On the bar-cobar duality

for algebras and operads

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Introduction

- The goal is to give the bar-cobar duality for algebras and operads a unified treatment
- We are using some aspects of Sweedler's theory of coalgebras extended to cooperads
- Our theory is purely algebraic

Overview

Part I:

- The bar-cobar duality for algebras recalled
- Enrichement of monoids over comonoids
- Twisting cochains
- The Maurer-Cartan algebra
- The bar-cobar duality for algebras revisited

Part II:

- The operadic bar-cobar duality recalled
- Enrichement of operads over cooperads
- Pre-Lie algebras
- Twisting cochains
- The Maurer-Cartan operad
- The operadic bar-cobar duality revisited

The bar-cobar duality for algebras recalled

The classical *bar construction* takes a pointed dg algebra A to a pointed dg coalgebra $\mathrm{B}(A)$, and the *cobar construction* takes a pointed dg coalgebra C to a pointed dg algebra $\Omega(C)$.

This defines two functors

$$\Omega : \mathbf{dgCoalg}_\bullet \to \mathbf{dgAlg}_\bullet \qquad\qquad \mathrm{B} : \mathbf{dgAlg}_\bullet \to \mathbf{dgCoalg}_\bullet$$

But the functor Ω is **not** left adjoint to the functor B.

There is only an adjunction

$$\Omega : \mathbf{nildgCoalg}_\bullet \longleftrightarrow \mathbf{dgAlg}_\bullet : \mathrm{B},$$

where $\mathbf{nildgCoalg}_\bullet$ denotes the category of conilpotent dg coalgebras (Brown, Prouté).

The functor $\Omega^\vee$

We shall see that the functor $\Omega : \mathbf{dgCoalg_\bullet} \to \mathbf{dgAlg_\bullet}$ has a right adjoint $\Omega^\vee$. Moreover,

- The category of conilpotent coalgebras is a full coreflexive subcategory of the category of pointed coalgebras;
- The coreflexion functor takes a pointed dg coalgebra C to its *coradical* $R(C)$;
- We have $\mathrm{B}(A) = R\Omega^\vee(A)$ for any pointed dg algebra A.

It follows that the adjunction

$$\Omega : \mathbf{nildgCoalg_\bullet} \longleftrightarrow \mathbf{dgAlg_\bullet} : \mathrm{B}$$

is obtained by composing the adjunctions

$$inc : \mathbf{nildgCoalg_\bullet} \longleftrightarrow \mathbf{dgCoalg_\bullet} : R$$

$$\Omega : \mathbf{dgCoalg_\bullet} \longleftrightarrow \mathbf{dgAlg_\bullet} : \Omega^\vee.$$

Closed category

Recall that a symmetric monoidal category $\mathcal{K} = (\mathcal{K}, \otimes, I)$ is said to be *closed* if the functor $X \otimes (-)$ has a right adjoint $\mathcal{K}(X, -)$ for every object $X \in \mathcal{V}$.

There is then a canonical isomorphism (the tensor-hom isomorphism),

$$\mathcal{K}(X \otimes Y, Z) \simeq \mathcal{K}(X, \mathcal{K}(Y, Z)).$$

Examples

- The category **Vect** of vector spaces over a field $\mathbb{F}$.
- The category **gVect** of $\mathbb{Z}$-graded $\mathbb{F}$-vector spaces.
- The category **dgVect** of complexes of $\mathbb{F}$-vector spaces.

Tensor and cotensor

Let $\mathcal{E}$ be a category enriched over a symmetric monoidal closed category $\mathcal{K}$.

We have $\mathcal{E}(A, B) \in \mathcal{K}$ for every objects $A, B \in \mathcal{E}$.

Recall that $\mathcal{E}$ is **tensored** by $\mathcal{K}$ if the functor $\mathcal{E}(A, -) : \mathcal{E} \to \mathcal{K}$ has a (strong) left adjoint $X \mapsto X \triangleright A$ for every object $A \in \mathcal{E}$.

Recall that $\mathcal{E}$ is **cotensored** by $\mathcal{K}$ if the contravariant functor $\mathcal{E}(-, A) : \mathcal{E}^{op} \to \mathcal{V}$ has a (strong) right adjoint $X \mapsto [X, A]$ for every object $A \in \mathcal{E}$.

We then have the hom-tensor-cotensor isomorphisms:

$$\boxed{\mathcal{K}(X, \mathcal{E}(A, B)) \simeq \mathcal{E}(X \triangleright A, B) \simeq \mathcal{E}(A, [X, B])}$$

The trinity

The enrichement of a $\mathcal{E}$ over $\mathcal{K}$ can be described **equivalently** by **any** of the following three functors:

- the hom functor

$$\mathcal{E}(-,-) : \mathcal{E}^{op} \times \mathcal{E} \to \mathcal{K}$$

- the tensor product functor

$$(-) \triangleright (-) : \mathcal{K} \times \mathcal{E} \to \mathcal{E}$$

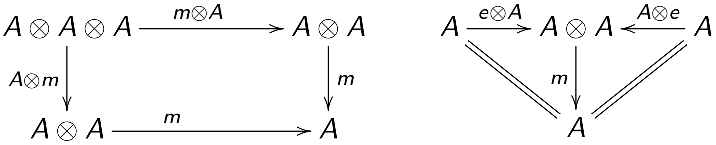
- the cotensor product functor

$$[-,-] : \mathcal{K}^{op} \times \mathcal{E} \to \mathcal{E}$$

We shall use the third functor (the cotensor) to define the enrichement of the category of algebras over the category of coalgebras.

Monoid

Recall that a *monoid* in a monoidal category $\mathcal{V} = (\mathcal{V}, \otimes, I)$ is an object A equipped with a multiplication $m : A \otimes A \to A$ and a unit $e : I \to A$ satisfying the following conditions:



The tensor product of two monoids A and B has the structure of a monoid $A \otimes B$.

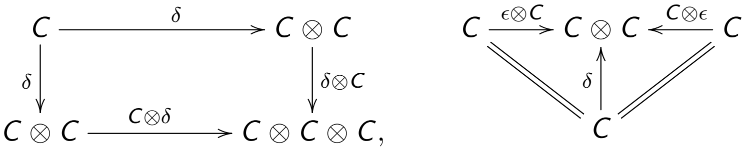
Hence the category $\mathbf{Mon}(\mathcal{V})$ of monoids in $\mathcal{V}$ is symmetric monoidal.

The unit object is the monoid I.

Comonoid

Recall that a *comonoid* in a monoidal category $\mathcal{V} = (\mathcal{V}, \otimes, I)$ is a monoid in the opposite category $\mathcal{V}^{op}$.

It is an object $C \in \mathcal{V}$ equipped with a comultiplication $\delta : C \to C \otimes C$ and a counit $\epsilon : C \to I$ satisfying the following conditions:



The tensor product of two comonoids C and D has the structure of a comonoid $C \otimes D$.

Hence the category **Comon**$(\mathcal{V})$ of comonoids in $\mathcal{V}$ is symmetric monoidal.

The unit object is the comonoid I.

The category **Comon**($\mathcal{V}$)

Theorem (Porst)

If the monoidal category $\mathcal{V}$ is closed and locally presentable, then so is the monoidal category **Comon**($\mathcal{V}$).

The hom object is denoted $HOM(C, D)$.

As a consequence, we have:

Corollary (Sweedler)

The monoidal category of coalgebras over a field is closed.

Corollary (Barr)

Let R be a commutative ring. Then the category of cocommutative R-coalgebras is cartesian closed.

A philosophical remark

We argue that the hom object $Hom(A, B)$ between two monoids **wants to be a comonoid**:

To see this, observe that a map $\phi : A \to B$ is a morphism of monoid iff the following two conditions are satisfied:

- $\phi(xy) = \phi(x)\phi(y)$
- $\phi(e_A) = e_B$.

The first condition is using the diagonal

$$Hom(A, B) \to Hom(A, B) \times Hom(A, B)$$

and the second condition is using the projection $Hom(A, B) \to I$.

We would like to define an enrichement

$$Hom : \mathbf{Mon}(\mathcal{V})^{op} \times \mathbf{Mon}(\mathcal{V}) \to \mathbf{Comon}(\mathcal{V})$$

The convolution monoid

$\mathcal{V}$ a symmetric monoidal closed category.

If $A = (A, m, e)$ is a monoid in $\mathcal{V}$ and $C = (C, \delta, \epsilon)$ is a comonoid, then $\mathcal{V}(C, A)$ has the structure of a monoid $[C, A]$,

- the product is the *convolution product* $\star$

$$[C, A] \otimes [C, A] \xrightarrow{can} [C \otimes C, A \otimes A] \xrightarrow{[\delta, m]} [C, A].$$

- the unit is the composite $e\epsilon : C \to I \to A$.

This defines a functor

$$[-, -] : \mathbf{Comon}(\mathcal{V})^{op} \times \mathbf{Mon}(\mathcal{V}) \to \mathbf{Mon}(\mathcal{V})$$

The category $\mathbf{Mon}(\mathcal{V})$

$\mathcal{V}$ a symmetric monoidal closed category.

Theorem (A-J)

If the category $\mathcal{V}$ is locally presentable, then the category $\mathbf{Mon}(\mathcal{V})$ is locally presentable, enriched and bicomplete over the category $\mathbf{Comon}(\mathcal{V})$.

- the cotensor product of a monoid A by a comonoid C is the convolution monoid $[C, A]$.
- the tensor product of A by C is the *Sweedler product* $C \triangleright A$ of A by C.
- the hom object between two monoids A and B is a comonoid denoted $\{A, B\}$.

We have the hom-tensor-cotensor isomorphisms:

$$\boxed{HOM(C, \{A, B\}) \simeq \{C \triangleright A, B\} \simeq \{A, [C, B]\}}$$

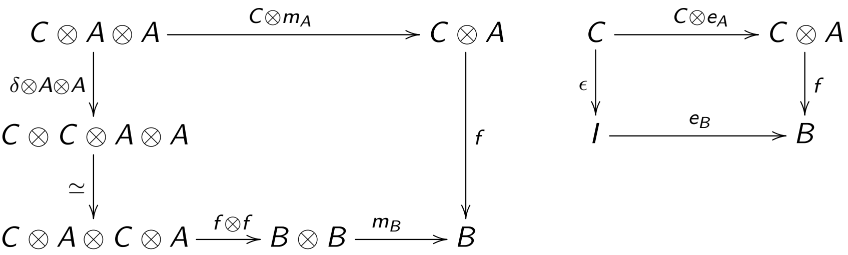
The Sweedler product $C \triangleright A$

A, B monoids, C a comonoid

Definition (Sweedler)

A map $f : C \otimes A \to B$ is a **measuring** if the corresponding map $A \to [C, B]$ is a morphism of algebras.

This condition means that the following two diagrams commute:



The Sweedler product $C \triangleright A$ is the target of a *universal measuring*

$$C \otimes A \to C \triangleright A.$$

The pointed variant

The enrichment of the category of monoids over the category of comonoids has a pointed variant.

A **pointed monoid** A is a morphism of monoids $\epsilon : A \to I$ (an augmentation)

A **pointed comonoid** is a morphism of comonoids $e : I \to C$ (a coaugmentation).

The smash product

In a symmetric monoidal closed category $\mathcal{V}$.

The **smash product** $C \wedge D$ of two pointed comonoids C and D is defined by the following pushout square

$$\begin{array}{ccc} C \sqcup D & \xrightarrow{(C \otimes e, e \otimes D)} & C \otimes D \\ {\scriptstyle (\epsilon,\epsilon)}\downarrow & & \downarrow \\ I & \longrightarrow & C \wedge D \end{array}$$

The smash product gives the category of pointed comonoids $\mathbf{Comon}_\bullet(\mathcal{V})$ a symmetric monoidal structure.

The unit object is the comonoid $I_+ = I \sqcup I$.

The category $\mathbf{Comon}_\bullet(\mathcal{V})$

Theorem (A-J)

If the monoidal category $\mathcal{V}$ is closed and locally presentable, then so is the monoidal category $\mathbf{Comon}_\bullet(\mathcal{V})$.

The hom object is denoted $HOM_\bullet(C, D)$.

The pointed convolution monoid

Let $A = (A, \epsilon)$ be a pointed monoid and $C = (C, e)$ be a pointed comonoid.

The **pointed convolution monoid** $[C, A]_\bullet$ is defined by the following pullback square of monoids:

$$\begin{array}{ccc} [C,A]_\bullet & \longrightarrow & [C,A] \\ \big\downarrow & & \big\downarrow{\scriptstyle ([C,\epsilon],[e,A])} \\ I & \xrightarrow{(\epsilon,e)} & [C,I] \times [I,A] \end{array}$$

This defines a functor

$$[-,-]_\bullet : \mathbf{Comon}_\bullet(\mathcal{V})^{op} \times \mathbf{Mon}_\bullet(\mathcal{V}) \to \mathbf{Mon}_\bullet(\mathcal{V})$$

The category $\mathbf{Mon}_\bullet(\mathcal{V})$

$\mathcal{V}$ a symmetric monoidal closed category.

Theorem (A-J)

If the category $\mathcal{V}$ is locally presentable, then the category $\mathbf{Mon}_\bullet(\mathcal{V})$ is locally presentable, enriched and bicomplete over the category $\mathbf{Comon}_\bullet(\mathcal{V})$.

- the cotensor product of a pointed monoid A by a pointed comonoid C is the pointed convolution monoid $[C, A]_\bullet$
- the tensor product of A by C is the **pointed Sweedler product** $C \triangleright_\bullet A$.
- the hom object between two pointed monoids A and B is a pointed comonoid $\{A, B\}_\bullet$.

Hom-tensor-cotensor isomorphisms:

$$\boxed{HOM_\bullet(C, \{A, B\}_\bullet) \simeq \{C \triangleright_\bullet A, B\}_\bullet \simeq \{A, [C, B]_\bullet\}_\bullet}$$

Pointed dg algebras and coalgebras

Corollary

The category $\mathbf{dgCoalg}_\bullet$ *is symmetric monoidal closed.*

- The tensor product between C and D is $C \wedge D$
- The hom object is denoted $HOM_\bullet(C, D)$

Corollary

The category $\mathbf{dgAlg}_\bullet$ *is enriched and bicomplete over the category* $\mathbf{dgCoalg}_\bullet$.

- The hom object is denoted $\{A, B\}_\bullet$
- The tensor product of a A by C is denoted $C \triangleright_\bullet A$.
- The cotensor product of A by C is denoted $[C, A]_\bullet$.

Twisting cochains

Definition

Let A be a dg algebra. An element $a \in A$ of degre -1 is said to be a *Maurer-Cartan element* if it satisfies the Maurer-Cartan-Brown equation:

$$\partial(a) + aa = 0.$$

Definition (Brown)

Let A be a dg algebra and C be a dg coalgebra. A linear map $\alpha : C \to A$ of degree -1 is said to be a *twisting cochain* if it is a Maurer-Cartan element of the convolution algebra $[C, A]$.

We shall denote the set of twisting cochains $C \to A$ by $Tw(C, A)$.

The Maurer-Cartan algebra

Definition

The **Maurer-Cartan algebra** MC is the dg algebra freely generated by a Maurer-Cartan element $u \in MC$.

In other words, for any dg algebra A and any Maurer-Cartan element $a \in A$, there exists a unique morphism of dg algebras $f : MC \to A$ such that $f(u) = a$.

MC has the structure of a dg Hopf algebra. The coproduct $\delta : MC \to MC \otimes MC$ is defined by putting $\delta(u) = 1 \otimes u + u \otimes 1$ and the counit $\epsilon : MC \to \mathbb{F}$ by putting $\epsilon(u) = 0$.

MC and twisting cochains

If A is a dg algebra and C is a dg coalgebra, then the set $Tw(C, A)$ of twisting cochains $C \to A$ is in bijection with the set of morphisms $MC \to [C, A]$ in the category of dg algebras.

By the hom-tensor-cotensor isomorphisms, we have two natural bijections :

$$\boxed{Hom(C, \{MC, A\}) \simeq Hom(C \triangleright MC, A) \simeq Tw(C, A).}$$

MC and pointed twisting cochains

If the algebra A is pointed and the coalgebra C is pointed, there is a notion of *pointed* twisting cochain $\alpha : C \to A$ ($\alpha e = 0 = \epsilon\alpha$).

The set $Tw_\bullet(C, A)$ of pointed twisting cochains $C \to A$ is in bijection with the set of morphisms $MC \to [C, A]_\bullet$.

By the hom-tensor-cotensor isomorphisms, we have two natural bijections :

$$\boxed{Hom(C, \{MC, A\}_\bullet) \simeq Hom(C \triangleright_\bullet MC, A) \simeq Tw_\bullet(C, A).}$$

The bar-cobar duality for algebras revisited

By combining the classical isomorphism

$$Hom(\Omega C, A) \simeq Tw_\bullet(C, A)$$

with the natural isomorphism

$$Hom(C \triangleright_\bullet MC, A) \simeq Tw_\bullet(C, A),$$

we obtain that $\Omega(C) = C \triangleright_\bullet MC$.

Let us put

$$\Omega^\vee(A) := \{MC, A\}_\bullet.$$

We then have an adjunction

$$\Omega : \mathbf{dgCoalg}_\bullet \leftrightarrow \mathbf{dgAlg}_\bullet : \Omega^\vee$$

A closing remark

The adjoint pair

$$\Omega : \mathbf{dgCoalg}_\bullet \leftrightarrow \mathbf{dgAlg}_\bullet : \Omega^\vee$$

is entirely determined by Maurer-Cartan algebra MC since we have $\Omega(C) = C \triangleright_\bullet MC$.

Conversely, the algebra MC is determined by the cobar construction Ω since we have

$$MC = I_+ \triangleright_\bullet MC = \Omega(I_+),$$

where I_+ is the unit object for the smash product.

Part II

The bar-cobar duality for operads and cooperads:

- The operadic bar-cobar duality recalled
- Enrichement of operads over cooperads
- Pre-Lie algebras
- Twisting cochains
- The Maurer-Cartan operad
- The operadic bar-cobar duality revisited

The operadic bar-cobar duality recalled

The *operadic bar construction* of Ginzburg and Kapranov takes a pointed dg operad A to a pointed dg cooperad $\mathrm{B}(A)$. The *operadic cobar construction* takes a pointed dg cooperad C to a pointed dg operad $\Omega(C)$.

This defines two functors

$$\Omega : \mathbf{dgCoop}_\bullet \to \mathbf{dgOp}_\bullet \qquad \mathrm{B} : \mathbf{dgOp}_\bullet \to \mathbf{dgCoop}_\bullet$$

But the functor Ω is **not** left adjoint to the functor B.

There is only an adjunction

$$\Omega : \mathbf{nildgCoop}_\bullet \longleftrightarrow \mathbf{dgOp}_\bullet : \mathrm{B}$$

where $\mathbf{nildgCoop}_\bullet$ denotes the category of conilpotent dg cooperads.

The functor $\Omega^\vee$

We shall see that the functor $\Omega : \mathbf{dgCoop}_\bullet \to \mathbf{dgOp}_\bullet$ has a right adjoint $\Omega^\vee$. Moreover,

- the category of conilpotent cooperads is a full coreflexive subcategory of the category of pointed cooperads;
- the coreflexion functor takes a pointed cooperad C to its *coradical* $R(C)$;
- we have $\mathrm{B}(A) = R\Omega^\vee(A)$ for any pointed dg operad A.

It follows that the adjunction

$$\Omega : \mathbf{nildgCoop}_\bullet \longleftrightarrow \mathbf{dgOp}_\bullet : \mathrm{B}$$

is obtained by composing the adjunctions

$$inc : \mathbf{nildgCoop}_\bullet \longleftrightarrow \mathbf{dgCoop}_\bullet : R$$

$$\Omega : \mathbf{dgCoop}_\bullet \longleftrightarrow \mathbf{dgOp}_\bullet : \Omega^\vee.$$

Symmetric sequences

Let $\mathcal{V} = (\mathcal{V}, \otimes, I)$ be a symmetric monoidal closed category.

Recall that a *symmetric sequence* $P = (P_n)$ in $\mathcal{V}$ is a sequence of objects $P_n \in \mathcal{V}$ equipped with an action $\Sigma_n \times P_n \to P_n$ of the symmetric group Σ_n.

We shall denote the category of symmetric sequences by $\mathcal{V}^{\Sigma_*}$.

The *Hadamar product* of two symmetric sequences P and Q is defined by putting

$$(P \otimes Q)_n = P_n \otimes Q_n$$

for every $n \geq 0$.

The Hadamar product gives the category $\mathcal{V}^{\Sigma_*}$ a symmetric monoidal closed structure.

The unit object is the *exponential symmetric sequence* E defined by putting $E_n = I$ for every $n \geq 0$.

Species

The groupoid $\Sigma_* = \bigsqcup \Sigma_n$ is equivalent to the groupoid $\mathbb{B}$ of finite sets and bijections.

A *species* in $\mathcal{V}$ is defined to be a functor $P = P[-] : \mathbb{B} \to \mathcal{V}$.

The category of species $\mathcal{V}^{\mathbb{B}}$ is equivalent to the category of symmetric sequences $\mathcal{V}^{\Sigma_*}$.

The *Cauchy product* of two species P and Q is defined by putting

$$(P \cdot Q)[S] = \bigsqcup_{S_1 \sqcup S_2 = S} P[S_1] \otimes P[S_2]$$

The Cauchy product gives the categories $\mathcal{V}^{\mathbb{B}}$ and $\mathcal{V}^{\Sigma_\star}$ a symmetric monoidal closed structure.

The unit object for the Cauchy product is the species $\mathbf{1}$ defined by putting $\mathbf{1}[\emptyset] = I$ and $\mathbf{1}[S] = \bot$ for $S \neq \emptyset$.

Invariants

The *invariant space* of a symmetric sequence P is defined by putting

$$Inv(P) = \prod_{n \geq 0} (P_n)^{\Sigma_n}.$$

If the category $\mathcal{V}$ is additive, then the functor

$$Inv : \mathcal{V}^{\Sigma_\star} \to \mathcal{V}$$

is lax monoidal with respect to the Cauchy product (Aguiar and Mahajan). There is thus a canonical morphism

$$Inv(P) \otimes Inv(Q) \to Inv(P \cdot Q).$$

Operads and cooperads

Recall that a *symmetric operad* in $\mathcal{V}$ is a symmetric sequence $P = (P_n)$ equipped with composition operations

$$P_n \otimes P_{k_1} \otimes \cdots \otimes P_{k_n} \to P_{k_1+\cdots+k_n}$$

satisfying certain identities (May).

Dually, a *symmetric co-operad* is a symmetric sequence $P = (P_n)$ equipped with decomposition operations

$$P_{k_1+\cdots+k_n} \to P_n \otimes P_{k_1} \otimes \cdots \otimes P_{k_n}$$

satisfying dual identities.

Grafting operations

The structure of an operad on a symmetric sequence $P = (P_n)$ can also be defined by using *grafting operations*

$$\gamma_{n,m} = \circ_{n+1} : P_{n+1} \otimes P_m \to P_{n+m}$$

(Markl-Shnider-Stasheff).

The operations can be assembled into a single **total grafting operation**

$$\gamma : P' \cdot P \to P$$

where $P' \cdot P$ is the Cauchy product of P' with P and where P' is the **derivative** of P defined by putting $P'_n = P_{n+1}$ (Joyal).

(In Loday-Vallette, $P' \cdot Q$ is noted $P \circ_{(1)} Q$.)

The category $\mathbf{Coop}(\mathcal{V})$

The Hadamar product of two cooperads C and D has the structure of a cooperad $C \otimes D$.

This defines a symmetric monoidal structure on the category $\mathbf{Coop}(\mathcal{V})$ of cooperads in $\mathcal{V}$.

Theorem (A-J)

If the monoidal category $\mathcal{V}$ is closed and locally presentable, then so is the monoidal category $\mathbf{Coop}(\mathcal{V})$.

The hom object is denoted $HOM(C, D)$.

The convolution operad

$\mathcal{V}$ a symmetric monoidal closed category.

If A is an operad in $\mathcal{V}$, and C is a cooperad, then the **convolution operad** $[C, A]$ is defined by putting

$$[C, A]_n = \mathcal{V}(C_n, A_n)$$

for every $n \geq 0$ (Berger-Moerdijk). The composition operation

$$[C_n, A_n] \otimes [C_{k_1}, A_{k_1}] \otimes \cdots \otimes [C_{k_n}, A_{k_n}] \to [C_k, A_k]$$

for $k = k_1 + \cdots + k_n$ is obtained from the composition operations of A and the decomposition operations of C.

This defines a functor

$$[-,-] : \mathbf{Coop}(\mathcal{V})^{op} \times \mathbf{Op}(\mathcal{V}) \to \mathbf{Op}(\mathcal{V})$$

The category $\mathbf{Op}(\mathcal{V})$

$\mathcal{V}$ a symmetric monoidal closed category.

Theorem (A-J)

If the category $\mathcal{V}$ is locally presentable, then the category $\mathbf{Op}(\mathcal{V})$ is locally presentable, enriched and bicomplete over the category $\mathbf{Coop}(\mathcal{V})$.

- the cotensor product is the convolution operad $[C, A]$.
- the tensor product is the **Sweedler product** $C \triangleright A$.
- the hom object is denoted $\{A, B\} \in \mathbf{Coop}(\mathcal{V})$.

Hom-tensor-cotensor isomorphisms:

$$\boxed{HOM(C, \{A, B\}) \simeq \{C \triangleright A, B\} \simeq \{A, [C, B]\}}$$

The pointed variant

The *unit operad* (cooperad) $\mathbb{I}$ is defined by putting

$$\mathbb{I}_n = \begin{cases} I & \text{if } n = 1 \\ \bot & \text{if } n \neq 1 \end{cases}$$

where $\bot$ is the initial object of the category $\mathcal{V}$.

A **pointed operad** is an operad A equipped with a morphism of operads $\epsilon : A \to \mathbb{I}$ (an augmentation).

A **pointed cooperad** is a cooperad C equipped with a morphism of cooperads $e : \mathbb{I} \to C$ (a coaugmentation).

The category $\mathbf{Coop}_\bullet(\mathcal{V})$

$\mathcal{V}$ a symmetric monoidal closed category.

The *smash product* $C \wedge D$ of pointed cooperads in $\mathcal{V}$ is a pointed cooperad.

The smash product gives the category $\mathbf{Coop}_\bullet(\mathcal{V})$ of pointed cooperads in $\mathcal{V}$ a symmetric monoidal structure. The unit object for the smash product is the pointed cooperad $E_+ = E \sqcup \mathbb{I}$.

Theorem (A-J)

If the closed monoidal category $\mathcal{V}$ is locally presentable, then the monoidal category $\mathbf{Coop}_\bullet(\mathcal{V})$ is closed and locally presentable.

The hom object is denoted $HOM_\bullet(C, D)$.

The category $\mathbf{Op}_\bullet(\mathcal{V})$

There is a notion of *pointed convolution operad* $[C, A]_\bullet$ for a pointed operad A and a pointed cooperad C.

Theorem (A-J)

If $\mathcal{V}$ is locally presentable, then the category $\mathbf{Op}_\bullet(\mathcal{V})$ of pointed operads in $\mathcal{V}$ is locally presentable, enriched and bicomplete over the category $\mathbf{Coop}_\bullet(\mathcal{V})$.

- The cotensor product of A by C is the pointed convolution operad $[C, A]_\bullet$.
- The tensor product is the *pointed Sweedler product* $C \rhd_\bullet A$.
- The hom object is a pointed cooperad $\{A, B\}_\bullet$

Hom-tensor-cotensor isomorphisms:

$$HOM_\bullet(C, \{A, B\}_\bullet) \simeq \{C \rhd_\bullet A, B\}_\bullet \simeq \{A, [C, B]_\bullet\}_\bullet$$

Pointed dg operads and cooperads

Corollary

The category pointed dg cooperads $\mathbf{dgCoop}_{\bullet}$ *is symmetric monoidal closed.*

- The tensor product between C and D is $C \wedge D$
- The hom object is denoted $HOM_{\bullet}(C, D)$

Corollary

The category of pointed dg operads $\mathbf{dgOp}_{\bullet}$ *is enriched and bicomplete over the category* $\mathbf{dgCoop}_{\bullet}$*.*

- The hom object is denoted $\{A, B\}_{\bullet}$
- The tensor product of a A by C is denoted $C \triangleright_{\bullet} A$.
- The cotensor product of A by C is denoted $[C, A]_{\bullet}$.

Graded pre-Lie algebra

We recall that a *graded pre-Lie algebra* is a graded vector space X equipped with a binary operation $\star : X \otimes X \to X$ satisfying

$$(x \star y) \star z - x \star (y \star z) = (-1)^{|y||z|}\big((x \star z) \star y - x \star (z \star y)\big)$$

A graded pre-Lie algebra $(X, \star)$ has the structure of a Lie algebra with the bracket operation $[-,-] : X \otimes X \to X$ defined by putting

$$[x, y] = x \star y - (-1)^{|x||y|} y \star x.$$

For the history of the notion of pre-Lie algebra, see Burde.

The pre-Lie algebra of a non-symmetric operad

The *total space* of a non-symmetric operad A is defined by putting

$$Tot(A) = \bigoplus_{n \geq 0} A_n.$$

We recall that the vector space $Tot(A)$ has the structure of a pre-Lie algebra with the Gerstenhaber operation $\circ : A_n \otimes A_m \to A_{n+m-1}$ defined by putting

$$\phi \circ \psi = \sum_{i=1}^{n} \phi \circ_i \psi$$

This contruction has an analog for symmetric operads.

The pre-Lie algebra of a symmetric operad

The invariant space $Inv(A)$ of a symmetric operad A has the structure of a pre-Lie algebra with the operation

$$\star : Inv(A) \otimes Inv(A) \to Inv(A)$$

obtained by composing the canonical maps

$$Inv(A) \otimes Inv(A) \to Inv(A') \otimes Inv(A) \to Inv(A' \cdot A) \to Inv(A),$$

where $Inv(A) \to Inv(A')$ is the natural projection and the last map is induced by the total grafting operation $\gamma : A' \cdot A \to A$ (see Loday & Valette).

Twisting cochains

If A is a dg operad, we say that an element $a = (a_n) \in Inv(A)$ of degree -1 is a *Maurer-Cartan element* if it satisfies the Maurer-Cartan equation:

$$\partial(a) + a \star a = 0.$$

If A is a dg operad and C is a dg cooperad, then a *twisting cochain* $\alpha : C \to A$ is defined to be a Maurer-Cartan element of the convolution operad $[C, A]$.

The Maurer-Cartan operad

Definition

The **Maurer-Cartan operad** $\mathcal{MC}$ is the dg operad freely generated by the components u_n of a Maurer-Cartan element $u = (u_n)$.

There is a bijection between the set $Tw(C, A)$ of twisting cochains $C \to A$ and the set of morphisms $\mathcal{MC} \to [C, A]$ in the category of dg operads.

By the hom-tensor-cotensor isomorphisms, we have two natural bijections

$$\boxed{Hom(C, \{\mathcal{MC}, A\}) \simeq Hom(C \triangleright \mathcal{MC}, A) \simeq Tw(C, A)}$$

We thus obtain an adjunction

$$(-) \triangleright \mathcal{MC} : \mathbf{dgCoop} \leftrightarrow \mathbf{dgOp} : \{\mathcal{MC}, -\}.$$

Pointed twisting cochains

There is also a notion of **pointed twisting cochain** $\alpha : C \to A$ ($\alpha e = 0 = \epsilon\alpha$) between a pointed cooperad $C = (C, e)$ and a pointed operad $A = (A, \epsilon)$.

There is then a bijection between the set $Tw_\bullet(C, A)$ of pointed twisting cochains $C \to A$ and the set of morphisms $\mathcal{MC} \to [C, A]_\bullet$ in the category of pointed dg operads.

By the hom-tensor-cotensor isomorphisms, we have two natural bijections :

$$\boxed{Hom(C, \{\mathcal{MC}, A\}_\bullet) \simeq Hom(C \triangleright_\bullet \mathcal{MC}, A) \simeq Tw_\bullet(C, A).}$$

We thus obtain an adjunction

$$(-) \triangleright_\bullet \mathcal{MC} : \mathbf{dgCoop}_\bullet \leftrightarrow \mathbf{dgOp}_\bullet : \{\mathcal{MC}, -\}_\bullet.$$

The operadic bar-cobar duality revisited

By combining the classical isomorphism

$$Hom(\Omega C, A) \simeq Tw_\bullet(C, A)$$

with the natural isomorphism

$$Hom(C \triangleright_\bullet \mathcal{MC}, A) \simeq Tw_\bullet(C, A),$$

we obtain that $\Omega(C) = C \triangleright_\bullet \mathcal{MC}$.

Let us put

$$\Omega^\vee(A) := \{\mathcal{MC}, A\}_\bullet.$$

We then have an adjunction

$$\Omega : \mathbf{dgCoop}_\bullet \leftrightarrow \mathbf{dgOp}_\bullet : \Omega^\vee.$$

The closing remark

The adjoint pair

$$\Omega : \mathbf{dgCoop}_{\bullet} \leftrightarrow \mathbf{dgOp}_{\bullet} : \Omega^{\vee}$$

is entirely determined by the Maurer-Cartan operad $\mathcal{MC}$, since we have $\Omega(C) = C \rhd_{\bullet} \mathcal{MC}$.

Conversely, the operad $\mathcal{MC}$ is determined by the operadic cobar construction Ω, since we have

$$\mathcal{MC} = E_{+} \rhd_{\bullet} \mathcal{MC} = \Omega(E_{+}),$$

where E_{+} is the unit object for the smash product.

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Thanks !