being able to write the invertibility of a map f by means of a topological relation. The best one can do with topological relations for an arbitrary map is to force it to become ∞-connected. This is in fact the new meaning of topological relations in the setting of ∞-logoi. The following conditions of generation are equivalent for a quotient of ∞-logoi:

- inverting some monomorphisms;
- forcing some maps to become covers;
- forcing some maps to become ∞-connected.

We shall say that a quotient is *topological* if it satisfies the above conditions, and that a quotient is *cotopological* if it can be presented by inverting a set R of ∞-connected maps. An example of a cotopological relation is $\mathrm{red} : \mathcal{PS}\mathrm{p} \to \mathcal{S}$, where all ∞-connected maps are inverted. Any quotient $\mathcal{E} \to \mathcal{E}/\!\!/R$ of ∞-logoi can be factored

[86] The situation is comparable with a well-known fact in algebraic geometry. Let a be an element of a ring A viewed as a function $Spec(A) \to \mathbb{A}$. The values of this function at a point p is the residue of a in the field $\kappa(p)$. Then, because of nilpotent elements, an element a of a ring A is not completely determined by its set of values. In fact, it seems a good idea to compare the subcategory spanned by ∞-connected objects of an ∞-logoi to the radical of a ring.

[87] This functor sends an object B in $\mathcal{S}$ to the constant diagram $B \to \mathcal{S}\mathrm{p}$ with value the null spectrum.

into a topological quotient followed by a cotopological one: the topological quotient forces the relations to become ∞-connected maps, then the cotopological quotient finishes the job by inverting these ∞-connected maps [23, 6.5.2]. Finally, we see that even thought the notion of site, that is, topological quotients, is insufficient to present all ∞-logoi, it is nonetheless a meaningful notion of the theory.

Examples of topological relations and factorizations

(i) The ∞-logos classifying n-connected objects is defined by

$$\mathcal{S}[X_{>n}] := \mathcal{S}[X] /\!\!/ \big(\forall k \le n+1, \Delta^k X \text{ is a cover}\big).$$

In particular, the ∞-logos classifying ∞-connected objects is

$$\mathcal{S}[X_{>\infty}] := \mathcal{S}[X] /\!\!/ (\forall n, \Delta^n X \text{ is a cover}).$$

A variation is the ∞-logos classifying *pointed* ∞-connected objects defined by

$$\mathcal{S}[X^\bullet_{>\infty}] := \mathcal{S}[X^\bullet] /\!\!/ (\forall n, \Delta^n X^\bullet \text{ is a cover}).$$

All of these are examples of topological quotients of $\mathcal{S}[X]$ or $\mathcal{S}[X^\bullet]$.

(ii) Recall the quotient

$$\mathcal{S}[X^\bullet] \to \mathcal{S}\big[X^{(1)}\big] := \mathsf{Set}[X^\bullet] /\!\!/ (\forall m, n,\ \Omega^m X^\bullet \vee \Omega^n X^\bullet \simeq \Omega^m X^\bullet \times \Omega^n X^\bullet)$$

classifying stably additive objects. Any stably additive object can be proved to be ∞-connected. This gives a factorization $\mathcal{S}[X^\bullet] \to \mathcal{S}[X^\bullet_{>\infty}] \to \mathcal{S}\big[X^{(1)}\big]$ that is the topological/cotopological factorization.

(iii) Recall the logos classifying *self-free groups* is $\mathsf{Set}[X^\bullet] /\!\!/ (X^\bullet \simeq \Omega\Sigma X^\bullet)$. Any self-free group is ∞-connected, and the factorization $\mathcal{S}[X^\bullet] \to \mathcal{S}[X^\bullet_{>\infty}] \to \mathcal{S}[X^\bullet] /\!\!/ (X^\bullet \simeq \Omega\Sigma X^\bullet)$ is the topological/cotopological factorization.

(iv) Recall that $\mathsf{Set}[X^\bullet] = [\mathcal{S}^\bullet_{\text{fin}}, \mathcal{S}]$. In particular, $\mathcal{S}\big[X^{(1)}\big]$ and $\mathsf{Set}[X^\bullet] /\!\!/ (X^\bullet \simeq \Omega\Sigma X^\bullet)$ are examples of ∞-logoi that cannot be presented by a topology on ${\mathcal{S}^\bullet_{\text{fin}}}^{op}$.

4.2.6 New relations with logic In the line of what we said in Section 3.4.2, ∞-logoi provide several important new elements. The almost representability of the universe $\mathbb{U}$ and the existence of enough partial universes authorize semantics for logical theories having a type of types, quantification on objects, or modalities on types. This feature is somehow behind the whole homotopical semantics of Martin–Löf type theory with identity types [13].

The existence of ∞-connected objects also has consequences from the logical point of view. Recall from Section 3.4.2 that topological relations correspond logically to forcing some existential statements. Then logical meaning of the impossibility to present all quotients of ∞-logoi by topological relations is the surprising fact that it is impossible to describe the invertibility of a map by means of geometric formulas. Related to this, the ∞-connected objects are also responsible for the failure of Deligne completion theorem for coherent topoi [25, Appendix A].

The notion of ∞-logoi also leads to the construction of classifying objects for some non trivial theories with only the point as a model in $\mathcal{S}$, namely, theories where the underlying objects are ∞-connected. We saw examples with stably additive objects and self-free group objects. These theories are somehow akin to theories without any models in Set or $\mathcal{S}$.

4.2.7 Homotopy theory of ∞-logoi We have explained in Section 3.2.15 how topos theory provides a nice theory of connectedness with the connected–disconnected factorization. The same definitions make sense in the setting of ∞-topoi, but changing the coefficients from Set to $\mathcal{S}$ has the effect for enhancing the theory of connectedness into a theory of contractibility. A morphism of ∞-topoi $\mathbf{Y} \to \mathbf{X}$ is called *contractible* if the corresponding morphism of ∞-logoi $\mathcal{S}\mathrm{h}_\infty(\mathbf{X}) \to \mathcal{S}\mathrm{h}_\infty(\mathbf{Y})$ is fully faithful. An ∞-topos $\mathbf{X}$ is contractible if the morphism $\mathbf{X} \to \mathbf{1}$ is. The *image* of a morphism of ∞-logoi $u^* : \mathcal{E} \to \mathcal{F}$ is defined as the smallest full sub-∞-category of $\mathcal{S}\mathrm{h}(\mathbf{Y})$ containing the image of $\mathcal{F}$ and stable under finite limits and colimits. The morphism u^* is said to be *dense* if its image is the whole of $\mathcal{F}$. A morphism of ∞-topoi $\mathbf{Y} \to \mathbf{X}$ is *uncontractible* if the corresponding morphism of ∞-logoi $\mathcal{S}\mathrm{h}_\infty(\mathbf{X}) \to \mathcal{S}\mathrm{h}_\infty(\mathbf{Y})$ is dense.[88] Any morphism of ∞-topoi $u : \mathbf{Y} \to \mathbf{X}$ factors as a contractible morphism followed by an uncontractible morphism:

$$\begin{tikzcd}
\mathbf{Y} \arrow[rr, "u"] \arrow[dr, "\text{contractible}"'] & & \mathbf{X} \\
& {|\mathbf{Y}|_{\mathbf{X}}} \arrow[ur, "\text{uncontractible}"'] &
\end{tikzcd}$$

We call the morphism $|\mathbf{Y}|_{\mathbf{X}} \to \mathbf{X}$ the *residue of the contraction* of $\mathbf{Y} \to \mathbf{X}$. This construction is an analogue for the whole homotopy type of the π_1 construction of Dubuc for topoi [9].

A morphism $u : \mathbf{Y} \to \mathbf{X}$ is *locally contractible* when u^* has a local left adjoint. In this case, the residue $|\mathbf{Y}|_{\mathbf{X}} \to \mathbf{X}$ is ∞-étale and associated to an object of $\mathcal{S}\mathrm{h}_\infty(\mathbf{X})$. When $\mathbf{X} = \mathbf{1}$, this object is called the *homotopy type* of the topos $\mathbf{Y}$. This generalizes to the whole homotopy type the situation of connected components of topoi. The set of connected components of a topos does not always exist as a set but always exists as totally disconnected space. Similarly, the whole homotopy type of an ∞-topos does not always exists as an ∞-groupoid, but always exists as an uncontractible ∞-topos.[89]

From locales, to topoi, to ∞-topoi, there is a progression in the kind of homotopy features for which the theory is convenient. Table 23 summarizes the situation.

Table 23: Degrees of homotopy theory

	Locale (0-topos)	*Topos*	*∞-Topos*
Coefficients	$\{0 \le 1\} = \mathcal{S}^{\le -1}$	$\mathsf{Set} = \mathcal{S}^{\le 0}$	$\mathcal{S}$
Algebraic morphism	$\mathcal{O}(X) \xrightarrow{u^*} \mathcal{O}(Y)$	$\mathcal{S}\mathrm{h}(\mathbf{X}) \xrightarrow{u^*} \mathcal{S}\mathrm{h}(\mathbf{Y})$	$\mathcal{S}\mathrm{h}_\infty(\mathbf{X}) \xrightarrow{u^*} \mathcal{S}\mathrm{h}_\infty(\mathbf{Y})$
u^* fully faithful	surjective morphisms	connected morphisms	contractible morphisms
u^* dense	embeddings	disconnected morphisms	uncontractible morphisms
u^* has a local left adjoint	open morphisms	locally connected morphisms	locally contractible morphisms
Convenient for	image theory (π_{-1})	connected components theory (π_0)	full homotopy type

4.2.8 Cohomology theory of ∞-topoi

The theory of ∞-topoi is also well suited for cohomology theory with coefficient in sheaves. The modern formulation of derived functors as functors between ∞-categories has reformulated the definition of sheaf cohomology as the computation of the global sections of sheaves of spectra. The cohomology of an ∞-topos $\mathbf{X}$ is then dependent on the ∞-category of sheaves of spectra $\mathcal{S}\mathrm{h}_\infty(\mathbf{X}, \mathcal{S}\mathrm{p})$. The nice descent properties of ∞-logoi provide a simple description of this category as a tensor

[88] These morphisms are called *algebraic* in [23, 6.3.6].
[89] This point of view goes around the theory of shape of [23, 15].

product of presentable ∞-categories:[90]

$$\mathcal{S}\mathrm{h}_\infty(\mathbf{X},\mathcal{S}\mathrm{p}) := \mathcal{S}\mathrm{h}_\infty(\mathbf{X})\otimes_{\mathcal{S}}\mathcal{S}\mathrm{p} = [\mathcal{S}\mathrm{h}_\infty(\mathbf{X}),\mathcal{S}\mathrm{p}]^{\mathrm{c}}.$$

The cohomology spectrum of $\mathbf{X}$ with values in a sheaf of spectra E is given simply by the global sections

$$\begin{aligned}\Gamma : \mathcal{S}\mathrm{h}(\mathbf{X},\mathcal{S}\mathrm{p}) &\longrightarrow \mathcal{S}\mathrm{p}\\ E &\longmapsto \Gamma(\mathbf{X},E).\end{aligned}$$

Then, the cohomology groups of $\mathbf{X}$ with coefficients in E are defined as the stable homotopy groups of the spectra $H^i(\mathbf{X},A) := \pi_{-i}\big(\Gamma(\mathbf{X},H(A))\big)$.

In terms of the analogy of logos theory with commutative algebra, the formula $\mathcal{S}\mathrm{h}(\mathbf{X},\mathcal{S}\mathrm{p}) = \mathcal{S}\mathrm{h}(\mathbf{X})\otimes\mathcal{S}\mathrm{p}$ says that the stabilisation operation is a change of scalar from $\mathcal{S}$ to $\mathcal{S}\mathrm{p}$ along the canonical stabilisation map $\Sigma^\infty_+ : \mathcal{S}\to\mathcal{S}\mathrm{p}$. The resulting ∞-category is not a logos, though, but a stable ∞-category.

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