

Lectures on ∞ -Topoi

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– Lecture I –

HOMOTOPY THEORY
AND
LOCALIZATION OF CATEGORIES

Today

I want to address the following questions (and some of the answers)

What is homotopy theory?		
study of the shape of spaces	techniques to work in localizations of categories	a way to work with higher categories by means of 1-category theory
a theory of ambiguity		a new way to think about the quotient operation

How can it be connected with logic?		
rewriting problems (the deepest connexion)	higher theories (tomorrow)	higher semantics (HoTT)

Summary

- ▶ Homotopy theory is about **localization of categories**
- ▶ Localization of categories = proper way to define **quotients of categories**
- ▶ The **ambiguity of identification** generates a higher structure on quotients
- ▶ This structure is that of **∞ -groupoid/ ∞ -category**
- ▶ all ∞ -categories can be **presented by a 1-category**
- ▶ this leads to a way to reduce **higher category to set theory**
- ▶ this leads also to a notion of **homotopical semantics** for logical theories

Plan

0. Timeline of homotopy theory
1. Congruence of categories
2. Localization of categories
3. Simplicial localization & ∞ -categories
4. Interaction with logic

I.0 – Timeline of homotopy theory

I.0 Timeline of homotopy theory

1945	Eilenberg- Steenrod	homology = functors
1956	Cartan- Eilenberg	homology = derived functors = resolutions
1963	Grothendieck- Verdier	homology = chain complexes up to quasi-iso = localization of categories , resolution = replacement
1967	Gabriel- Zisman	calculus of fractions
1967	Quillen	model categories = general framework for both homotopy and homology
1980	Dwyer-Kan	simplicial localization = localization in ∞ -categories
1983	Grothendieck	homotopy = ∞ - groupoids , need ∞ -categories
1990s	Joyal	quasi-categories: there exists a theory of higher categories theory and it has the same theorems as 1-category theory
2000s	Rezk, Lurie	homotopy = ∞ - topoi homology = stable ∞ - categories

I.0 The evolution of homotopy theory

1	Congruences	homotopy and (co)homology invariants	$[S^n, -] : CW_{/\sim} \rightarrow \text{Set}$ $H_n : CW_{/\sim} \rightarrow \text{Ab}$
2	Localizations of categories	derived categories model categories	$D(\mathbb{Z}) = \text{Ch}(\mathbb{Z})[Qis^{-1}]$ $D(\text{SSet}) = \text{SSet}[W^{-1}]$ $= D(\text{Top}) = \text{Top}[W^{-1}]$
3	Higher categories	homotopy types = ∞ -gpd simplicial cat = ∞ -cat Dwyer-Kan localization	$L(\text{Ch}(\mathbb{Z}), Qis)$ $L(\text{SSet}, W)$ $= L(\text{Top}, W)$
4	Higher structures	∞ -topoi stable ∞ -categories	$\mathcal{S}, [\text{Fin}^\bullet, \mathcal{S}], \text{Sh}(\mathcal{X})$ $\text{Sp}, \text{Sh}(\mathcal{X}, \text{Sp})$

I.1 – Congruences

I.1 Congruences of categories

Definition

A **congruence** on a category C is the data of

equivalence relations $\sim_{x,y}$ on each $C(x,y)$

which are compatible with composition

$$(f \sim f' \quad \text{and} \quad g \sim g') \implies gf \sim g'f'.$$

The quotients of these equivalence relations do form again a category

$$C/\sim$$

with the same objects as C , called a **quotient of C by the congruence**.

I.1 Congruences of categories

Definition

A map $f : x \rightarrow y$ in C is called a \sim -equivalence if there exists $g, h : y \rightarrow x$ such that $gf \sim 1_x$ and $fh \sim 1_y$.

For a map $f : x \rightarrow y$ in C , let $[f] : x \rightarrow y$ be its image in C / \sim .
Then,

$[f]$ is an isomorphism iff f is \sim -equivalence.

Proposition (Universal property of quotient)

A functor $C \rightarrow D$ factors through C / \sim iff it sends \sim -equivalent maps to equal maps,

I.1 Congruences – Example 1: Homotopy equivalence

In the category of CW complexes. Let $I = [0, 1]$ be the topological interval. The relation of **homotopy of maps** is:

$$f \sim g \quad \iff \quad \exists h, \text{ s.t.} \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ i_0 \downarrow & & \parallel \\ X \times I & \xrightarrow{h} & Y \\ i_1 \uparrow & & \parallel \\ X & \xrightarrow{g} & Y \end{array}$$

Quotient by this equivalence define the **homotopy category**

$$CW / \sim$$

where

$$\text{hom}(X, Y) := \text{Top}(X, Y) / \sim .$$

I.1 Congruences – Example 1: Homotopy equivalence

One application is **Eilenberg-Steenrod axioms** for homology theories. A **reduced homology theory** is a family of functors indexed by \mathbb{Z} .

$$\begin{array}{ccc} H_n : CW & \longrightarrow & Ab \\ X & \longmapsto & H_n(X) \end{array}$$

The first axiom is **invariance by homotopy**, which means that the H_n factor through CW^\bullet / \sim

$$\begin{array}{ccc} CW & \xrightarrow{H_n} & Ab \\ \downarrow & & \nearrow \\ CW / \sim & \text{---} & \end{array}$$

I.1 Congruences – Example 2: Syntactic categories

Let T be a **first order theory**.

The arrows of the syntactic category $C(T)$ of T are **equivalence classes of formulas** for provable equivalence.

This can be used to define $C(T)$ as the quotient of some other category D by the congruence generated by equivalences of formulas.

This is the seed of "homotopical logical": proofs of equivalence behave like homotopies.

I.1 Congruences – Example 3: Chain homotopy equivalences

A **chain homotopy** between two morphisms $f, g : C_* \rightarrow D_*$ of chain complexes is a family of morphisms of $h_n : C_n \rightarrow D_{n+1}$ such that

$$\partial_{n+1}^D h_n - h_{n-1} \partial_n^C = f_n - g_n$$

$$\begin{array}{ccccccc} \dots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}^C} & C_n & \xrightarrow{\partial_n^C} & C_{n-1} & \longrightarrow & \dots \\ & & \downarrow & \swarrow h_n & \downarrow & \swarrow h_{n-1} & \downarrow & & \\ & & f_{n+1} - g_{n+1} & & f_n - g_n & & f_{n-1} - g_{n-1} & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & D_{n+1} & \xrightarrow{\partial_{n+1}^D} & D_n & \xrightarrow{\partial_n^D} & D_{n-1} & \longrightarrow & \dots \end{array}$$

Projective or injective resolutions of abelian group are **unique up to chain homotopy**.

This provide another setting with the same structure as homotopy theory.

I.2 – Localizations

I.2 Weak equivalences

The definition of higher homotopy groups, prompted homotopy theory to focus on a weaker notion of equivalence. A map $f : X \rightarrow Y$ is **weak homotopy equivalence** if it induces bijections between all homotopy invariants:

$$f : X \rightarrow Y \text{ WHE} \iff \pi_n(f) : \pi_n(X) \simeq \pi_n(Y).$$

Any homotopy equivalence is a weak homotopy equivalence.

Whitehead: for CW-complexes, homotopy equivalence = weak homotopy equivalence.

But the constructions of algebraic topology, when defined on all spaces, need to be invariants by this larger class of maps.

1.2 Quasi-isomorphisms

On the algebraic side, the study of **chain complexes** also motivate weaker notion than **chain homotopy equivalence**. A morphism of chain complexes **quasi-isomorphism** if it induces isomorphisms of all homology groups:

$$f : A \rightarrow B \text{ Qis} \iff H_n(f) : H_n(A) \simeq H_n(B).$$

Two projective resolutions are chain homotopy equivalent. And so are their image by any functor. But a projective and a flat resolution need not be, they are only **quasi-isomorphic**. And so are their image by any functor (for which they are both admissible resolutions).

Grothendieck emphasized that constructions on arbitrary chain complexes must be **invariant under quasi-isomorphism** and not only chain homotopy equivalence.

I.2 Localization of categories

The abstraction of these situations is the following.

We have a **category** C and $W \subset C^{\rightarrow}$ a **family of arrows** in C

C	$W \subset C^{\rightarrow}$
Spaces	weak homotopy equivalences
Chain complexes	quasi-isomorphism

and we are interested in **functors** $C \rightarrow D$ sending **maps in W to isomorphisms** in D :

$$\begin{array}{ccc} C & \longrightarrow & D \\ W & \longmapsto & Iso(D). \end{array}$$

I.2 Localization of categories

Let $LOC(C, W)$ be the full subcategory of $C \downarrow CAT$ spanned by the functors $C \rightarrow D$ sending maps in W to isomorphisms in D

Definition

The **localization** of C along W is the initial object $C \rightarrow C[W^{-1}]$ in the category $LOC(C, W)$.

Theorem

The localization of C along W always exists.

We going to see an explicit description soon.

I.2 Motivation for localizations

	Homotopy (spaces)	Homology (chain complexes)
(Strong) equivalences	invertible up to homotopy	invertible up to chain homotopy
Weak equivalences	induce isomorphism on homotopy invariants π_n	induce isomorphism on homology invariants H_n
Coincide on some specific objects	CW complexes	complexes of projective or injective modules
Functors of interest must send WE to isomorphisms	(co)homology theories, classifying spaces	derived functors (Tor , Ext)
This prompted the notion of localization of categories as domain of the functors of interest	$Top[WE^{-1}] \xrightarrow{H_n} Ab$ $Top[WE^{-1}]^{op} \xrightarrow{H^n} Ab$	$Ch(\mathbb{Z})[Qis^{-1}] \xrightarrow{Tor_n} Ab$ $Ch(\mathbb{Z})[Qis^{-1}]^{op} \xrightarrow{Ext^n} Ab$

I.2 Motivation for localizations

$$\begin{array}{ccc} & CW / \sim & \xrightarrow{H_n} Ab \\ & \downarrow & \nearrow H_n \\ Top & \longrightarrow & Top[WE^{-1}] \end{array}$$

$$\begin{array}{ccc} \left\{ \begin{array}{l} \text{bounded cplx} \\ \text{of proj. mod.} \end{array} \right\} / \sim & \xrightarrow{Tor_n} Ab \\ & \nearrow Tor_n \\ Ch(\mathbb{Z}) & \longrightarrow & Ch(\mathbb{Z})[Qis^{-1}] \end{array}$$

nicer categories
to work in:
all limits/colimits

I.2 Localizations = quotients

The notion localization of category is badly named.

From the **point of view of arrows**, it is indeed a localization, analogous to the notion for monoids or rings.

But, from the **point of view of objects**, it is in fact a **quotient!**

Indeed, **the purpose of localizations is to force objects to become isomorphic.**

I.2 Localizations = quotients

Many features of localizations become clearer when thought in terms of quotient.

Quotients	
Sets	Categories
relation $R \subset X \times X$	class of arrows $W \subset C^{\rightarrow}$
reflexivity	W contains isomorphisms
transitivity	W stable by composition
symmetry	W has left and right cancellation
symmetry + reflexivity	W has the "2 out of 3" property
equivalence relation	W contains isomorphisms + has the "2 out of 3" property (pre-saturated)
quotient X/R	localization $C[W^{-1}]$

I.2 Computations of localizations

We are going to review several techniques to deal with the localization of a category

1. zig-zags
2. congruences
3. fractions
4. reflective subcategories
5. model structures

I.2.1 – Computation of localizations Zig-Zags

I.2.1 Computations of localizations – zig-zags

Localizations of categories are called this way because they are related to localizations of rings and monoids.

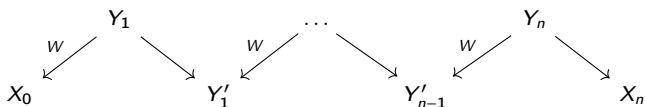
Their description can be done by introducing **formal symbols** w^{-1} for the inverse of map w and imposing **relations** $ww^{-1} = 1$ and $w^{-1}w = 1$.

An arrow in $C[W^{-1}]$ will then be words like $w^{-1}fv^{-1}gu^{-1}$, up to rewriting.

I.2.1 Computations of localizations – zig-zags

But with categories, we have a more **geometric picture**.

An arrow in $C[W^{-1}]$ is an equivalence class of **zig-zags**



where the maps going to the left are in W .

I.2.1 Computations of localizations – zig-zags

The relations between zig-zags (rewriting rules) are given by

1. contraction of composition

$$\dots X \xleftarrow{w} Z \xleftarrow{w'} Y \dots = \dots X \xleftarrow{ww'} Y \dots$$

$$\dots X \xrightarrow{f'} Z \xrightarrow{f} Y \dots = \dots X \xrightarrow{ff'} Y \dots$$

2. contraction of identities

$$\dots X \xleftarrow{w} Z \xlongequal{\quad} Z \xleftarrow{w'} Y \dots = \dots X \xleftarrow{ww'} Y \dots$$

$$\dots X \xrightarrow{f'} Z \xlongequal{\quad} Z \xrightarrow{f} Y \dots = \dots X \xrightarrow{ff'} Y \dots$$

3. contraction of equivalences

$$\dots X \xleftarrow{w} Z \xrightarrow{w} X \dots = \dots X \dots$$

$$\dots X \xrightarrow{w} Z \xleftarrow{w} X \dots = \dots X \dots$$

I.2.1 Computations of localizations – zig-zags

The contraction of equivalences can be given a geometric picture.

A morphisms of zig-zag is defined as a diagram

$$\begin{array}{ccccccc}
 X_0 & \longleftarrow & Y_1 & \longrightarrow & \dots & \longleftarrow & Y_n & \longrightarrow & X_n \\
 \parallel & & \downarrow u_1 & & & & \downarrow u_n & & \parallel \\
 X_0 & \longleftarrow & Z_1 & \longrightarrow & \dots & \longleftarrow & Z_n & \longrightarrow & X_n
 \end{array}$$

where the all vertical maps are in W .

In particular we have

$$\begin{array}{ccc}
 X & \xleftarrow{w} & Y & \xrightarrow{w} & X \\
 \parallel & & \downarrow w & & \parallel \\
 X & \xlongequal{\quad} & X & \xlongequal{\quad} & X
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 Y & \xrightarrow{w} & X & \xleftarrow{w} & Y \\
 \parallel & & \uparrow w & & \parallel \\
 Y & \xlongequal{\quad} & Y & \xlongequal{\quad} & Y
 \end{array}$$

$$ww^{-1} = id_X$$

$$w^{-1}w = id_Y$$

The contraction of equivalences can be replaced by the relation "being connected by a zig-zag of morphisms of zig-zags"

I.2.1 Computations of localizations – zig-zags

A morphism

$$\begin{array}{ccccc} X & \xleftarrow{w} & Z & \xrightarrow{f} & Y \\ \parallel & & \downarrow u & & \parallel \\ X & \xleftarrow{w'} & Z' & \xrightarrow{f'} & Y \end{array}$$

must be read as the relation

$$f'(w')^{-1} = f'uu^{-1}(w')^{-1} = f'u(w'u)^{-1} = fw^{-1}$$

1.2.1 Homotopical algebra

The description in terms of zig-zags is useful to prove that the localization exists, but not for computing explicitly the hom sets of $C[W^{-1}]$.

Many techniques have been invented to do so.

All are based on the same idea: **reducing the length of zig-zags**

congruences	length 1	\rightarrow
reflective localizations	length $1\frac{1}{2}$	$\rightarrow\leftarrow$
calculus of fractions	length 2	$\rightarrow\leftarrow, \leftarrow\rightarrow$
model structures	length 3	$\leftarrow\rightarrow\leftarrow$

Altogether these **localization techniques** form the topic of **homotopical algebra**, so named because of its origin in homotopy theory.

I.2.2 – Computation of localizations

Congruences

I.2.2 Computations of localizations – Congruences

An **interval** is defined as an object I with two disjoint points $i_0, i_1 : 1 \rightarrow I$.

If C has cartesian products, an interval defined a relation of homotopy and an associated congruence.

Let us say that a congruence is **homotopical** if it is given by an interval.

I.2.2 Computations of localizations – Congruences

Proposition

If a congruence on C is homotopical, then a functor $C \rightarrow D$ factors through C / \sim iff it *sends \sim -equivalences to isomorphisms*, i.e. the quotients C / \sim is a localization:

$$CW[HE^{-1}] = CW / \sim \quad \text{and} \quad Ch(\mathbb{Z})[CHE^{-1}] = Ch(\mathbb{Z}) / \sim .$$

Proof: $X \times I \rightarrow X$ is an \sim -equivalence. It is inverted by the quotient. Reciprocally, if $X \times I \rightarrow X$ is inverted, then \sim -equivalence maps are identified by the relation

$$\begin{array}{ccccc} X & \xlongequal{\quad} & X & \xrightarrow{f} & Y \\ \parallel & & \downarrow i_0 & & \parallel \\ X & \xleftarrow{p} & X \times I & \xrightarrow{h} & Y \\ \parallel & & \uparrow i_1 & & \parallel \\ X & \xlongequal{\quad} & X & \xrightarrow{g} & Y \end{array}$$

I.2.2 Computations of localizations – Congruences

Method 1: If $W = \sim$ -equivalence for a homotopical congruence, then

$$C[W^{-1}] = C / \sim .$$

I.2.3 – Computation of localizations

Reflective localizations

I.2.3 Reflective localizations

When C is a cocomplete category localization can be computed in a very efficient way

The category $LOC_{cc}(C, W)$ is the full subcategory of $C \downarrow CAT_{cc}$ spanned by the cocontinuous functors $C \rightarrow D$ sending maps in W to isomorphisms in D .

Definition (cc-Localization)

The cc-localization of C along W is the initial object $C \rightarrow C[W^{-1}]$ in the category $LOC_{cc}(C, W)$.

I.2.3 Reflective localizations

And now, my favorite theorem in category theory.

Theorem (Fundamental thm of CT)

If W is a small category of small objects in C^{\rightarrow} , the cc-localization $C \rightarrow C[W^{-1}]$ has a fully faithful right adjoint.

The localization can be computed as a *full subcategory!*

This is the **best** one can hope to compute localizations.

Recall that a localization is in fact a quotient, the theorem says that the quotient $C \rightarrow C[W^{-1}]$ has a **distinguished section** (terminal in the category of sections).

I.2.3 Reflective localizations

The image of $C[W^{-1}] \hookrightarrow C$ is the category C_W of **local objects**:

X is **local** if, for all $w : A \rightarrow B$ in W

$$\mathrm{hom}(B, X) \xrightarrow{\cong} \mathrm{hom}(A, X)$$

(= X believes that w is an isomorphism)

The localization coincide with the reflection

$$C \rightarrow C[W^{-1}] \quad = \quad P : C \rightarrow C_W$$

I.2.3 Reflective localizations

A morphism in $C[W^{-1}]$ is then a zigzag

$$X \longrightarrow PY \longleftarrow Y.$$

But since the map $Y \rightarrow PY$ is fixed once and for all, reflective localizations actually compute maps in $C[W^{-1}]$ as a **single map**

$$X \longrightarrow PY.$$

Notice that this is more efficient than congruences since no quotient is needed

$$\text{hom}_{C[W^{-1}]}(X, Y) = \text{hom}_C(PX, PY) = \text{hom}_C(X, PY).$$

I.2.3 Reflective localizations

Method 2: If the localization is taken in cocomplete categories and W small, then

$$\mathrm{hom}_{C[W^{-1}]}(X, Y) = \mathrm{hom}_C(X, PY)$$

where $P : C \rightarrow C_W$ is the **reflection into local objects**.

Reflective localizations are a particular case of calculus of fractions (see Gabriel-Zisman).

I.2.4 – Computation of localizations

Calculus of fractions

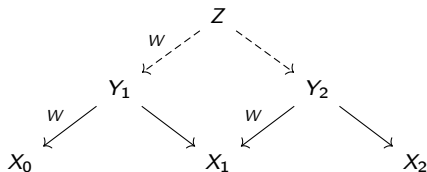
I.2.4 Calculus of fractions

The idea behind fractions is to use only zig-zags of length 2.

A **right fraction** is a diagram $X \xleftarrow{w} Z \xrightarrow{f} Y$ where w is an equivalence. It is read as a morphism $X \rightarrow Y$ and can be written suggestively fw^{-1} .

A **left fraction** is a diagram $X \xrightarrow{f} Z \xleftarrow{w} Y$ where w is an equivalence. It can be written $w^{-1}f$.

The axioms of the calculus of fractions are ways to **compose fractions**. Essentially, they amount to replace a left fraction by a right fraction.



I.2.4 Calculus of fractions

Gabriel-Zisman: the *homotopy category of simplicial sets* can be defined as

$$Ho(SSet) = SSet[AE^{-1}] = (SSet / \sim)[ae^{-1}]$$

where AE is the class of anodyne extensions. The intermediate category $SSet / \sim$ has a calculus of fractions for the image of AE .

Verdier: the *derived category of chain complexes* can be defined as

$$D(\mathbb{Z}) = Ch(\mathbb{Z})[Qis^{-1}] = (Ch(\mathbb{Z}) / \sim)[qis^{-1}]$$

where Qis is the class of quasi-isomorphism. The intermediate category $K(\mathbb{Z}) = Ch(\mathbb{Z}) / \sim$ has a calculus of fractions for the image of Qis .

I.2.4 Calculus of fractions

Method 3: If (C, W) has a calculus of fractions

$$\begin{aligned}\mathrm{hom}_{C[W^{-1}]}(X, Y) &= \mathrm{colim}_{w:Z \rightarrow X} \mathrm{hom}_C(Z, Y) \\ &= \pi_0(\text{category of fractions } X \leftarrow Z \rightarrow Y)\end{aligned}$$

$$\text{Morphism of fractions} = \begin{array}{ccccc} X & \xleftarrow{w} & Z & \xrightarrow{f} & Y \\ \parallel & & \downarrow u & & \parallel \\ X & \xleftarrow{w'} & Z' & \xrightarrow{f'} & Y \end{array}$$

Overall, calculus of fractions are rare.

Model structures provide a more flexible notion.

I.2.5 – Computation of localizations

Model structures

I.2.5 Model structures

Definition

1. A pair (C, W) is called **pre-saturated** if W has the "2 out of 3" property and contains all isomorphisms.
2. A **model structure** on a pre-saturated pair (C, W) is the choice of
 - 2.1 two classes of maps Cof , Fib of C
 - 2.2 such that $(Cof \cap W, Fib)$ and $(Cof, Fib \cap W)$ are weak factorization systems.

$$\begin{array}{ccc} X & \xrightarrow[\simeq]{Cof \cap W} & X_f \\ \text{Cof} \downarrow & \searrow f & \downarrow Fib \\ Y^f & \xrightarrow[\simeq]{Fib \cap W} & Y \end{array}$$

I.2.5 Model structures

A **model structure** on a pair (C, W) can be understood as a way to use only zig-zags of length 3.

A **double fraction** is a diagram $X \xleftarrow{u} Z' \xrightarrow{f} Z \xleftarrow{w} Y$ where w and u are equivalences. It can be written $w^{-1}fu^{-1}$.

Model categories consider **specific double fractions**:

$$\begin{array}{ccccccc} & & \text{cofibrant object} & & \text{fibrant object} & & \\ & & \underbrace{} & & \underbrace{} & & \\ X & \xleftarrow{u} & QX & \xrightarrow{f} & RY & \xleftarrow{w} & Y \\ & \underbrace{} & & & & \underbrace{} & \\ & \text{triv. fibration} & & & & \text{triv. cofibration} & \end{array}$$

I.2.5 Model structures

Method 4: If (C, W) has a model structure

$$\mathrm{hom}_{C[W^{-1}]}(X, Y) = \mathrm{hom}_C(QX, RY) / \sim$$

where QX is a **cofibrant replacement** of X

and RY is a **fibrant replacement** of Y .

(Since the method come back to congruence, it may look like zig-zags of length 1 only are used. But, in practice, it is important to let QX and RY vary, so this is really zig-zags of length 3.)

I.2.5 Model structures

The **strategy** of model structure is complex

1. define a **homotopy relation** and homotopy equivalence
2. define a class of **good objects** C^{cf} (fibrant + cofibrant objects)
3. over which homotopy equivalence and equivalence **coincide**
4. and such that every object is **equivalent to a good one**
5. then the localization is simply given by a **congruence on good objects**

$$C[W^{-1}] = C^{cf} / \sim$$

Nonetheless, model structures are quite frequent and are a very effective tool.

I.2 Summary

Methods to compute the hom sets $[X, Y]$ in a localization $C[W^{-1}]$

congruences	length 1	\rightarrow	$[X, Y] = \text{hom}(X, Y) / \sim$
reflective localizations	length $1\frac{1}{2}$	$\rightarrow\leftarrow$	$[X, Y] = \text{hom}(X, PY)$
calculus of (right) fractions	length 2	\leftrightarrow	$[X, Y] = \text{colim}_{w:Z \rightarrow X} \text{hom}(Z, Y)$
model structures	length 3	$\leftrightarrow\leftarrow$	$[X, Y] = \text{hom}(QX, RY) / \sim$

I.3 – Simplicial localization and ∞ -categories

I.3 Simplicial localization

Recall that a simplicial set $X = \{X_n\}$ has a set of connected components

$$\pi_0(X) = \text{quotient of } X_0 \text{ by relation image of } X_1 \rightarrow X_0 \times X_0$$

The **simplicial localization** of a pair (C, W) is a **category enriched over simplicial sets** $L(C, W)$ enhancing $C[W^{-1}]$ in the sense that

$$\pi_0(\underline{\text{hom}}_{L(C, W)}(X, Y)) = \text{hom}_{C[W^{-1}]}(X, Y).$$

I.3 Simplicial localization

The morphisms of zig-zags

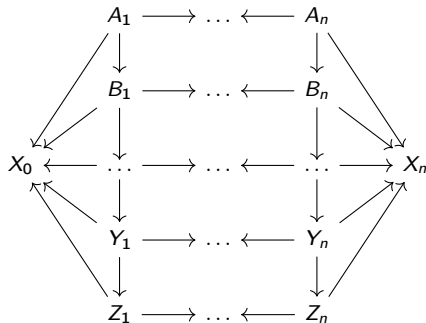
$$\begin{array}{ccccccc} X_0 & \longleftarrow & Y_1 & \longrightarrow & \dots & \longleftarrow & Y_n & \longrightarrow & X_n \\ \parallel & & \downarrow u_1 & & & & \downarrow u_n & & \parallel \\ X_0 & \longleftarrow & Z_1 & \longrightarrow & \dots & \longleftarrow & Z_n & \longrightarrow & X_n \end{array}$$

define a category of zig-zags $X_0 \rightsquigarrow X_n$ of length n .

I.3 Simplicial localization

By taking the nerve of this category, we get a **simplicial set**.

An n -simplex is a **hammock**



From these simplicial sets and the rewriting rules on zig-zags, Dwyer & Kan construct a **category enriched over simplicial sets**

$$L(C, W).$$

I.3 Simplicial localization

The category $L(C, W)$ contains all the information about the localization $C[W^{-1}]$ because

$$\pi_0(\underline{\mathrm{hom}}_{\mathcal{L}(C, W)}(X, Y)) = \mathrm{hom}_{C[W^{-1}]}(X, Y)$$

But it contains **more information** since the $\mathrm{hom}_{\mathcal{L}(C, W)}(X, Y)$ can have **non-trivial higher homotopy**.

Example: for $C = \{0 \rightrightarrows 1\}$, we have

$$\underline{\mathrm{hom}}_{\mathcal{L}(C, C \rightarrow)}(1, 1) \simeq S^1$$

I.3 Ambiguity and homotopy quotient

The simplicial structure $\underline{\text{hom}}(X, Y) := \underline{\text{hom}}_{\mathcal{L}(C, W)}(X, Y)$ encode the **ambiguity of rewritings** in the description of arrow by means of zigzags.

The fact that there is no canonical way to identify two zig-zags creates a structure.

We touch here another meaning of homotopy theory: **a better theory of quotients**.

I.3 Ambiguity and homotopy quotient

In practice, quotients are rarely given by means of equivalence relation $R \subset E \times E$, but rather by graphs $R \rightrightarrows E$.

The quotient is given the connected components of the graph.

But the ambiguity of identification is captured by the loops in the graph.

The **homotopy quotient** is defined as the homotopy type of the graph.

It recovers the classical quotient but it also remembers the loops.

I.3 Homotopy theory – The judiciary metaphor

A path in $\underline{\text{hom}}_{\mathcal{L}(C,W)}(X, Y)$ between two zig-zags

$$f \xrightarrow{u} g$$

is a **witness** of the identity of f and g .

Two witnesses u and v **agree** if there exists a witness of their agreement

$$u \xrightarrow{\alpha} v$$

in $\Omega_{f,g}\underline{\text{hom}}(X, Y)$.

Then the **homotopy classes of paths**

$$\pi_1(\underline{\text{hom}}(X, Y) ; f, g) := \pi_0(\Omega_{f,g}\underline{\text{hom}}(X, Y))$$

form the set of **discordances** between witnesses of the identity of f and g .

I.3 Homotopy theory – The judiciary metaphor

One can think a simplicial set as a **case file** recording all **testimonies**.

The homotopy type is **contractible** when **all witnesses agree**.

But, in general, **not all witnesses agree**.

The non-triviality of the homotopy type is the **obstruction to figure out the truth** among testimonies.

(For more remarks along those lines see my notes on *Homotopy quotient* and the *Geometry of ambiguity*.)

I.3 Calculus of fractions

When (C, W) has a **calculus of fractions** $L(C, W)$ can be computed easily

hom $_{L(C, W)}(X, Y) = \text{nerve of cat. of fractions } X \leftarrow Z \rightarrow Y$

Recall that morphism of fractions =

$$\begin{array}{ccccc} X & \xleftarrow{w} & Z & \xrightarrow{f} & Y \\ \parallel & & \downarrow u & & \parallel \\ X & \xleftarrow{w'} & Z' & \xrightarrow{f'} & Y \end{array}$$

I.3 Simplicial model structures

The axioms of model structures have been invented to compute $C[W^{-1}]$, not $L(C, W)$ (which did not exist at the time).

The computation of simplicial hom (mapping spaces) is difficult from Quillen's axioms (framings...).

Unless the category C is already enriched over $S\text{Set}$ and the model structure is compatible (**simplicial model structure**: $S\text{Set}$, Top , Cat ...)

In this case,

$$L(C, W) = C^{cf}$$

with the simplicial enrichment from C .

I.3 ∞ -groupoids and ∞ -categories

What does Dwyer-Kan construction means?

The answer was found when the following correspondance was understood (Grothendieck, Pursuing Stacks)

spaces and simplicial sets up to homotopy	∞ -groupoids
categories enriched in spaces or simplicial sets up to homotopy	$(\infty, 1)$ -categories

I.3 ∞ -groupoids and ∞ -categories

Fundamental intuition for ∞ -groupoids

1. objects = point of a space X
2. 1-morphism = paths in X
3. 2-morphism = homotopies between paths
4. 3-morphism = homotopies between homotopies
5. etc.

Fundamental intuition for ∞ -categories

- ▶ category enriched over ∞ -groupoids

The idea **homotopy types = ∞ -groupoids** led to a great **conceptual simplification** of the constructions of homotopy theory.

I.3 ∞ -groupoids and ∞ -categories

Let CAT_∞ be the ∞ -category of ∞ -categories.

Let $LOC_\infty(C, W)$ be the full sub- ∞ -category of $C \downarrow CAT_\infty$ spanned by functors $C \rightarrow D$ sending W to invertible maps in D .

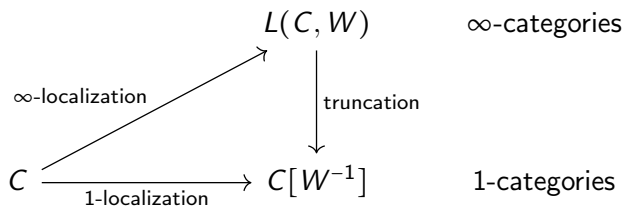
Definition

The ∞ -localization of C by W is the initial object in $LOC_\infty(C, W)$.

Theorem (Dwyer-Kan)

$C \rightarrow L(C, W)$ is the ∞ -localization of C by W .

I.3 ∞ -groupoids and ∞ -categories



Satisfying a universal property in a **bigger category**,
the object $L(C, W)$ is **more universal** than $C[W^{-1}]$.

I.3 ∞ -groupoids and ∞ -categories

Now here is the amazing theorem:

Theorem (Dwyer-Kan [2])

Any simplicial category is equivalent to some $L(C, W)$.

Any ∞ -category can be presented as a localization of a 1-category.

Any kind of *higher objects* can be presented as *equivalence classes of classical objects*,

i.e. as equivalence classes of diagrams of sets equipped with some structure.

I.3 ∞ -groupoids and ∞ -categories

Dwyer-Kan theorem becomes a bit less surprising when it is compared with a classical result for ∞ -groupoids.

Recall first that for a category C , the localization of all its arrows $L(C, C)$ is an ∞ -groupoid which coincides with the homotopy type of geometric realization of the nerve of C .

Theorem

Any ∞ -groupoid is equivalent to some $L(C, C)$ for C a category (or even a poset).

Proof.

Use a spatial model and consider the poset of contractible open subspaces. □

1.3 ∞ -groupoids and ∞ -categories

Dwyer-Kan theorem proposes to think the localization techniques of **homotopical algebra** as

techniques to work with higher objects by means of classical objects

techniques to work on higher categories by means of 1-categories.

I.3 ∞ -groupoids and ∞ -categories

This apply in particular to ∞ -categories themselves.

They can be defined by means of *structured diagrams of sets*.

simplicial categories	structured graph of simplicial sets
quasi-categories	structured simplicial sets
complete Segal spaces	structured bisimplicial sets or structured simplicial spaces

I.3 ∞ -groupoids and ∞ -categories

This apply also to ∞ -groupoids.

They can be define by means of structured diagrams of sets.

spaces	structured sets
Kan complexes	structured simplicial sets
Segal groupoids	structured bisimplicial sets or structured simplicial spaces

I.3 Toward naive ∞ -category theory

Dwyer-Kan localization theorem says that one can always work **with or within** ∞ -categories by means of objects in 1-categories.

In the 00's higher categories were only manipulated by means of model categories.

A number of works proved a number of model to be equivalent (simp. cat, Segal cat, CSS, quasi-cat).

Depending of what people wanted to prove they were choosing the most suitable model.

I.3 Toward naive ∞ -category theory

Because of the work of Joyal, Lurie and others the theory of ∞ -categories exists and the main constructions are proven to be sound in the model of [quasi-categories](#).

- ▶ adjunctions, equivalences, essentially surjective or fully faithful functor
- ▶ diagrams, (co)limits, (co)completion, Yoneda lemma, fibrations
- ▶ accessibility, presentability, SAFT
- ▶ localizations, reflective localizations
- ▶ Kan extensions, coends
- ▶ factorization systems
- ▶ monoidal structures, monads, operads
- ▶ ...

I.3 Toward naive ∞ -category theory

It is then possible to **forget this particular model** or any other reducing ∞ -categories to sets,

i.e. **not to work analytically** anymore and to **work synthetically** instead:

not defining the objects anymore and manipulating them through the various constructions on them that have been proven to be sound.

Nowadays, topologists are moving away from model categories. They start to use higher categories **naively** (in the sense of naive set theory).

I.4 – Higher structures

I.4 – Higher structures

I call a **classical structure** are structure defined on a set.

I call **higher structure** structure defined on 1-categories, ∞ -categories or ∞ -groupoids (monoidal categories, cocomplete categories...)

More about this **tomorrow** with the theory of logoi!

I.5 – Interaction with logic

I.5 Interaction with logic

Connection between homotopy theory and logic	
Rewriting problems	In my opinion, the deepest connexion: the hierarchy of rewriting processes is that of higher arrows in a higher category
Higher theories	Tomorrow
Higher semantics	HoTT

I.5 Interaction with logic

Given a logical theory, there is a notion of a **semantic in a category**.

But is there a notion of **semantic in an ∞ -category**?

What is the **internal language** of a ∞ -category?

It need to have

- ▶ type
- ▶ terms
- ▶ and **higher terms** (cf. rewriting).

But HoTT does not have higher terms.

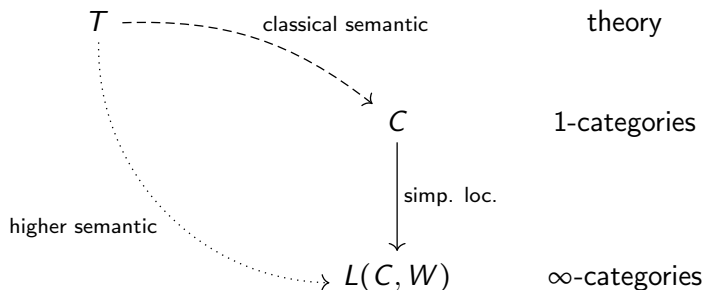
This seems to limit semantics for HoTT to be 1-categories and not ∞ -categories...

I.5 Interaction with logic

The **solution** is given by **Dwyer-Kan theorem**.

Since any ∞ -category \mathcal{C} is a $L(C, W)$.

It is enough to define a semantic of T in C and then compose by the localization $C \rightarrow \mathcal{C}$.



I.5 Interaction with logic

Problem: localization functors $C \rightarrow L(C, W)$ rarely preserve

- ▶ colimits ($0, +, \Sigma$, quotients, inductive types),
- ▶ limits ($1, \times, \Pi$),
- ▶ diagonal (*Id* types)
- ▶ or classifying objects (universes)

So if

$$T \dashrightarrow C$$

is a semantic in the classical sense, the composition

$$T \dashrightarrow C \rightarrow L(C, W)$$

may not be a semantic in any reasonable sense.

We need another kind of semantic, depending somehow on W :
a **homotopical semantic**.

I.5 Interaction with logic

This raises an important problem of homotopy theory.

How to compute limits and colimits in $L(C, W)$ in terms of limits and colimits of C ?

The answer of homotopy theorists is the notion of **homotopy limits and colimits**.

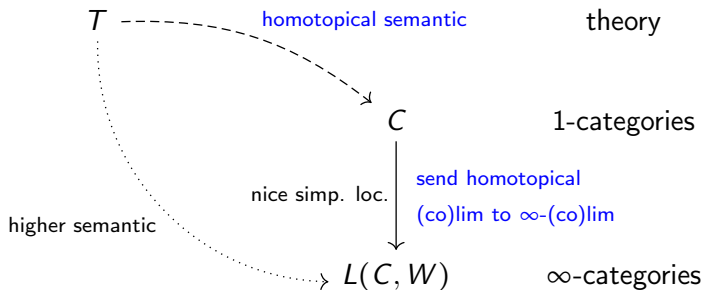
If the pair (C, W) is nice enough (combinatorial model structure), limits and colimits in $L(C, W)$ can be described in terms of **homotopical limits and colimits** in C .

If the pair (C, W) is even nicer (combinatorial simplicial model structure), homotopical limits and colimits can be described in terms of **weighted limits and colimits** in C (see Gambino).

I.5 Interaction with logic

	Classical semantic	Homotopical semantic
<i>logical constructor</i>	<i>categorical construction</i>	<i>homotopical construction</i>
substitution $\sigma : \Gamma \rightarrow \Delta$	fiber product σ^*	homotopy fiber product (= fiber product with a fibration)
dependent type $\Gamma \vdash X$	morphism $\Gamma.X \rightarrow \Gamma$	fibration $\Gamma.X \rightarrow \Gamma$
identity type	diagonal $X \rightarrow X \times X$	path space $X^I \rightarrow X \times X$ (= fibrant replacement of actual diagonal)
Π	right adjoint to σ^*	homotopy right adjoint to σ^* [no need of fibrant replacement if model structure is right proper]
Σ	left adjoint to σ^* (= composition)	homotopy left adjoint to σ^* (= composition) [no need of fibrant replacement because $\sigma : \Gamma \rightarrow \Delta$ is a fibration]

I.5 Interaction with logic



I.5 Interaction with logic

Voevodsky build a homotopical semantic of MLTT in $S\text{Set}$.

This should produce a higher semantic of MLTT in the ∞ -category \mathcal{S} (modulo the fact that no definition of such higher semantics exist).

The recent result of Shulman ensure that

given an ∞ -topos \mathcal{E} , there exists a presentation $\mathcal{E} = L(C, W)$ such that C is equipped with a homotopical semantic.






This should also produce a higher semantic of MLTT in \mathcal{E} .

Summary







- ▶ Homotopy theory is about **localization of categories**
- ▶ Localization of categories = proper way to define **quotients of categories**
- ▶ The **ambiguity of identification** generates a higher structure on quotients
- ▶ This structure is that of **∞ -groupoid/ ∞ -category**
- ▶ all ∞ -categories can be **presented by a 1-category**
- ▶ this leads to a way to reduce **higher category to set theory**
- ▶ this leads also to a notion of **homotopical semantics** for logical theories

That's all for today!



Bibliography on quasi-categories

-  Joyal – Notes on quasi-categories (2008)
-  Joyal – The theory of quasi-category – Volume I (2008)
-  Cisinski – Higher Categories and Homotopical Algebra
-  Rezk – Notes on ∞ -categories
-  Lurie – HTT book






Bibliography on higher topoi

-  Anel-Joyal – Topo-logie (2019)
-  Anel – notes from HoTTEST seminar (2019)
-  Anel – notes from NY oktoberfest (2018)
-  Anel – notes from YaMCATS & PSSL (2018)
-  Rezk – original note on homotopy topos
-  Lurie – HTT book

Bibliography on the homotopization of mathematics

-  Anel – The homotopy quotient
-  Anel – The geometry of ambiguity

Bibliography – Others

-  Dwyer-Kan – Constructing simplicial localization (1980)
-  Dwyer-Kan – Equivalences between homotopy theories of diagrams (1983)
-  Dwyer-Hirschhorn-Kan-Smith – Homotopy limit functors on model categories and homotopical categories
-  Gambino – Weighted limits in simplicial homotopy theory
-  Shulman – All $(\infty, 1)$ -toposes have strict univalent universes