Lectures on ∞-Topoi

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– Lecture I –

HOMOTOPY THEORY
AND
LOCALIZATION OF CATEGORIES

Today

I want to address the following questions (and some of the answers)

What is homotopy theory?		
study of the shape of spaces	techniques to work in localizations of categories	a way to work with higher categories by means of 1-category theory
a theory of ambiguity	a new way to think about the quotient operation	

How can it be connected with logic?		
rewriting problems (rewriting rules = higher cells)	higher theories (tomorrow)	higher semantics (HoTT)

Summary

- Homotopy theory is about localization of categories
- Localization of categories = way to define quotients of categories
- The ambiguity of identification generates a higher structure on quotients
- This structure is that of ∞-groupoid/∞-category
- all ∞-categories can be presented by a 1-category
- this leads to a way to reduce higher category to set theory
- this leads also to a notion of homotopical semantics for logical theories

Plan

0. Timeline of homotopy theory
1. Congruence of categories
2. Localization of categories
3. Simplicial localization & ∞-categories
4. Interaction with logic

I.0 – Timeline of homotopy theory

I.0 Timeline of homotopy theory

1945	Eilenberg-Steenrod	homology = **functors**
1956	Cartan-Eilenberg	homology = derived functors = **resolutions**
1963	Grothendieck-Verdier	homology = **chain complexes** up to quasi-iso = **localization of categories**, resolution = **replacement**
1967	Gabriel-Zisman	calculus of **fractions**
1967	Quillen	**model categories** = general framework for both homotopy and homology
1980	Dwyer-Kan	**simplicial localization** = localization in ∞-categories
1983	Grothendieck	homotopy = ∞-**groupoids**, need ∞-categories
1990s	Joyal	quasi-categories: there exists a **theory** of higher categories theory and it has the same theorems as 1-category theory
2000s	Rezk, Lurie	homotopy = ∞-**topoi** homology = **stable** ∞-**categories**

I.0 The evolution of homotopy theory

1	Congruences	homotopy and (co)homology invariants	$[S^n, -] : CW_{/\sim} \to Set$ $H_n : CW_{/\sim} \to Ab$
2	Localizations of categories	derived categories model categories	$D(\mathbb{Z}) = Ch(\mathbb{Z})[Qis^{-1}]$ $D(SSet) = SSet[W^{-1}]$ $= D(Top) = Top[W^{-1}]$
3	Higher categories	homotopy types $=$ ∞-gpd simplicial cat $=$ ∞-cat Dwyer-Kan localization	$L(Ch(\mathbb{Z}), Qis)$ $L(SSet, W)$ $= L(Top, W)$
4	Higher structures	∞-topoi stable ∞-categories	S, $[Fin^\bullet, S]$, $Sh(X)$ Sp, $Sh(X, Sp)$

I.1 – Congruences

I.1 Congruences of categories

Definition

A congruence on a category C is the data of

$$\text{equivalence relations } \sim_{x,y} \text{ on each } C(x,y)$$

which are compatible with composition

$$\left(f \sim f' \quad \text{and} \quad g \sim g'\right) \quad \Longrightarrow \quad gf \;\sim\; g'f'.$$

The quotients of these equivalence relations do form again a category

$$C/\sim$$

with the same objects as C, called a quotient of C by the congruence.

I.1 Congruences of categories

Definition

A map $f : x \to y$ in C is called a $\sim$-equivalence if there exists $g, h : y \to x$ such that $gf \sim 1_x$ and $fh \sim 1_y$.

For a map $f : x \to y$ in C, let $[f] : x \to y$ be its image in $C/\sim$. Then,

$$[f] \text{ is an isomorphism iff } f \text{ is } \sim\text{-equivalence.}$$

Proposition (Universal property of quotient)

A functor $C \to D$ factors through $C/\sim$ iff it sends $\sim$-equivalent maps to equal maps,

I.1 Congruences – Example 1: Homotopy equivalence

In the category of CW complexes. Let $I = [0,1]$ be the topological interval. The relation of homotopy of maps is:

$$f \sim g \quad \Longleftrightarrow \quad \exists h, \text{ s.t.} \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_0"'] & Y \arrow[d, equal] \\ X \times I \arrow[r, "h"] & Y \arrow[d, equal] \\ X \arrow[u, "i_1"] \arrow[r, "g"] & Y \end{tikzcd}$$

Quotient by this equivalence define the homotopy category

$$CW/\sim$$

where

$$\hom(X,Y) := Top(X,Y)/\sim .$$

I.1 Congruences – Example 1: Homotopy equivalence

One application is Eilenberg-Steenrod axioms for homology theories. A reduced homology theory is a family of functors indexed by $\mathbb{Z}$.

$$\begin{aligned} H_n : CW &\longrightarrow Ab \\ X &\longmapsto H_n(X) \end{aligned}$$

The first axiom is invariance by homotopy, which means that the H_n factor through $CW^\bullet/\sim$

$$\begin{tikzcd} CW \arrow[r, "H_n"] \arrow[d] & Ab \\ CW/\sim \arrow[ur, dashed, bend right] & \end{tikzcd}$$

I.1 Congruences – Example 2: Syntactic categories

Let T be a first order theory.

The arrows of the syntactic category $C(T)$ of T are equivalence classes of formulas for provable equivalence.

This can be used to define $C(T)$ as the quotient of some other category D by the congruence generated by equivalences of formulas.

This is the seed of "homotopical logical": proofs of equivalence behave like homotopies.

I.1 Congruences – Example 3: Chain homotopy equivalences

A chain homotopy between two morphisms $f, g : C_* \to D_*$ of chain complexes is a family of morphisms of $h_n : C_n \to D_{n+1}$ such that

$$\partial^D_{n+1} h_n - h_{n-1}\partial^C_n = f_n - g_n$$

$$\begin{tikzcd}
\dots \arrow[r] & C_{n+1} \arrow[r, "\partial^C_{n+1}"] \arrow[d, "f_{n+1}-g_{n+1}"'] & C_n \arrow[r, "\partial^C_n"] \arrow[dl, "h_n"'] \arrow[d, "f_n-g_n"] & C_{n-1} \arrow[r] \arrow[dl, "h_{n-1}"] \arrow[d, "f_{n-1}-g_{n-1}"] & \dots \\
\dots \arrow[r] & D_{n+1} \arrow[r, "\partial^D_{n+1}"'] & D_n \arrow[r, "\partial^D_n"'] & D_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

Projective or injective resolutions of abelian group are unique up to chain homotopy.

This provide another setting with the same structure as homotopy theory.

I.2 – Localizations

I.2 Weak equivalences

The definition of higher homotopy groups, prompted homotopy theory to focus on a weaker notion of equivalence. A map $f : X \to Y$ is weak homotopy equivalence if it induces bijections between all homotopy invariants:

$$f : X \to Y \text{ WHE} \quad \Longleftrightarrow \quad \pi_n(f) : \pi_n(X) \simeq \pi_n(Y).$$

Any homotopy equivalence is a weak homotopy equivalence.

Whitehead: for CW-complexes, homotopy equivalence = weak homotopy equivalence.

But the constructions of algebraic topology, when defined on all spaces, need to be invariants by this larger class of maps.

I.2 Quasi-isomorphisms

On the algebraic side, the study of chain complexes also motivate weaker notion than chain homotopy equivalence. A morphism of chain complexes quasi-isomorphism if it induces isomorphisms of all homology groups:

$$f : A \to B \text{ Qis} \quad \Longleftrightarrow \quad H_n(f) : H_n(A) \simeq H_n(B).$$

Two projective resolutions are chain homotopy equivalent. And so are their image by any functor. But a projective and a flat resolution need not be, they are only quasi-isomorphic. And so are their image by any functor (for which they are both admissible resolutions).

Grothendieck emphasized that constructions on arbitrary chain complexes must be invariant under quasi-isomorphism and not only chain homotopy equivalence.

I.2 Localization of categories

The abstraction of these situations is the following.

We have a category C and $W \subset C^{\rightarrow}$ a family of arrows in C

C	$W \subset C^{\rightarrow}$
Spaces	weak homotopy equivalences
Chain complexes	quasi-isomorphism

and we are interested in functors $C \to D$ sending maps in W to isomorphisms in D:

$$\begin{array}{ccc} C & \longrightarrow & D \\ W & \longmapsto & Iso(D). \end{array}$$

I.2 Localization of categories

Let $LOC(C, W)$ be the full subcategory of $C \downarrow CAT$ spanned by the functors $C \to D$ sending maps in W to isomorphisms in D

Definition

The localization of C along W is the initial object $C \to C[W^{-1}]$ in the category $LOC(C, W)$.

Theorem

The localization of C along W always exists.

We are going to see an explicit description soon.

I.2 Motivation for localizations

	Homotopy (spaces)	**Homology (chain complexes)**
(Strong) equivalences	invertible up to homotopy	invertible up to chain homotopy
Weak equivalences	induce isomorphism on homotopy invariants π_n	induce isomorphism on homology invariants H_n
Coincide on some specific objects	CW complexes	complexes of projective or injective modules
Functors of interest must send WE to isomorphisms	(co)homology theories, classifying spaces	derived functors (Tor, Ext)
This prompted the notion of localization of categories as domain of the functors of interest	$Top[WE^{-1}] \xrightarrow{H_n} Ab$ $Top[WE^{-1}]^{op} \xrightarrow{H^n} Ab$	$Ch(\mathbb{Z})[Qis^{-1}] \xrightarrow{Tor_n} Ab$ $Ch(\mathbb{Z})[Qis^{-1}]^{op} \xrightarrow{Ext^n} Ab$

I.2 Motivation for localizations

$$\begin{tikzcd}
 & CW/\sim \arrow[r, "H_n"] \arrow[d] & Ab \\
Top \arrow[r] & Top[WE^{-1}] \arrow[ur, "H_n"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \left\{\begin{array}{c}\text{bounded cplx}\\ \text{of proj. mod.}\end{array}\right\}/\sim \arrow[r, "Tor_n"] \arrow[d] & Ab \\
Ch(\mathbb{Z}) \arrow[r] & Ch(\mathbb{Z})[Qis^{-1}] \arrow[ur, "Tor_n"'] &
\end{tikzcd}$$

$$\underbrace{\phantom{Top \quad Ch(\mathbb{Z})}}_{\substack{\text{nicer categories}\\ \text{to work in:}\\ \text{all limits/colimits}}}$$

I.2 Localizations = quotients

The notion localization of category is badly named.

From the point of view of arrows, it is indeed a localization, analogous to the notion for monoids or rings.

But, from the point of view of objects, it is in fact a quotient!

Indeed, the purpose of localizations is to force objects to become isomorphic.

I.2 Localizations = quotients

Many features of localizations become clearer when though in terms of quotient.

Quotients	
Sets	**Categories**
relation $R \subset X \times X$	class of arrows $W \subset C^{\to}$
reflexivity	W contains isomorphisms
transitivity	W stable by composition
symmetry	W has left and right cancellation
symmetry + reflexivity	W has the "2 out of 3" property
equivalence relation	W contains isomorphisms + has the "2 out of 3" property (pre-saturated)
quotient X/R	localization $C[W^{-1}]$

I.2 Computations of localizations

We are going to review several techniques to deal with the localization of a category

1. zig-zags
2. congruences
3. fractions
4. reflective subcategories
5. model structures

I.2.1 – Computation of localizations
Zig-Zags

I.2.1 Computations of localizations – zig-zags

Localizations of categories are called this way because they are related to localizations of rings and monoids.

Their description can be done by introducing formal symbols w^{-1} for the inverse of map w and imposing relations $ww^{-1} = 1$ and $w^{-1}w = 1$.

An arrow in $C[W^{-1}]$ will then be words like $w^{-1}fv^{-1}gu^{-1}$, up to rewriting.

I.2.1 Computations of localizations – zig-zags

But with categories, we have a more geometric picture.

An arrow in $C[W^{-1}]$ is an equivalence class of zig-zags

$$\begin{tikzcd}[column sep=small]
 & Y_1 \arrow[dl, "w"'] \arrow[dr] & & \cdots \arrow[dl, "w"'] \arrow[dr] & & Y_n \arrow[dl, "w"'] \arrow[dr] & \\
X_0 & & Y_1' & & Y_{n-1}' & & X_n
\end{tikzcd}$$

where the maps going to the left are in W.

I.2.1 Computations of localizations – zig-zags

The relations between zig-zags (rewriting rules) are given by

1. contraction of composition

$$\dots X \xleftarrow{w} Z \xleftarrow{w'} Y \dots \quad = \quad \dots X \xleftarrow{ww'} Y \dots$$

$$\dots X \xrightarrow{f'} Z \xrightarrow{f} Y \dots \quad = \quad \dots X \xrightarrow{ff'} Y \dots$$

2. contraction of identities

$$\dots X \xleftarrow{w} Z = Z \xleftarrow{w'} Y \dots \quad = \quad \dots X \xleftarrow{ww'} Y \dots$$

$$\dots X \xrightarrow{f'} Z = Z \xrightarrow{f} Y \dots \quad = \quad \dots X \xrightarrow{ff'} Y \dots$$

3. contraction of equivalences

$$\dots X \xleftarrow{w} Z \xrightarrow{w} X \dots \quad = \quad \dots X \dots$$

$$\dots X \xrightarrow{w} Z \xleftarrow{w} X \dots \quad = \quad \dots X \dots$$

I.2.1 Computations of localizations – zig-zags

The contraction of equivalences can be given a geometric picture.

A morphisms of zig-zag is defined as a diagram

$$\begin{tikzcd} X_0 \arrow[d, equal] & Y_1 \arrow[l] \arrow[r] \arrow[d, "u_1"] & \dots & Y_n \arrow[l] \arrow[r] \arrow[d, "u_n"] & X_n \arrow[d, equal] \\ X_0 & Z_1 \arrow[l] \arrow[r] & \dots & Z_n \arrow[l] \arrow[r] & X_n \end{tikzcd}$$

where the all vertical maps are in W.

In particular we have

$$\begin{tikzcd} X \arrow[d, equal] & Y \arrow[l, "w"'] \arrow[r, "w"] \arrow[d, "w"] & X \arrow[d, equal] \\ X \arrow[r, equal] & X \arrow[r, equal] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y \arrow[d, equal] \arrow[r, "w"] & X & Y \arrow[l, "w"'] \arrow[d, equal] \\ Y \arrow[r, equal] & Y \arrow[u, "w"] \arrow[r, equal] & Y \end{tikzcd}$$

$$ww^{-1} = id_X \qquad\qquad w^{-1}w = id_Y$$

The contraction of equivalences can be replaced by the relation "being connected by a zig-zag of morphisms of zig-zags".

I.2.1 Computations of localizations – zig-zags

A morphism

$$\begin{tikzcd} X \arrow[d, equal] & Z \arrow[l, "w"'] \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[d, equal] \\ X & Z' \arrow[l, "w'"'] \arrow[r, "f'"] & Y \end{tikzcd}$$

must be read as the relation

$$f'(w')^{-1} = f'uu^{-1}(w')^{-1} = f'u(w'u)^{-1} = fw^{-1}$$

I.2.1 Homotopical algebra

The description in terms of zig-zags is useful to prove that the localization exists, but not for computing explicitely the hom sets of $C[W^{-1}]$.

Many techniques have been invented to do so.

All are based on the same idea: reducing the length of zig-zags

congruences	length 1	$\rightarrow$
reflective localizations	length $1\frac{1}{2}$	$\rightarrow\leftarrow$
calculus of fractions	length 2	$\rightarrow\leftarrow,\ \leftarrow\rightarrow$
model structures	length 3	$\leftarrow\rightarrow\leftarrow$

Altogether these localization techniques form the topic of homotopical algebra, so named because of its origin in homotopy theory.

I.2.2 – Computation of localizations

Congruences

I.2.2 Computations of localizations – Congruences

An interval is defined as an object I with two disjoint points $i_0, i_1 : 1 \to I$.

If C has cartesian products, an interval defines a relation of homotopy and an associated congruence.

An interval is self-contractible if $id : I \to I$ is homotopic to both $I \to 1 \xrightarrow{i_0} I$ and $I \to 1 \xrightarrow{i_1} I$

Let us say that a congruence is homotopical if it is given by such an interval.

I.2.2 Computations of localizations – Congruences

Proposition

If a congruence on C is homotopical, then a functor $C \to D$ factors through $C/\sim$ iff it sends $\sim$-equivalences to isomorphisms, i.e. the quotients $C/\sim$ is a localization:

$$CW[HE^{-1}] = CW/\sim \qquad \textit{and} \qquad Ch(\mathbb{Z})[CHE^{-1}] = Ch(\mathbb{Z})/\sim .$$

Proof: $X \times I \to X$ is an $\sim$-equivalence. It is inverted by the quotient. Reciprocally, if I is self-contractible, $X \times I \to X$ is inverted and $\sim$-equivalence maps are identified by the relation

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, "i_0"'] & Y \arrow[d, equal] \\
X \arrow[d, equal] & X \times I \arrow[l, "p"'] \arrow[r, "h"] & Y \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[u, "i_1"] \arrow[r, "g"] & Y
\end{tikzcd}$$

I.2.2 Computations of localizations – Congruences

Method 1: If $W = \sim$-equivalence for a homotopical congruence, then

$$C[W^{-1}] = C/\sim .$$

I.2.3 – Computation of localizations

Reflective localizations

I.2.3 Reflective localizations

When C is a cocomplete category localization can be computed in a very efficient way

The category $LOC_{cc}(C, W)$ is the full subcategory of $C \downarrow CAT_{cc}$ spanned by the cocontinuous functors $C \to D$ sending maps in W to isomorphisms in D.

Definition (cc-Localization)

The cc-localization of C along W is the initial object $C \to C[W^{-1}]$ in the category $LOC_{cc}(C, W)$.

I.2.3 Reflective localizations

And now, my favorite theorem in category theory.

Theorem (Fundamental thm of CT)

If W is a small category of small objects in $C^{\to}$, the cc-localization $C \to C[W^{-1}]$ has a fully faithful right adjoint.

The localization can be computed as a *full subcategory*!

This is the best one can hope to compute localizations.

Recall that a localization is in fact a quotient, the theorem says that the quotient $C \to C[W^{-1}]$ has a distinguished section (terminal in the category of sections).

I.2.3 Reflective localizations

The image of $C[W^{-1}] \hookrightarrow C$ is the category C_W of local objects:

X is local if, for all $w : A \to B$ in W

$$\hom(B, X) \xrightarrow{\simeq} \hom(A, X)$$

($= X$ believes that w is an isomorphism)

The localization coincide with the reflection

$$C \to C[W^{-1}] \qquad = \qquad P : C \to C_W$$

I.2.3 Reflective localizations

A morphism in $C[W^{-1}]$ is then a zigzag

$$X \longrightarrow PY \longleftarrow Y.$$

But since the map $Y \to PY$ is fixed once an for all, reflective localizations actually compute maps in $C[W^{-1}]$ as a single map

$$X \longrightarrow PY.$$

Notice that this is more efficient than congruences since no quotient is needed

$$\hom_{C[W^{-1}]}(X, Y) = \hom_C(PX, PY) = \hom_C(X, PY).$$

I.2.3 Reflective localizations

Method 2: If the localization is taken in cocomplete categories and W small, then

$$\hom_{C[W^{-1}]}(X, Y) = \hom_C(X, PY)$$

where $P : C \to C_W$ is the reflection into local objects.

Reflective localizations are a particular case of calculus of fractions (see Gabriel-Zisman).

I.2.4 – Computation of localizations

Calculus of fractions

I.2.4 Calculus of fractions

The idea behind fractions is to use only zig-zags of length 2.

A right fraction is a diagram $X \xleftarrow{w} Z \xrightarrow{f} Y$ where w is an equivalence. It is read as a morphism $X \to Y$ and can be written suggestively fw^{-1}.

A left fraction is a diagram $X \xrightarrow{f} Z \xleftarrow{w} Y$ where w is an equivalence. It can be written $w^{-1}f$.

The axioms of the calculus of fractions are ways to compose fractions. Essentially, they amount to replace a left fraction by a right fraction.

$$\begin{tikzcd}
 & & Z \arrow[dl, dashed, "w"'] \arrow[dr, dashed] & & \\
 & Y_1 \arrow[dl, "w"'] \arrow[dr] & & Y_2 \arrow[dl, "w"'] \arrow[dr] & \\
X_0 & & X_1 & & X_2
\end{tikzcd}$$

I.2.4 Calculus of fractions

This can be seen as a morphism of zig-zags reducing the length.

$$\begin{tikzcd}
X_0 \arrow[d, equal] & Z \arrow[l] \arrow[r, equal] \arrow[d] & Z \arrow[r, equal] \arrow[d] & Z \arrow[r] \arrow[d] & X_2 \arrow[d, equal] \\
X_0 & Y_1 \arrow[l] \arrow[r] & X_1 & Y_2 \arrow[l] \arrow[r] & X_2
\end{tikzcd}$$

So that, by iteration, every zig-zag is equivalent to a zig-zag of length two.

I.2.4 Calculus of fractions

Gabriel-Zisman: the *homotopy category of simplicial sets* can be defined as

$$Ho(SSet) = SSet[AE^{-1}] = (SSet/\sim)[ae^{-1}]$$

where AE is the class of anodyne extensions. The intermediate category $SSet/\sim$ has a calculus of fractions for the image of AE.

Verdier: the *derived category of chain complexes* can be defined as

$$D(\mathbb{Z}) = Ch(\mathbb{Z})[Qis^{-1}] = (Ch(\mathbb{Z})/\sim)[qis^{-1}]$$

where Qis is the class of quasi-isomorphism. The intermediate category $K(\mathbb{Z}) = Ch(\mathbb{Z})/\sim$ has a calculus of fractions for the image of Qis.

I.2.4 Calculus of fractions

Method 3: If (C, W) has a calculus of fractions

$$\hom_{C[W^{-1}]}(X, Y) = \operatorname*{colim}_{w:Z\to X} \hom_C(Z, Y)$$
$$= \pi_0\Big(\text{category of fractions } X \leftarrow Z \rightarrow Y\Big)$$

Morphism of fractions =
$$\begin{tikzcd} X \arrow[d, equal] & Z \arrow[l, "w"'] \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[d, equal] \\ X & Z' \arrow[l, "w'"'] \arrow[r, "f'"] & Y \end{tikzcd}$$

Overall, calculus of fractions are rare.
Model structures provide a more flexible notion.

I.2.5 – Computation of localizations

Model structures

I.2.5 Model structures

Definition

1. A pair (C, W) is called pre-saturated if W has the "2 out of 3" property and contains all isomorphisms.

2. A model structure on a pre-saturated pair (C, W) is the choice of

   2.1 two classes of maps Cof, Fib of C

   2.2 such that $(Cof \cap W, Fib)$ and $(Cof, Fib \cap W)$ are weak factorization systems.

$$
\begin{tikzcd}
X \arrow[r, tail, "Cof \cap W", "\simeq"'] \arrow[d, tail, "Cof"'] \arrow[dr, "f" description] & X_f \arrow[d, two heads, "Fib"] \\
Y^f \arrow[r, two heads, "\simeq", "Fib \cap W"'] & Y
\end{tikzcd}
$$

I.2.5 Model structures

A model structure on a pair (C, W) can be understood as a way to use only zig-zags of length 3.

A double fraction is a diagram $X \xleftarrow{u} Z' \xrightarrow{f} Z \xleftarrow{w} Y$ where w and u are equivalences. It can be written $w^{-1}fu^{-1}$.

Model categories consider specific double fractions:

$$X \underbrace{\xleftarrow{\quad u \quad}}_{\text{triv. fibration}} \overbrace{QX}^{\text{cofibrant object}} \xrightarrow{\quad f \quad} \overbrace{RY}^{\text{fibrant object}} \underbrace{\xleftarrow{\quad w \quad}}_{\text{triv. cofibration}} Y$$

I.2.5 Model structures

Method 4: If (C, W) has a model structure

$$\hom_{C[W^{-1}]}(X, Y) = \hom_C(QX, RY)/\sim$$

where QX is a cofibrant replacement of X

and RY is a fibrant replacement of Y.

(Since the method come back to congruence, it may look like zig-zags of length 1 only are used. But, in practice, it is important to let QX and RY vary, so this is really zig-zags of length 3.)

I.2.5 Model structures

The strategy of model structure is complex

1. define a homotopy relation and homotopy equivalence
2. define a class of good objects C^{cf} (fibrant + cofibrant objects)
3. over which homotopy equivalence and equivalence coincide
4. and such that every object is equivalent to a good one
5. then the localization is simply given by a congruence on good objects

$$C[W^{-1}] = C^{cf} / \sim$$

Nonetheless, model structures are quite frequent and are a very effective tool.

I.2 Summary

Methods to compute the hom sets $[X, Y]$ in a localization $C[W^{-1}]$

congruences	length 1	$\rightarrow$	$[X, Y] = \hom(X, Y)/\sim$
reflective localizations	length $1\frac{1}{2}$	$\rightarrow\leftarrow$	$[X, Y] = \hom(X, PY)$
calculus of (right) fractions	length 2	$\leftarrow\rightarrow$	$[X, Y] = \operatorname{colim}_{w:Z\to X} \hom(Z, Y)$
model structures	length 3	$\leftarrow\rightarrow\leftarrow$	$[X, Y] = \hom(QX, RY)/\sim$

I.3 – Simplicial localization and ∞-categories

I.3 Simplicial localization

Recall that a simplicial set $X = \{X_n\}$ has a set of connected components

$$\pi_0(X) = \text{quotient of } X_0 \text{ by relation image of } X_1 \to X_0 \times X_0$$

The simplical localization of a pair (C, W) is a category enriched over simplicial sets $L(C, W)$ enhancing $C[W^{-1}]$ in the sense that

$$\pi_0\big(\underline{\mathrm{hom}}_{L(C,W)}(X, Y)\big) \quad = \quad \mathrm{hom}_{C[W^{-1}]}(X, Y).$$

I.3 Simplicial localization

The morphisms of zig-zags

$$\begin{tikzcd}
X_0 \arrow[d, equal] & Y_1 \arrow[l] \arrow[r] \arrow[d, "u_1"] & \dots & Y_n \arrow[l] \arrow[r] \arrow[d, "u_n"] & X_n \arrow[d, equal] \\
X_0 & Z_1 \arrow[l] \arrow[r] & \dots & Z_n \arrow[l] \arrow[r] & X_n
\end{tikzcd}$$

define a category of zig-zags $X_0 \rightsquigarrow X_n$ of length n.

I.3 Simplicial localization

By taking the nerve of this category, we get a simplicial set.
An n-simplex is a hammock

$$
\begin{tikzcd}
 & A_1 \arrow[r] \arrow[d] \arrow[ddl] & \cdots & A_n \arrow[l] \arrow[d] \arrow[ddr] & \\
 & B_1 \arrow[r] \arrow[d] \arrow[dl] & \cdots & B_n \arrow[l] \arrow[d] \arrow[dr] & \\
X_0 & \cdots \arrow[l] \arrow[r] \arrow[d] & \cdots & \cdots \arrow[l] \arrow[r] \arrow[d] & X_n \\
 & Y_1 \arrow[r] \arrow[d] \arrow[ul] & \cdots & Y_n \arrow[l] \arrow[d] \arrow[ur] & \\
 & Z_1 \arrow[r] \arrow[uul] & \cdots & Z_n \arrow[l] \arrow[uur] &
\end{tikzcd}
$$

From these simplicial sets and the rewriting rules on zig-zags, Dwyer & Kan construct a category enriched over simplicial sets

$$L(C, W).$$

I.3 Simplicial localization

The category $L(C, W)$ contains all the information about the localization $C[W^{-1}]$ because

$$\pi_0\big(\underline{\mathrm{hom}}_{L(C,W)}(X, Y)\big) \quad = \quad \mathrm{hom}_{C[W^{-1}]}(X, Y)$$

But it contains more information since the $\mathrm{hom}_{L(C,W)}(X, Y)$ can have non-trivial higher homotopy.

Example: for $C = \{0 \rightrightarrows 1\}$, we have

$$\underline{\mathrm{hom}}_{L(C,C^{\to})}(1, 1) \simeq S^1$$

I.3 Ambiguity and homotopy quotient

The simplicial structure $\underline{\hom}(X, Y) := \underline{\hom}_{L(C,W)}(X, Y)$ encode the ambiguity of rewritings in the description of arrow by means of zigzags.

The fact that there is no canonical way to identify two zig-zags creates a structure.

We touch here another meaning of homotopy theory: a better theory of quotients.

I.3 Ambiguity and homotopy quotient

In practice, quotients are rarely given by means of equivalence relation $R \subset E \times E$, but rather by graphs $R \rightrightarrows E$.

The quotient is given the connected components of the graph.

But the ambiguity of identification is captured by the loops in the graph.

The homotopy quotient is defined as the homotopy type of the graph.

It recovers the classical quotient but it also remembers the loops.

I.3 Homotopy theory – The judiciary metaphor

A path in $\underline{\mathrm{hom}}_{L(C,W)}(X, Y)$ between two zig-zags

$$f \overset{u}{\text{---}} g$$

is a witness of the identity of f and g.

Two witnesses u and v agree if there exists a witness or their agreement

$$u \overset{\alpha}{\text{---}} v$$

in $\Omega_{f,g}\underline{\mathrm{hom}}(X, Y)$.

Then the homotopy classes of paths

$$\pi_1\big(\ \underline{\mathrm{hom}}(X, Y)\ ;\ \ f, g\big) := \pi_0\big(\Omega_{f,g}\underline{\mathrm{hom}}(X, Y)\big)$$

form the set of discordances between witnesses of the identity of f and g.

I.3 Homotopy theory – The judiciary metaphor

One can think a simplicial set as a case file recording all testimonies.

The homotopy type is contractible when all witnesses agree.

But, in general, not all witnesses agree.

The non-triviality of the homotopy type is the obstruction to figure out the truth among testimonies.

(For more remarks along those lines see my notes on *Homotopy quotient* and the *Geometry of ambiguity*.)

I.3 Calculus of fractions

When (C, W) has a calculus of fractions $L(C, W)$ can be computed easily

$$\underline{\hom}_{L(C,W)}(X, Y) = \text{nerve of cat. of fractions } X \leftarrow Z \rightarrow Y$$

Recall that morphism of fractions =
$$\begin{tikzcd} X \arrow[d, equal] & Z \arrow[l, "w"'] \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[d, equal] \\ X & Z' \arrow[l, "w'"'] \arrow[r, "f'"] & Y \end{tikzcd}$$

I.3 Simplicial model structures

The axioms of model structures have been invented to compute $C[W^{-1}]$, not $L(C,W)$ (which did not existed at the time).

The computation of simplicial hom (mapping spaces) is difficult from Quillen's axioms (framings...).

Unless the category C is already enriched over $SSet$ and the model structure is compatible (simplicial model structure: $SSet$, Top, Cat...)

In this case,

$$L(C,W) = C^{cf}$$

with the simplicial enrichment from C.

I.3 ∞-groupoids and ∞-categories

What does Dwyer-Kan construction means?

The answer was found when the following correspondance was understood (Grothendieck, Pursuing Stacks)

spaces and simplicial sets up to homotopy	∞-groupoids
categories enriched in spaces or simplicial sets up to homotopy	$(\infty, 1)$-categories

I.3 ∞-groupoids and ∞-categories

Fundamental intuition for ∞-groupoids

1. objects = point of a space X
2. 1-morphism = paths in X
3. 2-morphism = homotopies between paths
4. 3-morphism = homotopies between homotopies
5. etc.

Fundamental intuition for ∞-categories

- category enriched over ∞-groupoids

The idea homotopy types = ∞-groupoids led to a great conceptual simplification of the constructions of homotopy theory.

I.3 ∞-groupoids and ∞-categories

Let CAT_∞ be the ∞-category of ∞-categories.

Let $LOC_\infty(C, W)$ be the full sub-∞-category of $C \downarrow CAT_\infty$ spanned by functors $C \to D$ sending W to invertible maps in D.

Definition

The ∞-localization of C by W is the initial object in $LOC_\infty(C, W)$.

Theorem (Dwyer-Kan)

$C \to L(C, W)$ is the ∞-localization of C by W.

I.3 ∞-groupoids and ∞-categories

$$\begin{tikzcd}[column sep=huge, row sep=huge]
 & L(C,W) \arrow[d, "\text{truncation}"] & \text{∞-categories} \\
C \arrow[ur, "\text{∞-localization}"] \arrow[r, "\text{1-localization}"'] & C[W^{-1}] & \text{1-categories}
\end{tikzcd}$$

Satisfying a universal property in a bigger category, the object $L(C,W)$ is more universal than $C[W^{-1}]$.

I.3 ∞-groupoids and ∞-categories

Now here is the amazing theorem:

Theorem (Dwyer-Kan [2])

Any simplicial category is equivalent to some $L(C, W)$.

Any ∞-category can be presented as a localization of a 1-category.

Any kind of *higher objects* can be presented as *equivalence classes of classical objects*,

i.e. as equivalence classes of diagrams of sets equipped with some structure.

I.3 ∞-groupoids and ∞-categories

Dwyer-Kan theorem becomes a bit less surprizing when it is compared with a classical result for ∞-groupoids.

Recall first that for a category C, the localization of all its arrows $L(C, C)$ is an ∞-groupoid which coincides with the homotopy type of geometric realization of the nerve of C.

Theorem

Any ∞-groupoid is equivalent to some $L(C, C)$ for C a category (or even a poset).

Proof.

Use a spatial model and consider the poset of contractible open subspaces. $\square$

I.3 ∞-groupoids and ∞-categories

Dwyer-Kan theorem proposes to think the localization techniques of homotopical algebra as

techniques to work with higher objects by means of classical objects

techniques to work on higher categories by means of 1-categories.

I.3 ∞-groupoids and ∞-categories

This apply in particular to ∞-categories themselves.

They can be defined by means of *structured diagrams of sets*.

simplical categories	structured graph of simplicial sets
quasi-categories	structured simplicial sets
complete Segal spaces	structured bisimplicial sets or structured simplicial spaces

I.3 ∞-groupoids and ∞-categories

This apply also to ∞-groupoids.

They can be define by means of structured diagrams of sets.

spaces	structured sets
Kan complexes	structured simplicial sets
Segal groupoids	structured bisimplicial sets or structured simplicial spaces

I.3 Toward naive ∞-category theory

Dwyer-Kan localization theorem says that one can always work with or within ∞-categories by means of objects in 1-categories.

In the 00's higher categories were only manipulated by means of model categories.

A number of works proved a number of model to be equivalent (simp. cat, Segal cat, CSS, quasi-cat).

Depending of what people wanted to prove they were chosing the most suitable model.

I.3 Toward naive ∞-category theory

Because of the work of Joyal, Lurie and others the theory of ∞-categories exists and the main constructions are proven to be sound in the model of quasi-categories.

- adjunctions, equivalences, essentially surjective or fully faithful functor
- diagrams, (co)limits, (co)completion, Yoneda lemma, fibrations
- accessibility, presentability, SAFT
- localizations, reflective localizations
- Kan extensions, coends
- factorization systems
- monoidal structures, monads, operads
- ...

I.3 Toward naive ∞-category theory

It is then possible to forget this particular model or any other reducing ∞-categories to sets,

i.e. not to work analytically anymore and to work synthetically instead:

not defining the objects anymore and manipulating them through the various constructions on them that have been proven to be sound.

Nowadays, topologists are moving away from model categories. They start to use higher categories naively (in the sense of naive set theory).

I.4 – Higher structures

I.4 – Higher structures

I call a classical structure are structure defined on a set.

I call higher structure structure defined on 1-categories, ∞-categories or ∞-groupoids (monoidal categories, cocomplete categories...)

More about this tomorrow with the theory of logoi!

I.5 – Interaction with logic

I.5 Interaction with logic

Connection between homotopy theory and logic	
Rewriting problems	In my opinion, the deepest connexion: the hierarchy of rewriting processes is that of higher arrows in a higher category
Higher theories	Tomorrow
Higher semantics	HoTT

I.5 Interaction with logic

Given a logical theory, there is a notion of a semantic in a category.

But is there a notion of semantic in an ∞-category?

What is the internal language of a ∞-category?

It need to have

- type
- terms
- and higher terms (cf. rewriting).

But HoTT does not have higher terms.

This seems to limit semantics for HoTT to be 1-categories and not ∞-categories...

I.5 Interaction with logic

The solution is given by Dwyer-Kan theorem.

Since any ∞-category C is a $L(C, W)$.

It is enough to define a semantic of T in C and then compose by the localization $C \to C$.

$$\begin{tikzcd}
T \arrow[dr, dashed, bend left, "\text{classical semantic}"] \arrow[ddr, dotted, bend right, "\text{higher semantic}"'] & & \text{theory} \\
& C \arrow[d, "\text{simp. loc.}"] & \text{1-categories} \\
& L(C, W) & \infty\text{-categories}
\end{tikzcd}$$

I.5 Interaction with logic

Problem: localization functors $C \to L(C, W)$ rarely preserve

- colimits $(0, +, \Sigma$, quotients, inductive types),
- limits $(1, \times, \Pi)$,
- diagonal (Id types)
- or classifying objects (universes)

So if

$$T \dashrightarrow C$$

is a semantic in the classical sense, the composition

$$T \dashrightarrow C \longrightarrow L(C, W)$$

may not be a semantic in any reasonnable sense.

We need another kind of semantic, depending somehow on W:
a homotopical semantic.

I.5 Interaction with logic

This raises an important problem of homotopy theory.

How to compute limits and colimits in $L(C, W)$ in terms of limits and colimits of C?

The answer of homotopy theorists is the notion of homotopy limits and colimits.

If the pair (C, W) is nice enough (combinatorial model structure), limits and colimits in $L(C, W)$ can be described in terms of homotopical limits and colimits in C.

If the pair (C, W) is even nicer (combinatorial simplicial model structure), homotopical limits and colimits can be described in terms of weighted limits and colimits in C (see Gambino).

I.5 Interaction with logic

	Classical semantic	**Homotopical semantic**
logical constructor	*categorical construction*	*homotopical construction*
substitution $\sigma : \Gamma \to \Delta$	fiber product σ^*	homotopy fiber product (= fiber product with a fibration)
dependent type $\Gamma \vdash X$	morphism $\Gamma.X \to \Gamma$	fibration $\Gamma.X \to \Gamma$
identity type	diagonal $X \to X \times X$	path space $X^I \to X \times X$ (= fibrant replacement of actual diagonal)
Π	right adjoint to σ^*	homotopy right adjoint to σ^* [no need of fibrant replacement if model structure is right proper]
Σ	left adjoint to σ^* (= composition)	homotopy left adjoint to σ^* (= composition) [no need of fibrant replacement because $\sigma : \Gamma \to \Delta$ is a fibration]

I.5 Interaction with logic

$$
\begin{tikzcd}
T \arrow[dr, dashed, bend left, "\text{homotopical semantic}"] \arrow[ddr, dotted, bend right, "\text{higher semantic}"'] & & \text{theory} \\
& C \arrow[d, "\text{nice simp. loc.}"', "\substack{\text{send homotopical}\\ \text{(co)lim to } \infty\text{-(co)lim}}"] & \text{1-categories} \\
& L(C, W) & \infty\text{-categories}
\end{tikzcd}
$$

(see Cisinski [3])

I.5 Interaction with logic

Voevodsky build a homotopical semantic of MLTT in $SSet$.

This should produce a higher semantic of MLTT in the ∞-category S (modulo the fact that no definition of such higher semantics exist).

The recent result of Shulman ensure that

given an ∞-topos E, there exists a presentation $E = L(C, W)$ such that C is equipped with a homotopical semantic.

This should also produce a higher semantic of MLTT in E.

Summary

- Homotopy theory is about localization of categories
- Localization of categories = proper way to define quotients of categories
- The ambiguity of identification generates a higher structure on quotients
- This structure is that of ∞-groupoid/∞-category
- all ∞-categories can be presented by a 1-category
- this leads to a way to reduce higher category to set theory
- this leads also to a notion of homotopical semantics for logical theories

That's all for today!

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