The Geometry of Ambiguity

An Introduction to the Ideas of Derived Geometry

Mathieu Anel*

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*Department of Philosophy, Carnegie Mellon University. mathieu.anel@gmail.com
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1 Introduction

Derived geometry is a theory of geometry whose aim it to provide a better behavior of singular points than in algebraic, complex, or differentiable geometries. It is named after derived categories, derived functors, derived tensor products, and pretty much anything "derived", because it proposes a setting where the natural constructions of all the notions give directly the derived versions. Within derived geometry, nothing has to be derived anymore.

As often in the history of mathematics, most of the computing methods to deal with singular points were invented before the proper formalization of a theory organizing and justifying them (e.g., , Koszul resolutions, Chevalley complexes, equivariant methods). Perhaps what is really new in derived geometry is not so much its methods as the new understanding it proposes. Derived geometry has successfully interpreted *in terms of geometry* these previously ad hoc constructions. We shall come back to this in our conclusion.

Tangent Complexes The easiest way to introduce derived geometry is probably with the following analogy. Recall that homological algebra can be read as the enhancement of the theory of vector spaces into the theory of chain complexes; then derived geometry is to geometry (ordinary topological spaces, manifolds, schemes, etc.) what chain complexes are to vector spaces. This analogy is good enough because a number of features of chain complexes do have analogues in "derived spaces". For example, complexes in positive or negative degrees have corresponding derived spaces called, respectively, *stacks* and *derived schemes* (or *derived manifolds*, depending on the context). It is also possible to truncate derived spaces and extract analogues of Z_0 and H_0.

In fact, there exists a precise comparison between the two theories: it happens that the tangent spaces to derived spaces are naturally chain complexes and no longer vector spaces. The relationship between the different derived spaces and their tangent is summarized in Table 1, which is useful to keep in mind.

Table 1: Tangent spaces

Type of space	Structure of the tangent	
scheme/manifold	vector space	T_0
stacks	chain complex	$\cdots \to T_1 \to T_0$
derived scheme/manifold	cochain complex	$T_0 \to T_{-1} \to \dots$
general derived space	unbounded complex	$\cdots \to T_1 \to T_0 \to T_{-1} \to \dots$

It turns out that these tangent complexes are not so difficult to compute in practice. Examples involve Hochschild complexes, tangent sheaf cohomology and group cohomology. Actually, they are somehow so easy to compute that people stumbled upon them before realizing what they were. For example, it is a classical fact of deformation theory that the tangent space at a point of a moduli space[1] is given by a homology group of some complex, but the rest of these complexes were for a long time overlooked. It is only by realizing that they were more regular as a whole (they have a Lie algebra structure and are often perfect complexes) than the sole homology group of interest that people came up with the idea that moduli spaces would be more regular if they could be defined such that the whole complex would be the tangent space. This tangent Lie structure was used in deformation theory [24], emphasized by Drinfeld [15] and theorized by the "derived deformation theory" (DDT) [30, 29], and the "perfection" of tangent complexes can be understood as the meaning of the "hidden smoothness" of moduli spaces [12]. Altogether, these ideas sprung up mostly in the late 1980s to 1990s.

[1] A *moduli space* is a space classifying something, it could be the solutions to some equations but also a structure such as the space of curves, the space of vector spaces... Inspired by the example of elliptic curves, *moduli* is the general name for the coordinates on those spaces.

Singularities This new geometry conserves the old one: classical manifolds or schemes embed faithfully in derived spaces. The new features concerns in fact only singular points and thus spaces with singularities: smooth manifolds and smooth schemes behaved as they always did in this new geometry.

Singular points are classically defined as the points whose tangent space has a dimension bigger than expected, which is a way to say that not all tangent vectors can be integrated into a curve. They are essentially of two kinds: *quotient and intersections singularities*. For example, quotient singularities appear in a quotient by a group action when a point has a stabilizer under the action, and intersection singularities appear when the intersection is not transverse. We shall see in Section 2 how these two kinds of singularities create, respectively, a positive and a negative part in the tangent complexes. This is actually the first insight about tangent complexes: a point is singular if an only if its tangent complex has non-zero homology. This extra tangent structure measures the complexity of the singularity. Also, the whole formal neighborhood of a point can be reconstructed from the tangent complex equipped with its Lie structure (see Section 2.2.3).

Stacks The daring idea of a new kind of space whose tangent spaces would be chain complexes, probably best summarized in the introduction of [12], could only be imagined because of a very mature ground. First of all, homological algebra had spread in algebraic geometry, where it had been rebirthed with *derived categories* and *total derived functors*. Since the 1960s, every geometer is accustomed to the "derived philosophy" which asserts that objects could have more regular properties if they were replaced by some "derived" version, enhanced with some "hidden" structure.

But most significant was the influence of *stack theory*, which had risen from an obscure topic for descent and non-abelian cohomology [23] to a proper geometrical theory after the works of Deligne, Mumford, and Artin [4, 13] in the late 1960s to 1970s. Stacks are a notion of highly unseparated space where points can form categories instead of just sets or posets, and this was perfectly suited to encoding the singular structures of moduli spaces.[2]

Stacks were for some time a kind of necessary evil in algebraic geometry, something efficient but whose geometrical nature was unclear. The proper context to understand them is in fact ∞-categories and an analogy with homotopy theory. Implicit in [23], this analogy was fully devised in the 1980s by Grothendieck in *Pursuing Stacks* [26], then successfully formalized in the 1990s by Hirschowitz and Simpson [31, 63] by using the ideas and methods of algebraic topology (simplicial presheaves and model categories).[3]

The importance of stacks for derived geometry was that they were already proposing a notion of space with tangent complexes. Within stack theory, a point can have a symmetry group, and the Lie algebra of this group is a part of the tangent structure at the point [45] (see also Section 2.1.2). In other words, the tangent complex of a stack at some point encodes not only first-order deformations of this point but also the symmetries of these deformations (which are in bijection with first-order deformations of the identity endomorphism). The tangent complexes of stacks, a notion that would eventually be better formalized by Simpson [63] in the context of higher stacks, are concentrated in (homological) non negative degrees and have been eventually recognized to explain half of the tangent complexes of the DDT. In fact, stack theory has been recognized to be one-half of the pursued derivation of geometry, precisely the part regularizing the properties of quotient singularities [12, 29]. The other half, corresponding to intersection singularities and the negative part of the tangent complex, would demand new ideas.

The Good Formalism The first formalisms of derived geometry were purely algebraic: since they were made to extract tangent chain complexes, they were using algebraic structures on chain complexes (Lie dg-algebras, commutative dg-algebras, and even dg-coalgebras) together with the natural extension of commutative algebra to this setting.[4] This was efficient, but the spatial language was rather a psychological trick to justify the formal manipulations than yet a proper insight on a new geometry. Also, the formalisms

[2]It was folkloric at the time that the symmetries of points in moduli spaces were an obstruction to describe them locally by affine schemes or manifolds.

[3]These methods were actually available to Grothendieck, but he was not satisfied with them and refused to used them.

[4]Perhaps the first occurence of this is to be found in physics; see Section 3.5.

were either for formal neighborhoods only [29, 42] or specialized to some example only [12, 39], and a better theory was called for.

The good idea was eventually found in the working philosophy of algebraic geometers, which, from the formalization of algebraic groups [14] to the formalization of moduli problems and algebraic stacks [13, 63], were constructing the objects of algebraic geometry in two steps: first by defining a notion of affine scheme (encoded faithfully by a commutative ring of functions), and then by completing the category of affine schemes by the quotient stacks of (étale or smooth) groupoids (we shall detail this in Section 3.1). The recipe for derived geometry was then to do the same starting with some "derived rings" (commutative dg-algebras). Hinich formalized this in the infinitesimal scale [29]. The attempt of Behrend [6], although correctly conceived on a conceptual level, was wronged by the technical limitations due to the use of 2-category theory instead of higher category theory. It was finally Toën and Vezzosi, both trained in homotopical algebra and ∞-categories, who were the first to formalize a proper setting by working out algebraic geometry within model categories [70, 71, 72].

Later, Lurie, with a better background in categorical logic, understood how to apply this approach to differentiable and complex geometries [36, 50, 65, 57]. He also improved the presentation of the theory by working out fully the higher categorical background. This eventually led him to re-prove and improve a number of major theorems of algebraic geometry in the derived setting and to develop a tremendous amount of higher categorical notions on the way [46, 47, 48].

Nowadays, derived geometry is out there and its methods are spreading from algebraic geometry to other mathematical fields. The most important exportation of its ideas may yet be derived symplectic geometry, where it provides a definition of symplectic structures for singular spaces [7, 8, 9, 56, 69].

Derived Rings The nature of those "derived rings" with which to start derived geometry is actually an important degree of freedom of the theory, and this was the reason for the very general setting of Toën and Vezzosi in [72]. If the first motivation of derived geometry was to improve the tools of algebraic geometry to work on singular spaces, Toën and Vezzosi were also motivated by an original application in algebraic topology where a "brave new algebraic geometry" was emerging. The application was regarding elliptic cohomology [72, Chapter 2.4] and would eventually be worked out fully by Lurie [49].

The possibility of this choice of derived rings is also what authorizes the definition of derived differential geometry and derived complex geometry [36, 50, 57, 65]. We shall say more about this in Section 3.2.6.

Why Derivation? Why stacks? Why derived rings? How can we have a geometric intuition of these objects? We shall try to answer these questions throughout the text. Let us only say for now that the deep reasons for the necessity for the derivation of geometry (and all derivations) are not to be looked for within geometry or algebra but within an insufficiency of set theory and even of its extension category theory. We shall come back to this in our conclusion.

Other Texts Previous texts have been written to explain the ideas of derived geometry. The main ones would be Toën's surveys [66, 68] and Lurie's introductions to his DAG series and to his book [48]. The present text has been written as a complement to these texts; I have tried to emphasize the conceptual guidelines of the theory rather than the applications (which are detailed in the aforementioned texts) and to give a more global view on the matter of derivation.

Notations Through all the text, the word *space* shall be used in an informal way to refer to the general idea of space, independently of any mathematical formalization (topological space, topos, manifold, scheme, stack, etc.). A basic knowledge of category theory is assumed, in particular regarding limits and colimits.

For the Differential Geometer We shall limit the presentation of derived *algebraic* geometry, but all considerations can be transposed into differential and complex geometries. For the reader uncomfortable with the notion of scheme, it is enough to know that the notion of a scheme differs from that of a manifold by the fact that it is allowed to have singularities. In the whole text, the word "scheme" can be always be understood as "manifold with possible singularities". The expression "affine scheme" means a scheme that can be defined as the zeros of some functions in some affine space $\mathbb{A}^n$, "general" schemes are constructed by gluing affine schemes. Contrary to differential geometry, in which any manifold can be embedded in $\mathbb{R}^n$, not every scheme is affine, that is, a subspace of some $\mathbb{A}^n$ (the projective spaces, for example). This explains the double vocabulary of "affine/non affine" schemes in algebraic geometry, and in this text, it can be ignored by the differential geometer.

Acknowledgments I was fortunate to learn derived geometry from Bertrand Toën while he was developing it. My mathematical training had left me frustrated by the absence of principles justifying what looked to me to be homologic and homotopic computational nonsense; my discovery with him of the ideas of higher category theory was illuminating.[5] May he find here my gratitude for proving to me that these maths are not just a bunch of incomprehensible techniques but actually do make sense!

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2 Tangent Complexes

The easiest way to get familiar with derived algebraic geometry features is the computation of the so-called *(co)tangent complexes*. The tangent complex is an enhancement of the tangent vector space into a chain complex, it is defined at every point of a space and globally as a bundle. The cotangent complex is the corresponding algebraic object–it is to the tangent complex what the module of differential forms is to tangent spaces, and analogously, it is more suitable for computations.

The theory of cotangent complexes for affine schemes is just a more geometric name for André–Quillen cohomology of commutative rings. The motivation was the study and classification of extensions of rings by cohomological methods. The geometric interpretation is the theory of infinitesimal deformations, where, the cotangent complex helps to answer questions such as "can a tangent vector be integrated into a path?"[6]

Although the motivation was coming from geometry, the geometrical meaning of these complexes (as a whole) was not at all clear at the time (Grothendieck wonders about such an interpretation in the introduction of [25]). Derived algebraic geometry has given a clear answer to this question: within derived algebraic geometry, the natural notion of tangent *is* the tangent complex.

As we will show in examples, tangent complexes are easy enough to compute in practice. For smooth points, they are quasi-isomorphic to the tangent spaces, but they contain a lot more information at singular points. This latter fact is their whole interest. By focusing on the algebra of functions (rings) and not only on topology (algebra of open subsets), algebraic geometry had already established, with the notion of scheme, a good notion of space that could support singularities, but derived algebraic geometry goes further and provides a notion of space with an even better handling of singularities. For example, there is no longer any need for transversality lemmas to compute intersections in derived algebraic geometry (see Section 3.3.2).[7]

[5] In particular, the notion of homotopy colimit that I have chosen to put at the heart of this text (see Section 3.2.1).

[6] The problem is obvious on a smooth manifold, but not if singular points are allowed, as in schemes. A point is singular if its tangent space has a dimension bigger than the dimension of the space. This means precisely that not all tangent vectors are tangent to paths. The structure of the tangent complex, in particular its Lie algebra structure, helps to describe the subset (a cone) of vectors that can be integrated. This is the so-called cohomological obstruction calculus.

[7] Difficulty in mathematics might be transformed but never really cancelled; the problem of having transversal intersections is replaced by that of finding a nice resolution (e.g., a Koszul resolution) of the rings at hand. However, we have transformed a geometrical problem into an algebraic problem, which is always more suited for computations.

The most important (and the most intriguing, but see Section 2.2.2) property of tangent complexes, which has no counterpart in algebraic geometry, is that they have a natural Lie algebra structure. At smooth points, this structure is trivial (that is abelian), but otherwise it contains a lot of structure about the singularity. For example, the whole formal neighborhood of the singular point can be reconstructed from this structure (see Section 2.2.3). This is particularly useful to study moduli spaces. This Lie structure is computable in practice by means of L_∞-structure[8] but often reduces to a simple Lie dg-algebra structure. The reason to be of this Lie structure will be explained in Section 2.2.2.

We shall not give a proper definition of the tangent complex (this would require the introduction of too many technical notions), but a few words will be said in Section 3.2.5 about the cotangent complex. We refer to [68] for an introduction and references therein for precise definitions.

2.1 Examples

2.1.1 Subschemes and intersections

In this section we compute the tangent complex of an affine subscheme defined by some equations. If there is more than one equation, this case encompasses the intersection of subschemes.

Let $\mathbb{A}^n$ be the affine plane of dimension n, we consider the affine scheme Z of zeros of a polynomial function $f : \mathbb{A}^n \to \mathbb{A}^m$, Z is defined by the fiber product

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbb{A}^n \arrow[d, "f"] \\ 1 \arrow[r, "0"] & \mathbb{A}^m. \end{tikzcd}$$

The function f is given by a ring map $f : \mathbb{C}[y_1, \dots y_m] \to \mathbb{C}[x_1, \dots x_n]$. The ring of functions on Z is the ring A, quotient of $\mathbb{C}[x_1, \dots x_n]$ by the ideal generated by the elements $f(y_j)$.

Let x be a point of Z; it determines a point of $\mathbb{A}^n$, that we still call x, whose image by f is 0 in $\mathbb{A}^m$. The tangent space T_xZ to x in Z is defined as the kernel of the differential $df_x : T_x\mathbb{A}^n \to T_0\mathbb{A}^m$. The point x is called a *regular point* of Z if df_x is surjective. In this case, the dimension of T_xZ is $n - m$, and it can be proved that so is the dimension of Z around x. If df_x is not surjective, x is called a *singular point*. If we are thinking Z as the intersection of the $Z_j = \{f_j = 0\}$, a point x is singular precisely when the intersection is not transverse.

The *tangent complex* of Z at x is defined as the chain complex

$$\mathbb{T}_xZ \quad = \quad T_x\mathbb{A}^n \quad \longrightarrow \quad T_0\mathbb{A}^m,$$

where $T_x\mathbb{A}^n$ is in (homological) degree 0 and $T_0\mathbb{A}^m$ in degree -1. The tangent space T_xZ is the H_0 of this complex, and x is regular if and only if the homology of $\mathbb{T}_xZ$ is concentrated in degree 0. The H_{-1} measures the defect of submersivity of f at x, and the excess of dimension of T_xZ, that is the singularity of x. Also, the Euler characteristic of $\mathbb{T}_xZ$ is always the expected dimension of Z around x. In particular, it is locally constant as a function of x (and not only semicontinuous, as is the dimension of T_xZ).

As we mentioned, the main feature of $\mathbb{T}_xZ$ is that it is endowed with a L_∞-structure. This structure exists in fact on the shifted complex[9] $\mathbb{T}_xZ[1]$ concentrated in degrees -1 and -2 and is particularly simple to

[8] We shall not give the full definition of a L_∞-structure on a chain complex $\mathfrak{g}$ and we refer to [53] for details. It will be sufficient to know that L_∞-structure are essentially a Lie structure but where all equations have been relaxed to hold only up to homotopy. Such a structure is given by higher-order brackets $[-, \dots, -]_n : \mathfrak{g}^{\otimes n} \to \mathfrak{g}[2-n]$ having homological weight $2-n$. The bracket of arity 1 is a differential, the bracket of arity 2 is the Lie bracket, the bracket of arity 3 is called the jacobiator and produces a chain homotopy ensuring Jacobi identity, and higher brackets are encoding higher coherence conditions. The structure is that of Lie dg-algebra if the brackets in arity > 2 are zero. Altogether, the brackets can be written as the homogeneous components of a single map $S(\mathfrak{g}[-1]) \to \mathfrak{g}$, where $S(\mathfrak{g}[-1])$ is the symmetric algebra on $\mathfrak{g}[-1]$.

[9] The reason for this shift will be explained in Section 2.2.2.

make explicit in this example. It turns out that an L_∞-structure on a complex concentrated in such degrees is given by a single map (satisfying no condition),

$$S(T_x\mathbb{A}^n) \longrightarrow T_0\mathbb{A}^m,$$

where $S(T_x\mathbb{A}^n)$ is the symmetric algebra on $T_x\mathbb{A}^n$. Then the L_∞-structure is simply given by the Taylor series of f, and the brackets $[-,\dots,-]_n$ of the L_∞-structure are then given by the homogeneous components, that is the higher differential D^nf of f viewed as symmetric functions of n variables.[10]

2.1.2 Quotients

We shall now describe the tangent complex of the quotient of a smooth scheme by a group action.

Let G be a group with Lie algebra $\mathfrak{g}$, acting on a smooth scheme X. Let x be a point of X and $\bar{x}$ the corresponding point in the quotient X/G. The infinitesimal action induces a map $\mathfrak{g} \to T_xX$, where T_xX is the tangent space at x. The image of this map is the tangent to the orbit of x and the kernel is the Lie algebra of the stabilizer of x. Let us call x *regular* if $\mathfrak{g} \to T_xX$ is injective and *singular* if not.[11] If x is regular, the action of G is locally free[12] around x and it is possible to find a local transversal section to the orbits. This section can be used to define the local structure around $\bar{x}$ in the orbit space. In particular, it is of dimension $d = \dim X - \dim G$, and we get that $T_{\bar{x}}(X/G) = (T_xX)/\mathfrak{g}$.

The *tangent complex* of X/G at $\bar{x}$ is defined as the chain complex

$$\mathbb{T}_{\bar{x}}(X/G) \quad = \quad \mathfrak{g} \longrightarrow T_xX$$

where T_xX is in (homological) degree 0 and $\mathfrak{g}$ in degree 1. As before, x is regular if and only if the homology of $\mathbb{T}_x(X/G)$ is concentrated in degree 0, in which case H_0 is the tangent space in the quotient. The H_1 of this complex is the Lie algebra of the stabilizer of x; it measures the singular nature of $\bar{x}$. The Euler characteristic of $\mathbb{T}_{\bar{x}}(X/G)$ is always the expected dimension of X; again, it is locally constant as a function of x (and not only semi-continuous as the dimension of $T_{\bar{x}}(X/G)$).

This tangent complex can be proved to be the tangent space of the *quotient stack* $X/\!/G$. In this setting, the points of $X/\!/G$ form a groupoid and not a set. The tangent space at any point is also a groupoid but with an extra linear structure, and such an object is the same thing as a chain complex in homological degrees 0 and 1 [1, 22]. In this tangent groupoid, the H_0 is the part of encoding first-order deformation of $\bar{x}$ as an object in $X/\!/G$, and H_1 is the part encoding the symmetries of such deformations (which are equivalent to first-order deformations of the identity of $\bar{x}$ as a morphism in $X/\!/G$).

Again, the shifted complex $\mathbb{T}_x(X/G)[1]$, concentrated in degrees 0 and -1, is endowed with an L_∞-structure easy to describe explicitely. It turns out that an L_∞-structure on a complex concentrated in degrees 0 and -1 is given by a two maps,

$$d' : S(T_xX) \otimes \mathfrak{g} \longrightarrow T_xX \qquad \text{and} \qquad d'' : S(T_xX) \otimes \Lambda^2\mathfrak{g} \longrightarrow \mathfrak{g},$$

where $S(T_xX)$ is the symmetric algebra on T_xX. The brackets $[-,\dots,-]_n$ of the L_∞-structure are the homogeneous components of these maps, viewed as symmetric functions of n-variables.

Recall that the action of the Lie algebra $\mathfrak{g}$ on X can be encoded by a Lie algebroid [54]. A Lie algebroid structure is characterized by two maps: the anchor and a Lie bracket. The maps d' and d'' are given, respectively, by the Taylor series of the anchor map and of the Lie bracket at the point x. We claim that the equations of the L_∞-structure give exactly the conditions on the anchor and the bracket of a Lie algebroid.

[10] This example shows the versatility of L_∞-structures; they can be quite remote from actual Lie algebra structures. But this ability to interpolate between what could be called "formal neighborhood structures" and Lie algebra structures is somehow the main interest of L_∞-structures.

[11] Because of discrete groups, x can be regular in this sense but still be a singular point in the quotient (an orbifold point). However, this kind of singularity can be unfolded by an étale cover (a local diffeomorphism) and therefore does not count as a singularity from the point of view of infinitesimal calculus.

[12] By which we mean that a neighborhood of the identity of G is acting freely around x. In algebraic geometry, local Henselian rings need to be used.

In particular, if X is a single point $*$, the quotient stack $*//G$ is the stack BG (classifying G-torsors). Its tangent complex at $*$ reduces to $\mathfrak{g}$ in degree 1, and the L_∞-structure is simply given by the Lie algebra structure of $\mathfrak{g}$.[13]

2.1.3 Fiber products and triangles

The previous computations are actually a particular case of a general formula for fiber products of derived stacks: a Cartesian square of (pointed) derived stacks induces a Cartesian square of tangent complexes [72, Lemma 1.4.1.16]:

$$\begin{tikzcd} F \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "f"] \\ * \arrow[r] & Y \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} \mathbb{T}_x F \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbb{T}_x X \arrow[d, "df"] \\ 0 \arrow[r] & \mathbb{T}_{f(x)} Y \end{tikzcd}$$

In other words, $\mathbb{T}_x F \to \mathbb{T}_x X \to \mathbb{T}_{f(x)} Y$ is a distinguished triangle in the category of chain complexes, and the left (or right) object can be computed as the mapping cone (or the mapping co-cone) of the other two.[14] We recover this way the computation of Section 2.1.1 as a cone. In the case of a quotient by a group action as in Section 2.1.2, we use the fact that the definition of the quotient as a stack $X//G$ always provides a Cartesian square,

$$\begin{tikzcd} G \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "q"] \\ * \arrow[r, "\overline{x}"] & X//G \end{tikzcd}$$

where the top map is the parameterization of the orbit of x.[15] Then the construction of the tangent complex follows from a mapping co-cone construction. For the corresponding L_∞-structures, the reader can look in[53].

2.1.4 Tangent Complexes in Deformations

Perhaps the most common examples of tangent complex are the chain complexes that appear in deformation theory.[16] We shall give some details about the case of principal bundles and only mention a few others.

Let $P \to X$ be a principal G-bundle, on a fixed smooth scheme (or manifold) X. Such a bundle is classified by a 1-cocycle of X with values in G, and the first-order deformations of P can be shown to be classified by 1-cocycles of X in the adjoint bundle $ad(P) = P \times_G \mathfrak{g}$. This suggests introducing the cohomology complex $C^*(X, ad(P))$ or, better for our purpose, the chain complex $\mathbb{T}_P$ such that $(\mathbb{T}_P)_i = C^{1-i}(X, ad(P))$.[17] Remark that $\mathbb{T}_P[1]$ has a Lie dg-algebra structure inherited from the Lie structure of $ad(P)$.

The positive part of this bundle is easy to understand: $H_0(\mathbb{T}_P) = H^1(X, ad(P))$ is in bijection with the isomorphism classes of first-order deformations of P, and $H_1(\mathbb{T}_P) = H^0(X, ad(P))$ classifies the symmetries of such deformations. Let $Bun_G(X)$ be the stack classifying of G-bundles on X; then the truncated complex $(\mathbb{T}_P)_{\geq 0}$ can be proved to be the tangent complex of $Bun_G(X)$ at the point P. This is quite similar to the example of Section 2.1.2.

[13] The proximity of L_∞-structures with Lie structures is clearer in this example than in the previous one, but once again, the L_∞-structure encompasses the extra structure of a formal neighborhood.

[14] The reader not fluent in homological algebra can look at [2, 21] for the definitions of these notions.

[15] It is one of the nice features of quotient stacks $X//G$ that the fibers of the quotient map $X \to X//G$ are always isomorphic to G, even if the action is not free. Actually, the map $X \to X//G$ can be proved to be a G-torsor.

[16] Deformation theory deals with the problem of classifying infinitesimal deformations of a given object (a scheme, a bundle, a ring structure, a group representation, etc.). It is always possible to consider such an object as a point in the moduli space for the structure in question; then, the infinitesimal deformations of the object correspond to the study of the infinitesimal neighborhood of the point. In particular, first-order deformations correspond to tangent vectors in the moduli space. Moduli spaces are difficult to construct, but first-order deformations are relatively easy since they consist of solving some linear equations.

[17] The degrees are changed for convenience; see Section 2.2.1.

The interpretation of the negative part of $\mathbb{T}_P$ is more subtle. As in the example of Section 2.1.1 we can think of it as related to some non transverse intersection feature.[18] It can also be explained in terms of obstruction theory or, better, in terms of deformations parameterized by dg-algebras, but this requires elements of derived geometry. We shall not explain this here (see [68, 75] or the introduction of [48] for details), the only thing we need to know is that from the point P in $Bun_G(X)$, we have again produced a tangent complex with a Lie algebra structure.

We list a few other complexes related to deformation problems. If X is a scheme (or a manifold), the tangent at X to the moduli space of schemes is the H_0 of the tangent cohomology complex $(\mathbb{T}_X)_i = C^{i-1}(X, TX)$. Then $\mathbb{T}_X[1]$ has a structure of a Lie dg-algebra inherited from the Lie bracket of TX.

If Γ is a discrete group and M a linear representation of Γ, the tangent space to M in the space of representations (or character space) of Γ is the H_0 of the complex $(\mathbb{T}_M)_i = C^{i-1}(\Gamma, End(M))$ computing the cohomology of Γ in the adjoint representation $End(M)$ of M. Remark that the shifted complex $\mathbb{T}_M[1]$ has a structure of a Lie dg-algebra inherited from the Lie structure of $End(M)$. More generally, if X is a topological space (the previous case being $X = B\Gamma$) and M a local system on X, then $End(M)$ is again a local system, and the tangent space at M to the space of local systems on X can be enhanced into the complex $(\mathbb{T}_M)_i = C^{i-1}(X, End(M))$ of cohomology of X in M. And again, $\mathbb{T}_M[1]$ is a Lie dg-algebra.

If A is an associative algebra, the tangent space at A to the space of associative algebra is the H_0 of the (shifted) Hochschild complex $(\mathbb{T}_A)_i = C^{i-2}_{Ass}(A, A)$. An L_∞-structure can be proved to exist on $\mathbb{T}_A[1]$, but it is difficult to explicit. Similarly, if $\mathfrak{g}$ is a Lie algebra, the tangent space at $\mathfrak{g}$ to the space of Lie algebra is the H_0 of the (shifted) Chevalley complex $(\mathbb{T}_\mathfrak{g})_i = C^{i-2}_{Lie}(\mathfrak{g}, \mathfrak{g})$. Again, $\mathbb{T}_\mathfrak{g}[1]$ has a Lie algebra structure inherited from that of the coefficient $\mathfrak{g}$.

2.2 Geometry of Tangent Complexes

2.2.1 The Three Parts of the Tangent Complex

We have seen two notions of singularities: intersection and quotient singularities. We have seen in each case that the tangent structure could be encoded in a complex whose homology is concentrated in degree 0 if and only if the point is regular. The other homology groups are thus a reflection the singular structure of the point. If there exists non trivial positive homology groups, this means that some *bad quotient* was involved in the singular structure. Moreover, if the homology is non trivial for some $n > 0$, this means at least n bad quotients had to be involved to create the singularity. If there exist non trivial negative homology groups, this means that some *bad intersection* was involved in the singular structure. Moreover, if the homology is non trivial for some $n < 0$, this means at least n non transverse intersections had to be involved to create the singularity. Since a general space is constructed by taking both intersections and quotients (typically a symplectic reduction; see Section 3.5), its singularities will have both positive and negative tangent parts in general.

Altogether, our examples propose a picture of the tangent complex in three specific parts, as in Fig. 1.

Figure 1: Structure of the tangent complex

Positive part		*Zero part*		*Negative part*
(quotient singularity)				(intersection singularity)
$\ldots \to T_2 \to T_1$	$\to$	T_0	$\to$	$T_{-1} \to T_{-2} \to \ldots$
"stacky structure"		"real" tangent		"derived structure"
internal symmetries				outer intersection structure

[18] This intersection has two sources: first the cutoff given by the cocycle condition and also the wild limit (indexed by the category of refinement of atlases) that has to be taken when bundles are described in terms of Čech cocycles.

2.2.2 Lie Structure and Loop Stacks

We have seen also that tangent complexes $\mathbb{T}$ were always equipped with a kind of Lie structure on the shift $\mathbb{T}[1]$. It turns out that derived geometry provides a very nice and quite simple explanation for this fact.

First, recall that in homotopy theory, for a pointed space $x : * \to X$, the homotopy fiber product $* \times_X *$ is nothing but the loop space $\Omega_x X$ of X at x, which in particular is a group.[19] The interpretation is in fact the same in any ∞-category, so in particular in derived stacks where $\Omega_x X$ is the derived group stack of symmetries of x in X. Now, recall from Section 2.1.3 that a Cartesian square of pointed derived stacks provides a triangle of tangent complexes. Applied to the square

$$\begin{tikzcd} \Omega_x X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & * \arrow[d, "x"] \\ * \arrow[r, "x"] & X \end{tikzcd}$$

it gives $\mathbb{T}_{id_x} \Omega_x X = cone(0 \to \mathbb{T}_x X) = \mathbb{T}_x X[1]$, that is, the shifted tangent complex at x is nothing but the tangent complex to the loop stack at the identity of x. Now, since the loop stack $\Omega_x X$ is a group, this explains the Lie algebra structure on the tangent.[20]

2.2.3 Lie Structure and Formal Neighborhood

Perhaps the most bizarre consequence of the existence of the Lie structure is the ability to reconstruct the whole formal neighborhood of a derived stack (called a *formal stack* or a *formal moduli problem*) from the tangent complex and its Lie structure. This idea has a long history [15, 24, 29, 30, 59] and was fully formalized within derived geometry in [52].

If a point x in a scheme X is regular, the formal neighborhood of x is the same as the formal neighborhood of a point in the tangent space $T_x X$[21] (thus encoded by the algebra of power series $k[\![T_x^* X]\!]$ generated by the cotangent module at x). When x is not smooth, $T_x X$ is too big, and the formal neighborhood is a subspace of $T_x X$. The equation of this subspace can be written from the Lie structure of the cotangent complex $\mathbb{T}_x X$ by means of the Maurer–Cartan equation. Example Section 2.1.1 shows clearly that the L_∞-structure can reconstruct the formal neighborhood; this should also be clear enough in the example of Section 2.1.2. We shall not say more about it; the matter is detailed in [68].

3 Derived Spaces

In this section, we explain how to build the spaces of derived geometry. This will use ideas from algebraic geometry crossed with ideas from higher category theory.

3.1 A View on Algebraic Geometry

As we mentioned in the introduction, it is convenient to split algebraic geometry into two parts: the theory of *affine schemes* and the theory of *non affine spaces* (general schemes, algebraic spaces). Affine schemes are those spaces that can be described faithfully by a commutative ring of functions. Non affine

[19] The fiber product $A \times_C B$ of a diagram of sets $f : A \to C \leftarrow B : g$ is the set of pairs (a, b) such that $f(a) = f(b)$ in C. The homotopy fiber product $A \times_C B$ of a diagram of homotopy types $f : A \to C \leftarrow B : g$ is essentially the same thing but where the equality $f(a) = f(b)$ is taken up to homotopy, i.e., replaced by a path in C. More precisely, $A \times_C B$ is the homotopy type of the space of triplets (a, b, γ), where γ is a path in C from $f(a)$ to $g(b)$. When $f = g = x : * \to X$, this gives the loop space. See [16] for more details.

[20] Actually, this interpretation is not fully proved yet. Although Lie algebra structures have been proved to exist on tangent complexes [28, 38, 52], the above interpretation relies on a theory of Lie algebras of derived Lie groups that has not been developed yet.

[21] This is the algebraic analog of the local homeomorphism between the tangent space and the neighborhood of a point in a manifold.

spaces are those spaces without enough functions and must be described by other means (functor of points, atlases).[22]

The theory of affine schemes consists of an almost perfect dictionary between geometric properties (points, open and closed subsets, étale and proper maps, connectedness, separatedness, bundles, dimensions, vector fields, etc.) and features of commutative rings (fields, localizations, quotients and ideals, separable and finite extensions, idempotents, valuations, modules, generating families and presentations, derivations, etc.). The tools involved in this first part are those of commutative algebra: proofs about affine schemes are ultimately proofs about modules over rings. Commutative algebra is greatly computational and improves the tools of the sole topology: the full power of this algebra is notably used in the definition of the infinitesimal calculus (Kähler differentials but also iterated powers of ideals); it is also used in the definition of closed subspaces (defined as quotients of rings), or intersections (defined as tensor products of rings). These ideas lead in particular to a nice formalization of intersection multiplicities, which topology alone is not able to get.[23] All this will generalize well to derived rings and derived affine schemes.

The second part of algebraic geometry consists in using affine schemes to build more sophisticated spaces. It happens that not every space of interest in algebraic geometry is an affine scheme: projective spaces, Hilbert schemes, and other moduli spaces do not in general have enough functions in the affine line to be described faithfully by a commutative ring.[24] However, practice proves that these spaces can often be constructed by pasting affine schemes, in the same way a manifold is a pasting of charts.[25] This is the theory of *(non affine) schemes* and of *algebraic spaces* and of general *sheaves of sets*[26] [73]. Contrary to affine schemes which are defined individually by means of a ring, these new objects cannot be defined individually, but only relatively to the previously defined affine objects. The tools here are those of category theory rather than commutative algebra (limits and colimits of diagrams, presheaves, universal properties, etc.). They are the tools to work on a collection of objects rather than on objects individually. These methods and notions will be derived into the theory of stacks.[27]

We shall present now a more conceptual understanding of these two parts. Recall that a commutative ring is always a quotient of a free ring by some system of equations. Geometrically, this says that affine schemes are constructed from affine spaces $\mathbb{A}^n$ (the affine schemes corresponding to free rings) as levels sets of functions, that is by fiber products, or by categorical limits. Then schemes and algebraic spaces are constructed from affine schemes by pasting, that is by categorical colimits. The two steps of the construction of the spaces of algebraic geometry can therefore be read as the following procedure: start with affine spaces (free rings), then add some limits, then add some colimits.

Having this in mind, we can say that derived algebraic geometry is built with the same procedure but where we are going to change the way to compute level sets and quotients, that is limits and colimits. Changing the way to compute limits transforms the theory of affine schemes into that of *derived affine schemes* (technically, it is done by changing the way quotients are computed in rings; see Section 3.2.3),

[22] This general structure has vocation to be used in differential and complex geometries. Rings need simply to be replaced by the appropriate notions: C^∞-rings for differential geometry [36, 55, 65]; and a similar notion for complex geometry [51, 57]. See also Section 3.2.6.

[23] This power of the algebra over geometry is one of the motivations for the extension of differential geometry with C^∞-rings methods [55], where there is an infinitesimal theory handier than in manifolds. In particular, such an extension allows one to apply the ideas and methods of synthetic differential geometry.

[24] Methods of commutative algebra can be extended to projective spaces (through graded rings), but not all schemes are projective, so other methods are needed.

[25] Which is to say that they may not have enough globally defined functions but always have enough locally defined functions (coordinates).

[26] Let us recall that sheaves have two different uses in topology: the most common is to use sheaves of Abelian groups as coefficients for cohomology, but sheaves of mere sets can be used as generalized spaces. In fact, from this point of view, sheaves (and stacks in a better way) are useful to solve the following conundrum: how to define a space that is not a manifold, i.e., that does not have an atlas (like orbifold) or, even worse, a space that is not a topological space, i.e., which does not have enough open subspaces (like an unseparated quotient)? Sheaves and stacks give a setting where to define such spaces, provided we know what a map from a manifold (or an affine scheme, or a topological space, or any other "basic block") to this space is, i.e., provided we can define a functor of points. This is particularly suited to moduli problems.

[27] A good introduction to these ideas is Toën's course [67].

and changing the way to compute colimits is the replacement of sheaves by *stacks* (see Section 3.4). Both changes will require the introduction of higher categories.

3.2 Homotopy Quotients and Derived Rings

Algebraic geometry has long enhanced commutative algebra by introducing homological or homotopical methods (chain complexes, dg-algebras, simplicial modules and algebras, etc.). Classical books on the matter justify these enhancements by the powerful computations they allow (essentially, the existence of long exact and spectral sequences) but not so much by principles.[28] The question of these underlying principles was actually a very difficult question for a long time; it has been solved only with the mutation of homotopical algebra into higher category theory in the 1990s. We shall explain only the part of the story that has to do with the operation of taking quotients.

3.2.1 Quotients of Sets

Quotients of sets are classically dealt with via equivalence relations. However, in practice, equivalences relations are often derived from other structures (graphs, group action, etc.) where two elements may have several ways to be identified (in a graph, there might be more than one edge between two vertices x and y; in a group action, there might be more than one element sending an element x to another y if the action is not free). The associated equivalence relation remembers merely the *existence* of an identification between two elements and forgets about the potential *ambiguity* (in the sense of non canonicity) of identifications.

It turns out that forgetting the multiplicity of identifications can generate irregularities. For example, in the case of a group action, it is not true that working equivariantly is the same thing as working over the quotient if the action is not free.

Another notion of quotient has been invented, that takes into account the potentially multiple identifications of elements. However, this operation has values not in sets but in homotopy types.[29] In consequence, it is called the *homotopy quotient*. It has been developed in homotopy theory, where it is one of the most basic tool, under the name of *homotopy colimit* [10, 16, 20, 32]. Technically, the homotopy colimit is constructed as a simplicial set such that the classical quotient (the set of equivalence classes) can be identified with the set of connected components.[30] The construction procedure is fairly simple, but the statement of its universal property requires some advanced homotopical algebra.[31]

The principle of the construction is to identify two elements in a set by putting an edge between them instead of equalizing them. Then three elements are identified by putting a triangle, four with a tetrahedron, and so on. It should be clear how this produces a simplicial set and that its set of connected components is indeed the classical quotient. The homotopy colimit of a diagram of sets is formally defined as the homotopy type of this simplicial set. We shall see some examples.

Let us consider first, the case of the diagram describing a graph with two edges a and b between two vertices x and y,

$$\{a, b\} \overset{s}{\underset{t}{\rightrightarrows}} \{x, y\},$$

where the two maps are the source and target maps from the set of edges with values in the set of vertices (here s sends a and b to x and t sends them to y). The colimit of this diagram is the quotient of the set $\{x, y\}$ by the relation $x \simeq y$, and the classical quotient is a singleton. But a and b provide two different

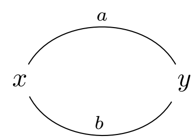
[28] Who never wonder about the necessity of homological/homotopical apparatus and how to make sense of all these constructions?

[29] Homotopy types are equivalence classes of topological spaces for the (weak) homotopy equivalence relation. They can be viewed as a generalization of the notion of groupoid and are sometimes called ∞-groupoids [58].

[30] For the reader unfamiliar with simplicial sets, it is sufficient to know that they look like triangulated topological spaces, but defined in a purely combinatorial way. In particular, they have a homotopy theory.

[31] This may explain why this construction is not more widely known. The best formulation of this universal property needs ∞-category theory: within this setting, the notion of homotopy colimit is simply the notion of colimit but computed in an ∞-category. We shall talk a bit more about this in Section 3.2.4.

identifications between x and y, and the homotopy quotient is simply the homotopy type of the graph, equivalent to the homotopy type of a circle



Its set of connected components is, as expected, in bijection with the classical quotient. If the set $\{a, b\}$ is replaced by a set with $n+1$ elements, the homotopy quotient can be computed to have the homotopy type of a wedge of n circles.[32]

Another important example is the homotopy quotient of the trivial action of a group G on a one-element set $*$. The recipe of the homotopy quotient gives the so-called *simplicial nerve* of the group, whose geometric realization is a classifying space for G [16, §4]. In consequence, the homotopy quotient is the homotopy type of a classifying space for G. It is connected but has a non trivial π_1 isomorphic to G.[33]

Finally, in the general case of a group G acting on a set E, the construction produces the nerve of the associated *action groupoid* $G \times E \rightrightarrows E$, and the homotopy quotient is the homotopy type of the classifying space $E//G$ of this groupoid. If the action is free, this space is homotopy equivalent to the quotient set and the homotopy quotient coincides with the classical one, but if the action is not free, the groupoid remembers the stabilizer of a point as symmetries for this point, thus creating some π_1 in the quotient.[34]

Our examples have only non trivial homotopy invariants in degree 0 and 1, but homotopy colimits of more complicated diagrams give rise to homotopy types with non trivial homotopy invariants in any degree. In fact, any homotopy type can be described as the homotopy colimit of a diagram of sets.[35]

The main fact about the homotopy quotient is precisely that its higher homotopy invariants need not be trivial. In this sense, it encodes strictly more structure than the classical quotient. This extra structure is related to some kinds of syzygies:[36] the π_1 groups remember the ambiguity to identify elements (redundancy of relations), the π_2 groups remember the ambiguity to identify identifications between elements (redundancy of relations between relations), and so forth. Altogether, the homotopy quotient has the universal property to encode not only equivalence classes of elements, but also the whole ambiguity about the identifications on those elements.[37]

The main reason for considering homotopy quotients is their computational advantages over classical quotients. Let us mention a few. First, it is always true that working equivariantly is equivalent to working over the homotopy quotient (a very important property called *effectivity of groupoid quotients*[38]). Also, a

[32]If both $\{a, b\}$ and $\{x, y\}$ are replaced by singletons, the homotopy colimit is a single vertex with a single edge, i.e., a circle. More generally, leaping from diagrams of sets to diagrams of homotopy types, if the set of edges $\{a, b\}$ is replaced by a homotopy type X of edges, the homotopy colimit is simply the unreduced suspension of X.

[33] The quotient map is a map $* \to BG$. It is funny to remark that with this new notion of quotient, the point has many non trivial quotients. This example shows that BG is the quotient of a point by the trivial action of the group G, but, in fact, any connected homotopy type can be viewed as a quotient of a single point. This is an important point of view on homotopy types: they are in a sense a structure "below" sets. For example, if G is finite, the cardinality of BG is defined as $1/\#G$, which is indeed below $*$. The reader curious about cardinality of homotopy types can look at [40] and references therein.

[34] The notion of homotopy type can sometimes be replaced by the simpler notion of groupoid (taken up to equivalence of categories) to compute the homotopy quotients. This is the case when the homotopy quotients have trivial π_n for $n \geq 2$, for example, in a group action. Groupoids have a definition simpler that that of homotopy types, and this explains why part of the literature focuses on them instead of full homotopy types. However, groupoids are limited by the fact that homotopy quotients of groupoids may have non trivial π_2. So it is better, albeit more sophisticated, to work directly with full homotopy types.

[35]Any simplicial set is a diagram of sets, and the homotopy colimit of this diagram is the so-called "geometric realization" of the simplicial set. Any homotopy type can be described as a simplicial set (via the nerve of a contractible open covering; see [21]).

[36]This is an analogy with what will be told in Section 3.2.3.

[37] With this idea in mind, it is possible, and quite fruitful, to read the whole of homotopy theory as an enhancement of the theory of sets incorporating the "ambiguity of identifications". We shall come back to this idea in the conclusion. (This notion of ambiguity is compatible with the way the word is used in Galois theory. I have actually chosen the term *ambiguity* in reference to Galois theory.)

[38] Let R be an equivalence relation on a set E with quotient $Q = E/R$ and quotient map $q : E \to Q$. The map q (or the

fundamental computational tool is the long exact sequence of homotopy groups associated to any fibration sequence. And finally, they possess nice numerical invariants: when it is defined, the Euler characteristic provides a nice generalization of cardinality. For example, the Euler characteristic of $E//G$ is always (for *any* action of a finite group on a finite set) the rational number $\#E/\#G$, a formula that is false for classical quotients.[39]

The moral about homotopy quotients is that when no canonical identification exists between two elements to be identified, it is best to keep track of all identifications (which is the only canonical thing to do) by mean of a homotopy type. This extra structure not only satisfies a stronger universal property but gives rise to an object with more regular properties. Projecting the identification data onto a mere equivalence relation truncates the structure and kills the nice computational regularities.

These facts actually make sense, not only for sets, but in a large variety of mathematical contexts, as we shall explain below for linear algebra, commutative algebra, and geometry.

3.2.2 Quotients of Vector Spaces

The issues with quotients of sets are inherited by quotients of any structure based on sets, although the technologies to take care of them may differ. For example, in linear algebra, chain complexes and quasi-isomorphisms turn out to be handier than simplicial objets and homotopy equivalences. In consequence, homotopy colimits of diagrams of vector spaces are, rather, described as chain complexes.[40]

Let us consider the example of a map of vector spaces $d : E_1 \to E_0$. The classical quotient is the vector space $E_0/\mathrm{im}(d)$, which can be defined as the colimit of the diagram

$$E_1 \underset{0}{\overset{d}{\rightrightarrows}} E_0,$$

where 0 is the zero map. The homotopy quotient of d, which is the homotopy colimit of this diagram, is the construction called the *mapping cone* (see [2, 21, 76] for details). In our example, the mapping cone is simply $E_1 \to E_0$ viewed as a chain complex E_*, concentrated in (homological) degrees 1 and 0. In particular, $H_0(E_*)$ is the classical quotient, and $H_1(E_*) = \ker d$ remembers the multiple identifications between elements of E_0. The space $H_1(E_*)$ plays the role of the π_1 of homotopy quotients of sets; it classifies non trivial syzygies.[41]

In particular, we can understand now that the computation of the tangent complex of a quotient in Section 2.1.2 is nothing but the homotopy quotient of the map $T_eG \to T_xX$ between the tangent spaces. The non trivial syzygies are essentially given by the stabilizer of the point. We shall see in Section 3.4.1 that the quotient stack $X//G$ is nothing but the homotopy quotient of the group action, so space and tangent are computed the same way.

Homotopy colimits of more complicated diagrams give rise to chain complexes with larger homology amplitude. In fact, any chain complex in homological non negative degree can be obtained as the homotopy colimit of a diagram of vector spaces, and the homology groups H_i have exactly the same interpretation in terms of non trivial syzygies as the homotopy groups π_i for homotopy quotients of sets.[42] In the same way

relation R) is said to be *effective* if R can be reconstructed from q by as $R = E \times_Q E$. Let $G : G_1 \rightrightarrows G_0$ be a groupoid; it can be thought as a generalized equivalence relation on G_0 (where elements have several ways to be equivalent). There exists a notion of quotient for groupoids that we shall not define here. Let $q : G_0 \to Q$ be the quotient map; q (or G) is said to be *effective* if G_1 can be reconstructed from q as $G_1 = G_0 \times_Q G_0$. Since the whole set of arrows G_1 of the groupoid G can be reconstructed from the quotient, this means that the quotient contains the information about the multiplicity of identification between elements of G_0. In the category of sets, quotients of equivalences relations are always effective, but not quotients of groupoids. However, quotients of groupoids are effective when computed in homotopy types. The precise statement of the property of effectivity of groupoid quotients demands a definition of a groupoid object that we shall not give here (but see [71, 46]).

[39] This last point may seem anecdotic, but for objects more sophisticated than sets, this kind of invariant is related to the so-called virtual dimensions, virtual K-theory classes, etc.

[40] It is possible to use simplicial methods in linear algebra, but the Dold–Puppe–Kan equivalence [37] proves that the two languages are in fact equivalent (provided we consider only complexes in non negative homological degrees).

[41] Again, this is an analogy with what will be said in Section 3.2.3.

[42] The chain complexes of non positive degree can be understood as generated by considering the dual notion of *homotopy limits*. The full category of unbounded chain complexes has a more subtle definition; it is generated by considering both

that homotopy types can be understood as homotopy colimits of sets, chain complexes can be (and should be) understood as homotopy colimits of vector spaces.

The reason to consider homotopy quotients in the context of linear algebra is classically the computational power of long exact sequences and their obstruction theory. Another reason is the effectivity of groupoid quotients, which, in this context, is reformulated into the property that the mapping co-cone of a mapping cone is the identity.[43] We shall not expand on the homotopy theory of chain complexes. It is enough for this text to know that homotopy colimits of vector spaces are given by chain complexes up to quasi-isomorphism.

3.2.3 Quotients of rings

Quotients of rings raise the same issues as with sets and vector spaces. The classical theory characterizes a quotient $A \to A'$ of a ring A by the data of an ideal $I \subset A$. In practice though, quotients of A are not so much given this way but by systems of equations which are then interpreted as generators for an ideal and a quotient. The replacement of equations by an ideal is an operation of the same nature than truncating a graph or a group action into an equivalence relation and bears the same defects.

Let us consider the case of a system of equations $E = \{a_1 = 0, \ldots, a_n = 0\}$ for some a_i in A. The quotient of A by the system E can be presented as the colimit, in the category of rings, of the diagram

$$A[x_1, \ldots, x_n] \underset{x_i \mapsto 0}{\overset{x_i \mapsto a_i}{\rightrightarrows}} A.$$

It is classical that the quotient behaves well if the family of a_i is a *regular sequence* (neither element is a zero divisor relative to the previous ones). Let us recall why. An ideal I of a ring A that is generated by more than one element a_i is never free as an A-module, since there always exists the relation $a_i a_j - a_j a_i = 0$. Relations between generators of an A-module are called *syzygies*. Let us call *trivial syzygies* the relations that can be derived from the $a_i a_j - a_j a_i = 0$. The sequence of a_i is regular if and only if there are no non trivial syzygies. Such syzygies exist when, for example, some a_{i_0} could be a zero-divisor, or equal to another a_{i_1} (repetition of equations), and this phenomenon can be understood as creating an ambiguity in the description of the elements of the ideal generated by E in terms of linear combinations of generators a_i.

For an element a in A, let $K(a) : A \to A$ be the chain complex concentrated in (homological) degree 0 and 1, where the differential is given by the multiplication by a. The complex $K(a)$ is a dg-algebra, called the *Koszul dg-algebra* of a (the multiplication of elements of degree 1 is nilpotent, and the other components are given by the multiplication of A; see [17, 21] for details). The group $H_0(K(a))$ is the classical quotient $A/(a)$, and the group $H_1(K(a))$ consists of all b in A such that $ba = 0$, it is non trivial if and only if a is a zero-divisor. Hence, the dg-algebra $K(a)$, considered up to quasi-isomorphism, encodes the quotient but also keeps track of the regularity of the element a.[44]

More generally, the *Koszul dg-algebra* of a family of elements $E = \{a_i\}$ is defined to be the dg-algebra $K(E) = K(a_1) \otimes \cdots \otimes K(a_n)$. The full combinatorics of trivial relations between the a_i is encoded by the differential of $K(E)$: trivial syzygies (in a given degree) are defined as the image of this differential, and general syzygies as the kernel; the (classes of) *non trivial syzygies* are defined as the homology of $K(E)$. The sequence E is regular if and only if $K(E)$ has only homology in degree 0, if and only if $K(E)$ is quasi-isomorphic to (or is a resolution of) the classical quotient $A/(a_1, \ldots, a_n)$.

Although this was not at all clear (or obvious) when it was invented, the construction $K(E)$ is now understood to be the homotopy quotient of A by the equations E. There exists a canonical map $A \to K(E)$

homotopy colimits and limits of vector spaces, but with a constraint of commutation called *stability* imposing that finite homotopy limits and finite homotopy colimits should commute; see [47] and Footnote 87.

[43] More precisely, for a map $f : E_1 \to E_0$ of (unbounded) chain complexes, let $F(f)$ be the mapping cone of f; there is a canonical map $g : F(f) \to E_1$ whose mapping co-cone $C(g)$ is quasi-isomorphic to E_0. In other words, homotopy Cartesian squares are also co-Cartesian squares in chain complexes. The same property is true in the category of non negative chain complexes, but the map f needs to be surjective.

[44] Geometrically, a function a on a space X is a zero-divisor (or irregular) essentially when the subset $Z = \{a = 0\}$ has a nonempty interior. In such a case, the dimension of Z would not be $\dim X - 1$ as expected, hence the term irregular.

which image by H_0 is the classical quotient map $A \to H_0(K(E))$. The regularity of the sequence is the hypothesis under which the classical and the homotopy quotients agree. As for non-free group actions, irregular sequences produce higher homotopical structure in the quotient, and the higher homology groups $H_i(K(E))$ have the same interpretation in terms of syzygies or ambiguity as the homotopy groups π_i for homotopy quotients of sets. Also, this notion of homotopy quotient in dg-algebras does satisfy the property of effectivity of groupoid quotients. We shall come back to this in Section 3.2.5.

The theory of dg-algebras provides also a good setting in which to define and compute cotangent complexes. Coming back to the example of Section 2.1.1, let $A = k[x_1, \ldots, x_n]$ and E be the system of m functions determined by f. Then, the derived fiber Z is encoded by $K(E)$, which is freely generated (as a graded algebra) by n generators x_i in degree 0 and m generators y_j in degree 1. Recall that the module of Kähler differential of a free ring is a free module on the same generators. The situation is the same for $K(E)$, and the cotangent complex at some point $x : K(E) \to k$ is a free module on the same generators (renamed dx_i and dy_j); the only difference is a differential that can be computed to be the Jacobian matrix df of the function f, giving back the result of Section 2.1.1.

Finally, a word should be said about how homotopy colimits are related to better numerical invariants. The previous construction explained in terms of quotients can also be understood in terms of tensor products of rings (geometrically, this corresponds to describing a subspace as an intersection): $K(E)$ is also a model for the derived tensor product $A/a_1 \otimes^{\mathbb{L}}_A \cdots \otimes^{\mathbb{L}}_A A/a_n$, and $H_0(K(E))$ is the underived tensor product.[45] From this point of view, the higher homology of $K(E)$ is nothing but Tor modules. In the situation where A is local and of dimension n, the Serre intersection formula says that the correct multiplicity of the intersection is the Euler characteristic of the complex $K(E)$ and not the dimension of the crude $H_0(K(E))$.

The Serre formula is commonly used but totally ad hoc in classical algebraic geometry. It is only when one considers that the theory of algebras has to be enhanced into the homotopy theory of dg-algebras, which implies that colimits have to be replaced by homotopy colimits, that this formula finds its natural context.

3.2.4 Derivation and ∞-Categories

Let us organize what we have said so far. In the three contexts of sets, vector spaces, and rings, we have described a new operation of quotients: the homotopy colimit. It turns out that the proper setting to understand homotopy colimits (and in fact all homotopical phenomena) is higher category theory. Intuitively, if a category is thought as an enhancement of a set by allowing morphisms between elements, then an ∞-category is the enhancement of a category where morphisms (renamed 1-morphisms) have morphisms between them (named 2-morphisms), and 2-morphisms have 3-morphisms between them, ad infinitum. We shall not need much of the theory of ∞-categories; it will suffice to know a few things: (1) any category is an ∞-category; (2) the difference between the two notions is essentially that the arrows between two objects of an ∞-category do form a homotopy type and not a mere set;[46] and (3) the theory of ∞-categories is essentially the same as the theory of 1-categories: all the notions of adjoint functors, diagrams, colimits, and so on, make sense in this new context and behave the same way.[47]

There exists an ∞-category $\mathcal{S}$ of homotopy types; this is essentially because morphisms between two homotopy types do form a homotopy type and not a set (the 2-arrows are given by homotopies of maps, the 3-arrows by homotopies between homotopies, etc.). Let Set be the category of sets; there exists an embedding $Set \to \mathcal{S}$ by viewing sets as discrete homotopy types. With this in mind, we can explain that the homotopy colimit of a diagram of sets is nothing but the colimit of this diagram computed in the ∞-category $\mathcal{S}$.

[45] The tensor product of commutative rings is a particular colimit (a pushout), the derived tensor product is nothing but the corresponding notion of *homotopy* pushout of commutative dg-algebras.

[46] The easiest way to picture an ∞-categories is to think of a category enriched in topological spaces, where these spaces are considered only up to homotopy equivalence. The 2-arrows are given by homotopies, 3-arrows by homotopies between homotopies, etc. All higher arrows are invertible; we shall not consider here higher categories with non invertible higher arrows.

[47] The references for the formal definitions of ∞-categories theory are [35, 46]. The introduction of [46] explains the relations of ∞-category theory and classical homotopical features. Another good introduction to these ideas is [5].

The same thing happens for vector spaces and rings. The homotopy theory of chain complexes and dg-algebras can be understood (and should) as the fact that they are naturally objects of ∞-categories rather than ordinary categories. Let *Vect* and *Ring* be the categories of vector spaces and rings, and let $\mathcal{C}plx_{\geq 0}$ and $dg\mathcal{A}lg_{\geq 0}$ be the ∞-categories of chains complexes and dg-algebras in homological non negative degrees;[48] there exist embeddings

$$Vect \to \mathcal{C}plx_{\geq 0} \qquad \text{and} \qquad Ring \to dg\mathcal{A}lg_{\geq 0},$$

the notion of homotopy colimit in *Vect* and *Ring* is simply the natural notion colimit in the ∞-categories $\mathcal{C}plx_{\geq 0}$ and $dg\mathcal{A}lg_{\geq 0}$.

This kind of embedding of an ordinary category into an ∞-category, transforming automatically the way colimits (and also limits) are computed, can be taken as the formal meaning of the term "derivation" used in a homotopical/homological context. From this point of view, there is something arbitrary in the choice of the ∞-category used to derive a given category, and we could as well have considered the ∞-categories $\mathcal{C}plx$ and $dg\mathcal{A}lg$ of unbounded chain complexes and dg-algebras. We could also have considered, as in [72], the notion of simplicial algebras instead of dg-algebras, or, as in [48], the notion of E_∞-ring spectra.

The purpose of derivation (as I see it) is to embed a category C into a ∞-category $\mathcal{D}$ having better behaved operations of colimit and limit. The choice of $\mathcal{D}$ depends on the kind of properties we are interested in. We already mentioned the property of effectivity of groupoid quotients, that is the possibility to work on quotients of groupoids by equivariant methods. This property is also what allows the construction of classifying objects (e.g. for any kind of bundles in geometry or for extensions in ring theory; see Section 3.2.5).[49] [50]

Another nice property for $\mathcal{D}$ is to be generated by homotopy colimits from C. This explains the hypothesis of support for chain complexes: for example, $\mathcal{C}plx_{\geq 0}$ is generated by homotopy colimits by *Vect* but not $\mathcal{C}plx$, and $dg\mathcal{A}lg_{\geq 0}$ is also generated as such from *Ring* but not $dg\mathcal{A}lg$. Moreover, these examples of completion by colimits are free in a certain sense (see Footnote 55).[51]

3.2.5 Classifying Objects and Cotangent Complexes

In this section, we explain how the property of effectivity of groupoid quotients in the ∞-category of dg-algebras brings a nice feature to the theory of commutative rings.

Let us recall first that the category of homotopy types (and this would also be true in stacks) has classifying spaces and universal bundles. Let $p : P \to X$ be a principal G-bundle then there exists a map $X \to BG$ (unique up to homotopy) and a (homotopy) Cartesian square

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & * \arrow[d] \\ X \arrow[r] & BG. \end{tikzcd}$$

where the map $* \to BG$ is the one of Footnote 33. The interpretation of this result is that $* \to BG$ is the universal G-bundle and BG is the classifying space for G-bundles.[52] The important remark for the sequel is that such a construction cannot exist in sets: even if G, X, and P are sets, that is discrete homotopy

[48] If ∞-categories are pictured as categories enriched in spaces, the ∞-categories $\mathcal{C}plx_{\geq 0}$ and $dg\mathcal{A}lg_{\geq 0}$ are classically constructed as Dwyer–Kan simplicial localizations of model categories.

[49] The weaker property of *effectivity of equivalence relations* only (and not all groupoids; see Footnote 38) is one of the axioms of Grothendieck topoi, hence it has many instances within ordinary category theory. The stronger effectivity axiom becomes important once it is realized that most equivalence relations considered in practice are built by truncation of groupoids. Effectivity of groupoid quotients is one of the axioms of ∞-topoi. The strength of the effectivity axiom is precisely the difference between ordinary topoi and ∞-topoi, [46, 60] and the chapter [3].

[50] In relation with homotopy type theory, the property of effectivity of groupoid quotients can be thought as a stronger form of the univalence axiom (it implies univalence since it implies the existence of classifying objects). However, since no good theory of quotients exists in HoTT, the effectivity property cannot be stated in this language.

[51] A natural question about derivation is the necessity of ∞-categories. Why is it necessary to go beyond the notion of ordinary category? We shall come back to this in our conclusion.

[52] Classically, this map is denoted $EG \to BG$ and built in topological spaces, with EG a contractible space, but since we are working with homotopy types and not spaces, there is no difference between EG and the contractible homotopy type $*$.

types, the classifying space BG is a non discrete homotopy type. The possibility to construct such classifying objects is the main application of the property of effectivity of groupoid quotients.

Something similar exists in rings. Let A be a ring and M an A-module; then $A_M := A \oplus M$ has the structure of a ring where the product of two elements of M is zero. Such a ring is called a *trivial first-order extension* of A. It has a surjective map $A_M \to A$ with kernel M. More generally, any quotient of commutative rings $A' \to A$ with kernel M is called a *first-order extension* of A by M if the product of two elements of M is zero.[53] It is classical that such extensions can be classified by cocycles of A with values in M.

The work of Quillen on cohomology of commutative rings brought a nice reformulation of these notions in more "geometrical" terms. He showed that trivial first-order extensions of A are exactly the bundles of abelian groups over A (defined in the category of rings) and that the first-order extensions of A by M are precisely the torsors over the group bundle $A_M \to A$.

Now, within dg-algebras with their effectivity of groupoid quotients, it is possible to build classifying objects and universal fibrations for these situations. If A and M are fixed, the classifying dg-algebra for A_M-torsors is the trivial first-order extension $A \oplus M[1]$, but where M is put in degree 1, and the universal extension by M is the canonical map $A \to A \oplus M[1]$. More precisely, a map of dg-algebras (up to quasi-isomorphisms) $C \to A \oplus M[1]$ classifies an extension of C by M (where M is viewed as a C-module through the induced map $C \to A$) given by the (homotopy) Cartesian square

$$\begin{tikzcd} D \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & A \arrow[d] \\ C \arrow[r] & A \oplus M[1]. \end{tikzcd}$$

In other words, cocycles of rings can be represented by maps to some classifying object, but this object exists only in dg-algebras.[54]

This property, once interpreted geometrically, is the heart of the obstruction theory used in deformation theory, which was one of the motivations to define derived geometry. We shall not expand on this in this text; we refer to [68, 75] for a more detailed account.

The definition of first-order extensions of rings generalizes in the obvious way to dg-algebras. In particular, it is possible, even for a ring, to consider extensions where M is a complex of A-modules. In particular, it is possible to prove the following result: any dg-algebra A can be built from $H_0(A)$ as the limit of a tower of first-order extensions (called the Postnikov tower of A)

$$A \to \cdots \to A_{\leq n} \to A_{\leq n-1} \to \cdots \to A_{\leq 1} \to H_0(A)$$

where the extension $A_{\leq n} \to A_{\leq n-1}$ is by the module $H_n(A)[n]$.

The work of Quillen actually went a bit further. If $A \oplus M$ is a trivial first-order extension, then a map $C \to A \oplus M$ is simply a derivation of $C \to M$ (where M is viewed as a C-module through the induced map $C \to A$). Classically, such maps are in bijection with maps of C-module $\Omega_C \to M$, where Ω_C is the module of differentials of C. In the context of dg-algebras where M can be a chain complex, Ω_C is not enough, and derivations $C \to M$ are now represented by a complex of C-module $\mathbb{L}_C$, called the *cotangent complex*, such that $H_0(\mathbb{L}_C) = \Omega_C$. This proves that the cotangent complex is nothing but the module of differentials but in the context of dg-algebras. In particular, the maps $C \to A \oplus M[1]$ classifying extensions by M are equivalent to maps of chain complexes $\mathbb{L}_C \to M[1]$, so $\mathbb{L}_C$ "controls" the theory of extensions.

Geometrically, this says something new: it says that infinitesimal thickenings of a derived affine scheme are controlled by its tangent bundle. We refer again to [68, 75] for more details.

[53] Recall that, geometrically, first-order and, more generally, any nilpotent extensions, $B \to A$ corresponds to infinitesimal thickenings.

[54] In comparison with the situation of homotopy types, the role of A is played by the contractible type $*$, and the fact that M is in degree 1 in $A \oplus M[1]$ is similar to BG, where G is the π_1.

3.2.6 Other Contexts of Derived Rings

We have presented the ∞-category of dg-algebras as a convenient context to enhance the theory of rings. However, this is not the only context that is used to derive the theory of commutative rings and the corresponding geometry. We briefly review a list of other contexts.

In order to derive algebraic geometry, simplicial rings can also be used. They should be thought as a commutative ring with an underlying homotopy type instead of a set. This is the original setting for derived algebraic geometry presented in [72]. The theory of simplicial rings is equivalent to that of dg-algebras in characteristic zero, but better suited in positive characteristic. One can also use *unbounded dg-algebras*, but the theory turns out to be a bit awkward (essentially the infinitesimal theory is more subtle) some details are given in [72].

Lurie is using yet another model, E_∞-ring spectra (that is multiplicative cohomologies theories), which can be thought as commutative rings with an underlying spectrum (in the sense of stable homotopy theory). These objects are what become commutative rings when the full structure of associativity and commutativity of both the product and the addition are relaxed homotopically (in comparison, simplicial rings are such that the addition and commutation of the product are kept strict). This context, which was an original motivation for the whole theory, is well suited to transform into geometry the results of stable homotopy theory. This compares to the way arithmetic properties have been translated into geometry with the theory of schemes (see the introduction of [48]).[55]

The fruitful (and elegant) algebraization of the geometry of polynomial and rational functions has always inspired attempts to turn the geometries of differentiable and holomorphic functions into algebra. An important problem to do so has been the extra-structure that the rings of those functions should have. The first idea to do so is probably to add some topology to these rings (in order to take care of convergence questions). But Lurie, in his development of derived geometry, has advertised another way to do so.

For the purposes of differential geometry, the theory of simplicial C^∞-rings seems well suited.[56] [57] Some foundational work has been done in [36, 65], but the theory is not much developed yet. An obstruction is certainly the importance in the field of geometrical and analytical methods (in particular problems of convergence) over algebraic methods. Another might be the natural focus of the theory on smooth objects rather than singular ones, whereas scheme theory (derived or not) is intended for a study of singular objects. However, the recent development of derived symplectic geometry, and the potential application to Floer homology, could bring some motivations for the development of the theory.

Finally, Lurie propose also a similar theory for the purposes of complex geometry. The idea is the same–try to avoid topological rings and find an algebraic structure encoding holomorphic functions–but the definition is a bit more subtle than for C^∞-rings.[58] However, the theory seems to work nicely enough to have results such as a GAGA theorem [57].

[55] The plurality of these generalizations of commutative rings can be explained with the notion of algebraic theory (in the sense of Lawvere). The problem to define "homotopy rings" can be set into the problem of finding an algebraic theory such that the models in the category of sets are commutative rings; then the models in the category of homotopy types will be "homotopy rings". Dg-algebras, simplicial rings, and E_∞-ring spectra are different such theories. Moreover, if the algebraic theory is encoded by an ordinary category, the category of models in homotopy types is always a colimit completion of the category of models in sets.

[56] And probably the "dg-" version also, although I don't know if anybody has looked into this.

[57] The idea of a C^∞-ring is the following: in a classical ring A, the operations of sum and product can be combined to give more elaborated n-ary operations $A^n \to A$ indexed by any polynomial in $\mathbb{Z}[x_1, \dots x_n]$. With this in mind, C^∞-rings are rings with n-ary operations given not only by polynomial functions but by any C^∞ function of n variables; for example, in a C^∞-ring, it makes sense to compute $\cos(e^x + y)$ for any elements x and y. An important difference with classical rings is that these operations cannot be generated from the sum and product only. The reader interested in learning more should look at [55] and learn about Lawvere theories.

In contrast to topological rings, C^∞-rings, by enhancing the addition and product of the ring with operations indexed by C^∞ functions, provide a *purely algebraic* setting where methods from algebraic geometry can be adapted. In particular, the tensor product of C^∞-rings is the good one with no need for completion.

[58] The trouble is essentially the fact that not every open subset is the complement of zeros of a holomorphic function. This forces us to encode within the ring of holomorphic functions the data of these extra open subsets. For this reason, the definition of a *holomorphic ring* is a bit more involved than the definitions of C^∞-rings and ordinary rings (see [51, 57]).

3.3 Geometry of Derived Affine Schemes

We have explained how the theory of rings could benefit from its enlargement into the theory of dg-algebras. Although it should be clear enough that most results of commutative algebra have analogues for dg-algebras, it is not immediately clear that dg-algebras have also a geometrical side. The geometric objects corresponding to dg-algebras are named *derived affine schemes*, but what do they look like, and what are their new features with respect to ordinary affine schemes?

As we mentioned, classical commutative rings are related to geometry because it is possible to interpret the language and structure of geometry (points, open subsets, fibrations, etc. their expected structural relations) in the opposite category of commutative rings. We shall sketch how it is possible to do the same for dg-algebras.

3.3.1 Comparison with Ordinary Schemes and Infinitesimal Structure

If A is dg-algebra (in non negative homological degrees), there always exists a quotient map $A \to H_0(A)$ (killing higher degree elements). This suggests that the derived affine scheme X corresponding to A is related to the classical affine scheme X_0 corresponding to $H_0(A)$ by a map $X_0 \to X$. X_0 is called the *classical part* or *truncation* of X.

It is reasonable to think that the points of this geometry will still be in correspondance with fields in dg-algebras. But, for reasons of degree, such fields have to be concentrated in degree 0, so there are no new points in the new geometry. Actually more is true: a map $A \to k$ from a dg-algebra A to a field k has to factor through $H_0(A)$. This says that the map $X_0 \to X$ induces a bijection on points.

It is also reasonable to expect that the correspondance between ring quotients and closed immersions will stay true in the new context. So $X_0 \to X$ would be a closed immersion inducing a bijection on points. Classically, this would say that X is an infinitesimal thickening of X_0. And this is the way it turns out to be: a derived affine scheme is an ordinary affine scheme endowed with an infinitesimal thickening, although this thickening can be of a new kind.[59]

3.3.2 Self-Intersections

One of the interests of schemes is that they give rise to spaces of infinitesimally closed points. For example, the ring $\mathbb{C}[x]/x^n$ is the ring of function on the subspace of the affine line $\mathbb{A}^1$ formed by $n+1$ points infinitesimally close. Derived geometry also gives rise to new spaces, for example, spaces of self-intersections.

Let us consider the simplest case of the self-intersection of a point in the line. The dg-algebra corresponding to this is the (derived) tensor product $\mathbb{C} \otimes_{\mathbb{C}[x]} \mathbb{C}$, which can also be written as a quotient of $\mathbb{C}[x]$ by the irregular system of equations $\{x = 0, x = 0\}$. With our previous notations, if $A = \mathbb{C}[x]$ and $E = \{x, x\}$, the computation of $K(E)$ gives the dg-algebra $\mathbb{C} \oplus \mathbb{C}[1]$ (with trivial differential). This algebra has to be understood as such: it is essentially $\mathbb{C}$, but with a non trivial loop at any number (given the the generator of $\mathbb{C}[1]$).

Let X be the geometric object corresponding to $K(E)$; its classical part X_0 is a point x since $H_0(\mathbb{C} \oplus \mathbb{C}[1]) = \mathbb{C}$. What is the thickening of the point giving X? It is difficult to say: the structure is "purely derived" and not easy to describe geometrically, particularly in classical terms. However, a few things can be said. First, recall from Section 2.2.2 that, for the pointed scheme $(\mathbb{A}^1, 0)$, the homotopy fiber product $* \times_{\mathbb{A}^1} *$ is nothing but the loop space of $\mathbb{A}^1$ at 0. Since $* \times_{\mathbb{A}^1} *$ is also the self-intersection X, the non triviality of X means that the affine line has a non trivial loop space at 0!

This situation is not specific to $\mathbb{A}^1$: unless the point is isolated, the self-intersection $* \times_X *$ of a point x in any derived affine scheme X will be non trivial. Then, since a loop space is a group, we can understand its Lie algebra and its actions. The Lie algebra of $* \times_X *$ can be proved to be $\mathbb{T}_x X[1]$, the tangent complex

[59] Formally, such a thickening is still encoded by nilpotent extensions, as mentioned in Section 3.2.5, but they are harder to interpret geometrically when the extension module has positive homology. With classical infinitesimal thickenings, the limit of two points converging to each other is understood as two points infinitesimally closed, i.e., a vector. But derived infinitesimal thickenings encompasses also the limit of the intersection of two lines (say in the plane) when the lines become the same. What then become of the intersection point at the limit? I don't know how to understand this in geometrical terms.

of X at x with degree shifted by -1 (see Section 2.2.2). As for the action of $* \times_X *$, let us say only that in the same way the loop group of a homotopy type acts on the fiber of any covering space, $* \times_X *$ acts on the fiber at x of any A-module (where A is the dg-algebra corresponding to X) [38, 28].

Finally, it can be proved that the classifying space (or better, the classifying scheme) of this group is the formal completion of X at x (see [68] for more on the matter).[60]

3.3.3 Intrinsic Geometry

The definition of geometric notions in derived algebraic geometry is done essentially the same way as in algebraic geometry. We refer to [47, 48, 72] for details.

Points correspond to fields (viewed as dg-algebras concentrated in degree zero), and the set of points of a derived scheme X is defined as in algebraic geometry by equivalence classes of maps from points. It is possible to define a notion of localization for dg-algebras and a corresponding notion of *Zariski open immersion*. It is also possible to define the notion of *smooth* and *étale* maps for dg-algebras (either by lifting properties of infinitesimal thickening or by using the cotangent complex). Moreover, all these notions satisfy the expected properties (stability by composition, base change, locality, etc.).

Once the idea that the extra structure of a derived affine scheme is infinitesimal is accepted, it is easy to reduce certain features of X/A to features of $X_0/H_0(A)$. For example, since open subsets contain the infinitesimal neighborhood of their points, the Zariski open subsets of X are in bijection with Zariski open subsets of its truncation X_0 in the classical sense. Also, using the infinitesimal lifting property, the category of étale maps over X can be shown to be equivalent to that of étale maps over X_0. As a consequence, a dg-algebra A is local (resp. Henselian) if and only if $H_0(A)$ is local (resp. Henselian). Another consequence is that the extra derived structure does not create more Zariski or étale topology, with the consequence that the Zariski and étale spectra of A and $H_0(A)$ will coincide.

There exists also a notion of *closed immersion*, corresponding to (homotopy) quotients of dg-algebras, and this is a notion with quite a different behavior than the classical one. Essentially, the main difference could be summarized by saying that closed immersions $Y \to X$ are not immersions.[61] If $Y \to X$ is a closed immersion, then it is not a monomorphism, that is the canonical diagonal map $Y \to Y \times_X Y$ is not an isomorphism, as we saw in Section 3.3.2.

It is possible to define also notions of *maps of finite presentation*, *flat maps*, *separated maps*, and *proper maps* (although this notion is more interesting for non affine schemes). Altogether, the whole formal structure of algebraic geometry (in particular, all the aforementioned classes of maps) has been successfully transposed into derived algebraic geometry.

3.3.4 Geometrization via Spectra

Derived schemes are presented in [48] and [68] as particular ringed spaces. I have voluntarily chosen to delay this presentation because it is not intrinsic. Indeed, such a presentation using topological spaces is implicitly based on the notion of the Zariski spectrum of a dg-algebra (generalizing the same notion for rings) but other notions of spectra exist (étale, Nisnevich, etc.) and there is no reason to prefer the Zariski one to the others (in fact, there are good reasons not to).

From the class of Zariski open immersions into a derived scheme X, it is possible to construct a topological space $Spec_{Zar}(X)$ called the *Zariski spectrum of* X (or of A if X is dual to A). Similarly, from the étale maps with target X, it is possible to construct a topos[62] $Spec_{et}(X)$ called the *étale spectrum of* X. Both these spectra functors are not faithful (a lot of dg-algebras have the same spectrum) but it is possible to

[60] This relationship between the formal neighborhood of a point and its loop stack, called *infinitesimal descent* in [68], is still a mysterious feature of derived geometry for me.

[61] Algebraically, this is somehow related with the fact that the notion of ideal is no longer suited to describing homotopy quotients of dg-algebras: ideals only encode quotients by equivalence relations, but more is needed to describe all homotopy quotients.

[62] Topoi are what become topological spaces if the poset of specialization maps between points is allowed to be replaced by a category. They provide a notion of highly unseparated spaces (way below T_0-spaces) well suited to study spaces such as bad quotients of group actions and moduli spaces; see [3].

enhance them into faithful functors by adding a structure sheaf. Moreover, together with their structure sheaf, the spectra can be interpreted as moduli spaces such that the structure sheaf becomes the universal family: the Zariski spectrum of A is the moduli space of localizations of A, and the étale spectrum of A is the moduli space of strict henselisations of A.[63] This nice point of view, inherited from classical considerations on spectra (since at least [27]), is fully developed for derived geometry by Lurie in [48, 50].

Spectra are nice because they are genuine spaces and not the abstract objects of the opposite category of rings or dg-algebras. They are definitely useful for a geometric intuition: for example, the Zariski spectrum of a ring can convince anybody immediately that rings indeed have something to do with geometry. However, the representation of the spatial nature of a ring via its Zariski spectrum (or any spectrum in fact) can also be misleading. The reason is that some geometrical features can be well understood in terms of the underlying space of the spectrum, but others still hide within the structure sheaf. For example, a scheme can have many closed subschemes, but its Zariski spectrum can be a single point. Also, the underlying space of the Zariski spectrum does not commute with products (there are more points in $\mathbb{A}^2$ than pairs of points in $\mathbb{A}^1$). Hence, the Zariski topological space reflects quite poorly the intrinsic geometry of affine schemes (already in the underived setting). The étale spectrum is better[64] but also suffer the problems listed above. Another drawback of both Zariski and étale spectra, specific to derived geometry, is that they do not reflect the non trivial nature of loop spaces at a point described in Section 3.3.2, and one is forced to hide this bit of geometry within the structure sheaf.[65]

In fact, each notion of spectrum should be taken as a reflection, or a projection, of the true, or intrinsic, geometric nature of affine schemes (derived or not); and these projections are preferred one to another, as Poincaré would say, for the sake of convenience (typically to define or to control cohomology theories). In my opinion, the most convincing fact that dg-algebras (or, already, rings) have a geometrical side is not so much their spectral theories as the successful interpretation of the whole language and expected structure of geometry they provide. To be able to talk about points, open subsets, étale maps, proper maps, fiber bundles, and other geometrical features is more important to doing geometry than to be able to faithfully describe a ring or a dg-algebra as a genuine space with a structure sheaf.[66]

3.4 Derived stacks

3.4.1 Quotients of spaces and stacks

The construction of non affine spaces in algebraic geometry is done by taking affine schemes as elementary building blocks and by defining the new objects to be formal pastings, that is formal colimits, of these blocks. Let *Aff* be the category of affine schemes, the construction of all formal colimits from objects of *Aff* is the meaning of the category $\mathrm{Pr}(\mathit{Aff})$ of presheaves of sets over *Aff*.[67] However, this construction is "too free" because it does not paste the pieces of a Zariski (or étale) atlas of an affine scheme X to X itself.[68] This leads to distinguishing certain pastings of affine schemes (Zariski, étale, or any other class of

[63] This point of view is also suited for the Nisnevich spectrum, which is the space enlarging the étale spectrum by classifying all henselizations of A and not only the strict ones.

[64] An important drawback of the Zariski spectrum is that it does not reflect étale maps of rings into étale maps of topological spaces, which forbids from using it to define cohomology theories with étale descent. This problem was the motivation to introduce the étale spectrum in order to define étale cohomology, but the toll was to work with spaces even more unseparated than Zariski spectra. These spaces, which cannot be defined as topological spaces, were themselves the motivation for introducing topoi. The situation is the same in derived geometry, where ∞-topoi must be used.

[65] Is there a notion of spectrum that could reflect this "infinitesimal descent" (as Toën calls it in [68]) in terms of classical descent in topoi? See also Section 3.3.2 and Footnote 60.

[66] Such an interpretation of the language of geometry is precisely what is missing for noncommutative rings, essentially because there is no good notion of locality for noncommutative rings (which leads Kontsevich to joke about "noncommutative non-geometry"). In consequence, there exists no noncommutative geometry in the sense of building a spectral theory for noncommutative rings. There are, however, a number of features of geometrical objects that can still be defined for noncommutative objects (properness, smoothness, etc.) [18].

[67] It is a classical result of category theory that the Yoneda embedding $C \to \mathrm{Pr}(C)$ of a category C into its presheaves of sets, sending an object to its so-called *functor of points*, is the free completion of C for the existence of colimits.

[68] Recall that an étale atlas for an affine scheme X is a family of étale maps $U_i \to X$ covering X. In the category *Aff*, X can be recovered as the quotient of an equivalence relation on the disjoint union $\coprod_i U_i$ (subcanonicity of étale topology). However,

atlases) and to considering rather the completion $\mathrm{Sh}(\mathit{Aff})$ of Aff that preserves these pastings. Technically, the distinguished atlases define a Grothendieck topology, and the completion is the category of sheaves of sets for this topology.[69]

Now, following the philosophy presented in Section 3.2.1, in order to have a better behavior of colimits, we should replace them by homotopy colimits in the construction of sheaves. This is precisely the definition of stacks: *stacks are to sheaves what homotopy types are to sets*. Or, in other terms, the category of stacks is the free homotopy colimit completion of the category of affine schemes (preserving atlases).[70] Technically, presheaves of sets are replaced by presheaves of homotopy types, called prestacks, and the definition of stacks copies that of sheaves by imposing that the homotopy colimits of atlases of X be X itself.[71] Intuitively, this means that we need to change our idea of a space with a underlying set of points to a notion of space with an underlying homotopy type (or ∞-groupoid) of points. The resulting morphisms between the points are of the same nature as the specialization morphisms in classical topology.

As with sets, complexes, and dg-algebras, a fundamental property of the category of stacks is the effectivity of groupoid quotients. This property is actually the essential reason to consider stacks and stacky quotients: if a group G acts on space X, then anything defined over the quotient stack $X/\!/G$ (function, bundle, sheaf, etc.) can equivalently be described by something over X that is equivariant for the action of G.

Finally, *derived stacks* are defined exactly the same way starting from the ∞-category $d\mathcal{A}\mathit{ff}$ of derived affine spaces instead of Aff, relatively to one of the Grothendieck topologies existing on them (usually the étale topology) [48, 50, 72].

3.4.2 Geometric Stacks

Derived stacks are formal, or free, homotopy colimits of derived affines schemes in a special sense. The category of stacks is useful because of this completion property, but, as often with completions, it turns out that the general object of this category is quite wild and improper to the purposes of geometry. The problem is the local structure of these objects: for example, the tangent spaces may not have an addition.[72] Concretely, in order to have a tamed, or *geometric*, local structure (with tangent spaces, infinitesimal neighborhoods, and everything), we would like to describe the neighborhood of a point in a stack, as for affine schemes,

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the quotient $|U_\bullet|$ of this equivalence relation in the category $\mathrm{Pr}(\mathit{Aff})$ of presheaves (of sets) will not be (the image by Yoneda embedding of) X. Indeed, because colimits are "freely" added in $\mathrm{Pr}(\mathit{Aff})$, the Yoneda embedding $\mathit{Aff} \to \mathrm{Pr}(\mathit{Aff})$ does not preserve the colimits existing on Aff. All we get is a canonical morphism $|U_\bullet| \to X$. The category of sheaves is then defined by imposing that the maps $|U_\bullet| \to X$, for all étale coverings of all affine schemes, be isomorphisms.

Technically, such an operation is a localization of categories and corresponds to a quotient of $\mathrm{Pr}(\mathit{Aff})$, but it can also be described as a full subcategory of $\mathcal{P}(\mathit{Aff})$ (this is similar to the fact that a quotient can be described as a subset by imposing a gauge condition). Precisely, a presheaf F is called a *sheaf* (for the étale topology) if it "sees" all the maps $|U_\bullet| \to X$ as isomorphisms, that is if all the maps $\mathrm{Hom}(X, F) \to \mathrm{Hom}(|U_\bullet|, F)$ are isomorphisms. This condition is often called the *descent condition* [16, 64, 67].

[69] For the reader unfamiliar with sheaves, this completion is analogous to the more classical completion of $\mathbb{Q}$ into $\mathbb{R}$: a sheaf is very close to a Dedekind cut and sheaves are defined as "formal colimits" in the same way real numbers are defined as formal limits of sequences.

[70] Stacks were invented and used before this universal property was understood. This elegant description of the category of stacks (which only is true if stacks means higher stacks or ∞-stacks) is the result of a long interaction between algebraic geometry (where stacks were defined first), algebraic topology (were all the necessary homotopical techniques were invented) and the philosophy of higher category theory (where the notion of homotopy colimit completion was conceived).

[71] This is done as in Footnote 68. Recall the inclusion $\mathit{Set} \to \mathcal{S}$ of the category of sets into the ∞-category of homotopy types from Section 3.2.4. Let $\mathcal{P}(\mathit{Aff})$ be the category of functors $\mathit{Aff}^{op} \to \mathcal{S}$ (called *prestacks*). The inclusion $\mathit{Set} \to \mathcal{S}$ induce an inclusion $\mathrm{Pr}(\mathit{Aff}) \to \mathcal{P}(\mathit{Aff})$ and a Yoneda embedding $\mathit{Aff} \to \mathcal{P}(\mathit{Aff})$. The category of stacks (for the étale topology) is defined from $\mathcal{P}(\mathit{Aff})$ by inverting the maps $|U_\bullet| \to X$, for all étale coverings of all affine schemes. It can be described as sub-category of prestacks: a prestack F is called a *stack* if the maps $\mathrm{Hom}(X, F) \to \mathrm{Hom}(|U_\bullet|, F)$ are all equivalences of homotopy types [16, 67, 64].

[72] It is only possible to define a tangent cone. The situation is the same with diffeologies [34]. This is also related to the *microlinearity* condition in synthetic differential geometry.

by means of a local ring (maybe Henselian or more).[73] [74] However, this is not possible in general. Any stack Y can be described as the homotopy colimit of some diagram of derived affine schemes X_i, and the local structure of Y can be tamed only if the neighborhoods of points of Y can be lifted to neighborhoods of points in some X_i for some diagram presenting Y, that is if the quotient map $\coprod_i X_i \to Y$ has a lifting property for neighborhoods.

A map $f : X \to Y$ is said to have the *lifting property for étale neighborhoods* if, for any scheme Z, corresponding to a strict Henselian local ring A, with unique closed point named z (such a Z is an étale neighborhood of z) and for any commutative square as follows, there exists a diagonal lift

$$\begin{tikzcd} z \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Z \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

A sufficient condition for a map to admit a such lifting property is to be smooth (submersive); moreover, the lift is unique if the map is étale.[75] This leads to considering those diagrams such that the quotient map $\coprod_i X_i \to Y$ is smooth or étale. This is done by restricting to specific diagrams, called smooth or étale *internal groupoid objects*, and by considering only colimits that are quotients of those groupoid objects. If X is the quotient of a groupoid G, then G is called an *atlas* for X. The choice of smooth or étale maps distinguishes two classes of geometric stacks called, respectively, *Artin stacks* and *Deligne–Mumford stacks*. For such stacks, it is possible to define local features such as tangent complexes, and we get back all computations of Section 2. We shall not say more here and refer to [66, 68] for more explanation and to [48, 59, 72] for the details. Table 2 summarizes all the types of geometrical objects that we have mentioned.

3.4.3 Geometry of Derived Stacks

We have finally arrived at the notion of derived geometric stack, which are the spaces of derived algebraic geometry.[76] It is convenient to compare them with classical schemes or sheaves by mean of a diagram (which I borrow from Toën and Vezzosi):

$$\begin{tikzcd} \mathit{Ring} = \mathit{Aff}^{op} \arrow[rr, "\text{sheaves (schemes, alg. sp.)}"] \arrow[drr, "\text{(DM, Artin) 1-stacks}"] \arrow[ddrr, "\text{(DM, Artin) } \infty\text{-stacks}"] \arrow[dd] & & \mathit{Set} \arrow[d] \\ & & \{\text{1-groupoids}\} \arrow[d] \\ \mathit{dgAlg} = \mathit{dAff}^{op} \arrow[rr, "\text{(DM, Artin) derived stacks}"'] & & \mathcal{S} = \{\text{Homotopy types}\} \arrow[u, bend right, "\pi_1"'] \arrow[uu, bend right=60, "\pi_0"'] \end{tikzcd}$$

The arrows from the left to the right side are the *functor of points* of the differents objects of the theory. On the left side are the elementary blocks, or affine objects, and on the right are the categories of values for the *points* of the objects. It is convenient to read this as "the points of a scheme (or an algebraic space) form

[73] This problem is already present with sheaves; it has nothing to do with the homotopical nature of stacks.

[74] In the analogy of Footnote 69 between sheaves (or stacks) and real numbers, the wild nature of general sheaves and stacks compares well to that of general real numbers. To work with a given real number, it is better to give it more structure, like the property to be algebraic. Algebraic numbers compare well with "tamed", or "geometric", sheaves and stacks.

[75] Several kinds of neighborhoods exists in (classical or derived) algebraic geometry. Let X be an affine scheme, A its ring of functions, k a field, and x a point of X given by a map $A \to k$. The most common neighborhoods of x in X are the Zariski, Nisnevich, étale (related to the corresponding Grothendieck topologies) and formal neighborhoods. They are defined, respectively, as the local ring of A at x, the Henselian local ring of A at x, the strict Henselian local ring of A at x, and the formal completion of A at x. Depending on the kind of local structure we are interested, the lifting property for maps can be defined with respect to any of these notions. The most common choices are the *smooth* and *formally smooth* maps defined with the étale and formal neighborhoods, respectively. All this can be transcribed in analytic and differential geometry, where there are even more choices of neighborhood notions (e.g., Gevrey classes).

[76] Their definition can be transposed in differential or complex geometry with no trouble.

Table 2: Types of spaces

<table>
<tr><th></th><th>Basic objects</th><th>Objects with Zariski atlas</th><th>Objects with étale atlas</th><th>Objects with smooth atlas</th></tr>
<tr><td>Sheaves (ordinary AG)</td><td rowspan="2">affine schemes</td><td rowspan="2">schemes</td><td colspan="2">algebraic spaces</td></tr>
<tr><td>Stacks (AG + derived colimits)</td><td>Deligne–Mumford stacks</td><td>Artin stacks</td></tr>
<tr><td>Derived stacks (AG + derived limits and colimits)</td><td>derived affine scheme</td><td>derived schemes</td><td>derived D-M stacks</td><td>derived Artin stacks</td></tr>
<tr><td>Tangent complex</td><td>in d° ≤ 0</td><td>in d° ≤ 0</td><td>in d° ≤ 0</td><td>unbounded</td></tr>
</table>

a set, the points of a 1-stack form a groupoid, and the points of a derived stack form a homotopy type (or ∞-groupoid)".

We mentioned in the introduction that derived stacks could be truncated like chain complexes; this is done by means of the previous diagram. If $X : d\mathcal{A}\!f\!f \to \mathcal{S}$ is a derived stack, its *derived truncation* is the classical ∞-stack $Z_{\geq 0}X = \mathcal{A}\!f\!f \to d\mathcal{A}\!f\!f \to \mathcal{S}$, and its *coarse space* is the sheaf $\pi_0 Z_{\geq 0}X = \mathcal{A}\!f\!f \to Set$. At the level of the tangent complexe $\mathbb{T}_x X$, for some point x in X, these operations correspond to the positive truncation $Z_{\geq 0}\mathbb{T}_x X$ and to $H_0(\mathbb{T}_x X)$.

Finally, let us mention that all the notions developed in Section 3.3.3 for derived schemes (points, open immersion, étale, smooth and proper maps, etc.) can be generalized for derived stacks, so the language of geometry finds yet another model. In particular, it is possible to define Zariski and étale spectra of stacks. A very nice theorem of Lurie [50] states that Deligne–Mumford derived stacks can be described faithfully by their étale spectra (which is an ∞-topos) together with the canonical structure sheaf. We refer to [48, 72] for the definitions of geometric features of derived stacks.

3.5 Derived geometry and the BRST construction

Applications of derived geometry are largely explained in [66, 68], we shall say only a few things about it related to the connection with symplectic geometry and BRST construction.

We mentioned in the introduction that the first formalisms of derived geometry were algebraic, using dg-algebras (commutative or Lie) as opposed to geometric. Perhaps the origin of this is to be found in physics when the BRST construction was developed in the 1970s. We shall briefly tell how to interpret the algebraic construction of [43] in terms of derived geometry.

Recall first that a symplectic reduction is constructed in two steps: from a symplectic manifold, we extract a subspace (possibly with intersection singularities) and then take a quotient by a group action (possibly creating further quotient singularities). From everything that we have tried to explain in this text, it should be clear that derived geometry is the perfect setting in which to compute such a reduction. And indeed, such a reduction is one of the main operations of the new field of *derived symplectic geometry*, where it is proved that the resulting object, however singular, is still symplectic; see [7, 8, 56] for more details.

Now, in order to make the connection with [43], we are going to construct a functor $\mathcal{O} : d\mathcal{S}tack \to dg\mathcal{A}lg$ that sends a derived stack X to an unbounded dg-algebra $\mathcal{O}(X)$ playing the role of the algebra of (globally defined) functions in X. Let $d\mathcal{S}tack$ be the ∞-category of derived stacks, and recall that the Yoneda embedding $d\mathcal{A}\!f\!f \to d\mathcal{S}tack$ is a homotopy colimit completion; then the construction of $\mathcal{O}$ is done using this universal property: if X is a derived affine scheme corresponding to a dg-algebra A, we put $\mathcal{O}(X) = A$, and for a general derived stack X, we write it as a colimit of derived affine schemes $X = \operatorname{colim} X_i$, and we put $\mathcal{O}(X) = \lim A_i$, where the (homotopy) limit is computed in the ∞-category $dg\mathcal{A}lg$ of unbounded dg-algebras. The resulting functor $\mathcal{O} : d\mathcal{S}tack \to dg\mathcal{A}lg$ is far from faithful, so it is not true that derived geometry can be replaced by a study of unbounded dg-algebras. This setting is more the poor man's approximation to derived geometry.

Coming back to [43], let us start with a symplectic algebraic variety[77] M with a group action $G \times M \to M$ and a moment map $\mu : M \to \mathfrak{g}^*$. The system of equations $\mu = 0$ (usually irregular) can be used to define the level set $\mu^{-1}(0)$ as a derived affine scheme X, encoded by a Koszul dg-algebra $K(\mu = 0)$ (see Section 3.2.3), which is $\wedge\mathfrak{g} \otimes \mathcal{O}(M)$ with $\mathfrak{g}$ in (homological) degree 1 and the Koszul differential d. Now recall that, by design, $\mathcal{O}$ send colimits of derived stacks to limits of dg-algebras; in particular, if M_0 is the quotient of X by the canonical action of G, $\mathcal{O}(M_0)$ will be described as a limit construction involving $K(\mu = 0)$ and functions on G. This limit is the meaning of the Chevalley complex construction built from $\mathfrak{g}^*$ on $K(\mu = 0)$, which is $\wedge\mathfrak{g} \otimes \mathcal{O}(M) \otimes \wedge\mathfrak{g}^*$ (where $\mathfrak{g}^*$ in degree -1) with total differential $d + \delta$, where the Koszul differential d is twisted by the Chevalley differential δ. This corresponds exactly to the two steps of the construction of [43] and interprets the resulting dg-algebra as the algebra of functions on the derived symplectic reduction M_0.

4 Conclusion

4.1 Success and prospects

The background for the development of derived geometry was a number of techniques used in geometry (algebraic, complex or differential) but not fully justified by geometrical reasons, they were ad hoc computational tools, efficient but somehow mysterious. Here is a short list:[78]

- homological features in commutative algebra (derived tensor products of rings, Serre formula, Koszul resolutions, cotangent complexes, virtual classes, etc.);
- equivariant techniques in geometry to work on bad quotients;
- cohomology with coefficients in bundles (or sheaves), derived categories and functors;
- the chain complexes with their Lie structure controlling deformation problems.

We have tried to explain how derived geometry has successfully found *geometrical principles* behind these constructions:

- Once commutative rings (affines schemes) are enhanced into dg-algebras[79] (derived affine schemes), derived tensor products, Serre formula, Koszul resolutions, cotangent complexes, and virtual classes become the proper notions of tensor product, intersection formula, quotient, Kähler differential, and fundamental classes (see Section 3.2.3 and Section 3.2.5). Moreover, this enhancement is of geometric nature (albeit surprising) because the full language and structure of geometry make sense in derived affine schemes (see Section 3.3.3).
- Equivariant techniques correspond indeed to work on the quotient if groupoid quotients are effective, which forces to compute it in the higher categories of stacks (see Section 3.4.1). Moreover, as above, this extension of affine objects by stacks is of a geometric nature because the full language and structure of geometry extends to stacks (see Section 3.4.3).

[77] The setting for symplectic manifolds has not yet been fully developed.

[78] To which one could add the use of geometric methods to understand some features of stable homotopy theory (the stack of formal groups for cobordisms, the stack of elliptic curves for topological modular forms), which was also ill justified by the theory. But I'm going to forget this aspect and restrict to more classical geometry. See [48] for more on the matter.

[79] Or simplicial algebras, or E_∞-ring spectra.

- Once accepted the context of dg-algebras, the natural categories of modules are formed by chain complexes and not only modules, even for ordinary rings. From this point of view, the old abelian categories of modules are no longer regarded as fundamental objects and the problem of derivation of functors that goes along with them somehow disappears. Derived functors are replaced by the natural functors between ∞-categories of chain complexes.[80] By extension, the categories of modules on a derived stack are also ∞-categories of chain complexes.[81]
- The chain complexes of deformation problems are simply the tangent spaces of derived geometry. The Lie structure is explained by the loop stacks (see Section 2.2.2), the obstruction theory by the structure theory of dg-algebras (see Section 3.2.5).

Altogether, this presents derived geometry as a new theory of geometry, not only encompassing the classical geometry, but having much better computational properties with respect to the whole homological/homotopical apparatus. The heart of this new geometry is to keep track of the inherent ambiguity to identify things when computing quotients (whether they be quotients of rings or quotients of spaces) by considering always homotopy quotients.[82] This is incredibly well suited for the study of moduli spaces as can be checked by the list of examples in [66, 68].

Having said so, the above tools and techniques certainly do not exhaust the methods existing in geometry, and derived geometry is not the answer to all problems.[83] As we have tried to explain, it is only an enhancement taking better care of intersection and quotient singularities. But since singular spaces appear also in differential and complex geometries, derived geometry does have something to say in these contexts. For example, because of the more regular tangential structure at singularities given by the tangent complexes and not only the tangent spaces, it is possible to define on the whole of singular spaces (and not only on the smooth locus) a proper notion of differential forms and of symplectic or Poisson structures. These definitions have led to a huge extension of the notion of symplectic variety and to a very nice algebra of operations producing symplectic spaces from other ones (symplectic reductions of course, but also intersections of Lagrangian maps and mapping spaces) [7, 8, 9, 56, 69]. So far this extension has only been done in the algebraic setting, but no doubt something similar could be done in differential and complex geometries .

4.2 Higher Categorical Mathematics

Some ideas subsuming derived geometry are not actually specific to geometry, they are ideas of higher category theory. The cross-breeding, in the 1990s to 2000s, of algebraic geometry, algebraic topology, and category theory, which gave birth at the same time to higher stack theory and higher category theory, may very well be one of the most fruitful of mathematics because it lays the groundwork for a deep revolution that we would like to advertise.[84]

We left pending in Section 3.2.4 and Footnote 51; the question of the necessity of ∞-categories in the derivation process. It is indeed a natural question to ask why ∞-categories have become so important. The answer, that we can only sketch here, is simple and deep: ∞-categories provide computational properties inaccessible to ordinary categories. For example, we have underlined several times in this text the role played by the important property of *effectivity of groupoid quotients*, that is the ability to work on quotients of groupoids by equivariant methods (see Section 3.2.4). This property of a category has the most remarkable property: if it is true in an ordinary category, then this category is the trivial category with one object.[85]

[80] This approach to the derivation of functors does even specify the role played by Abelian categories where they become *hearts* of *t-structures*. Ironically, in derived geometry, it is the notion of abelian category that is *derived*.

[81] Strictly speaking, when presented this way, only non negative chain complexes should be considered. The consideration of unbounded chain complexes is motivated because they have the nice extra property of *stability* (see Footnote 87).

[82] Instead of derived geometry, I think this should be more appropriately called the *geometry of ambiguity*, hence the title of this text.

[83] Notably, derived geometry has little to say about spaces of infinite dimension.

[84] Actually, derived geometry may very well be the first field born out of higher categorical ideas. Other fields could be stable category theory [47], homotopy type theory [33, 62], derived symplectic geometry [8].

[85] More precisely, $(n-1)$-groupoids can be effective in n-categories, but the effectivity of n-groupoids in an n-category forces it to be trivial. Only when $n = \infty$ can n-groupoids be effective in an n-category.

This means than this property can only have non trivial models in proper ∞-categories (and it does–we have seen a few)![86] Another very important such property, although outside the scope of this text, is the axiom of *stability*, which enhances and simplifies the theory of triangulated categories [47].[87] [88]

A second, quite natural question about ∞-categories is, why are homotopy types (or ∞-groupoids) so important? Here again there exists a simple and deep answer: because they can be seen as a notion more primitive than sets. At first, this might seem silly, because all mathematical objects, including homotopy types, are classically defined using sets as a primitive notion. But, assuming the notion of homotopy types, it is also possible to define sets as *discrete homotopy types*, that is, as homotopy types with a specific property.

Following this line of thought proposes to redefine the whole of mathematical structures with an underlying homotopy type instead of an underlying set. The first motivation for such a bold idea is the fact that examples of such notions of "structures with underlying homotopy type" exists (as we have tried to advertise in this text). But the main motivation is the fact that homotopy types provide a notion of identification for elements (through paths) more suited to talking about certain structures. We already mentioned how homotopy types could be seen as an enhancement of sets incorporating the ambiguity of identifications (see Section 3.2.1 and Footnote 37). Another situation is the following, of a more logical flavor. With respect to the manipulation of vector spaces, the set of all vector spaces is less natural than the groupoid of all vector spaces: since all constructions on vector spaces are expected to be invariant by isomorphisms, they should be defined with respect to the latter and not only the former. Developing a language based on homotopy types instead of sets would provide automatically that any construction be invariant by isomorphism.[89]

Since the 1960s, the best practical approach has been Quillen's model categories (and their variations) which underlie all approaches to higher categories. However, model categories are a way to reduce higher category theory to ordinary category theory, and the feeling is that there should be a proper theory for higher categories. Recently, homotopy type theory [33, 62] has proposed an interesting syntactic approach to homotopy types/∞-groupoids that could provide a foundational language for mathematical alternatives to set theory axioms. This is well suited to getting some aspects of homotopy types but largely insufficient yet for the purposes of the working mathematician. Another promising attempt is the development of tools to work directly in the quasi-category of quasi-categories [11].

4.3 Toward a New Axiomatisation of Geometry?

The objects of derived geometry are more complex than ordinary manifolds or even schemes, but they enjoy better properties than their classical counterparts. So what should we prefer? This is actually an illustration of a tension that exists within mathematics about whether to define its objects individually by some intrinsic structure (the affine plane as $\mathbb{R}^2$) or in a family by an algebra of operations on the family (planar geometry with its figures and incidence rules). Algebraic geometry, particularly under the influence of A. Grothendieck, has rather favored the simplicity of properties over the simplicity of its objects: points at infinity (projective geometry), points with complex coordinates (complex geometry), and multiple points (schemes) were in particular motivated by the regularization of the intersection properties of the plane. Derived geometry, with its treatment of singularities, is but the latest step in the same direction. Its objects might be strange in their individual nature, and they may have new properties, but, as a whole, they behave more regularly, and remarkably, the language to talk about them is still the same.

[86] This is in particular the case of all the so-called ∞-topoi and of all the ∞-categories of models of algebraic theories taken in an ∞-topos. For example, chain complexes or dg-algebras are models of algebraic theories in the topos $\mathcal{S}$ if homotopy types, hence their effectivity property.

[87] *Stability* is the property of a pointed category that any commutative square is a pushout if and only if it is a pullback, or equivalently that finite colimits commute with finite limits. Again, only the trivial ordinary category with one object is stable, but there are many non trivial stable ∞-categories: ∞-categories of (unbounded) chain complexes are stable, so are ∞-categories of spectra and parameterized spectra. See [47] for details and examples.

[88] From a logical point of view, one would say that there exists logical theories (or syntaxes) that have non trivial semantics only in higher categories, for example, an object X such that $\Omega\Sigma X \simeq X$, where Ω and Σ are the loop space and suspension functors.

[89] This would be a strong version of Leibniz's principle of indiscernables. This strengthening is false with ZFC and axioms of the like. However, this issue is one of the motivations for Martin-Löf to have introduced its identity types [33]; within this syntax, all propositions about, say, groups are automatically stable by isomorphism.

We have insisted several times on the fact that the geometric nature of objects like rings, dg-algebras, sheaves, and stacks was in the successful interpretation of the language of geometry (together with some expected structural properties) such as points, étale and smooth maps, and proper maps. (see Section 3.3.3, Section 3.3.4, Section 3.4.3). Actually, in front of the many settings where this language makes sense (topological spaces, topoi, manifolds, and schemes, and derived stacks being the latest), it is tempting to try to axiomatize an abstract setting for *geometry* as a category together with classes of maps corresponding to all the aforementioned classes. The notion of geometry of Lurie [50] is a first attempt in this direction, and so is Schreiber's setting for "differential cohesive homotopy theory" [61]. Such an axiomatization of geometry, emphasizing the structure of the relations between the objects of the geometry rather than the structure of the objects themselves, would be the XXI century version of Euclid's axioms.

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