

The spaces of logic

- logic for a mathematician
- geography of logic
 - geography of space
 - spatial semantics
 - spaces of models
- the underlying space of a logic

logic for a mathematician

logic can be considered

- 1 - as a foundation for mathematics
- 2 - as a particular kind of mathematical object-

I will talk from p.-o.v 2.

(only bad things to say about 1)

logic for a mathematician

from within maths a logical theory is a kind of algebraic structure (like groups, rings...)

We have , collection of objects (sorts , terms, formula types , propositions)
• operators
• deduction rules (telling how to apply the operators on

logic is a branch of algebra sorts, terms, formulas)

Geography of logic

	without dependent types or relations	with dependent types	
		dependent formula only	general dependent types
without variables sort/types	untypes λ -calculus λ, app linear logic $\otimes, \oplus, \&, \delta, \multimap, ()^*$	propositional theories type op. : \perp form. op. : $=, \top, \wedge, \perp, \vee, \Rightarrow, \neg$	dependent type theories
with variables 1st order	algebraic theories (GAT, EAT) type op. : λ, x form. op. : $=, \top$	1st order theories type op. : λ, x form. op. : $=, \perp, \wedge, \perp, \vee, \Rightarrow, \neg, \exists, \forall$	
with variables higher order	typed λ -calculus type op. : $\lambda, x, \rightarrow, \mathbb{N}$ form. op. : $=, \lambda$	higher order theories type op. : $\lambda, x, \rightarrow, \Sigma, \Pi, \mathbb{N}, \mathbb{U}$ form. op. : $=, \top, \wedge, \perp, \vee, \Rightarrow, \neg, \exists, \forall$	(MLTT, realizability...) $=, \perp, x, 0, +, \rightarrow, \Sigma, \Pi,$ \mathbb{N}, \mathbb{U}

type op. = operators to construct types
 form. op. = ————— formulas

two kinds of logic :

- formula oriented
prop logic
- operator = AND, OR, THEN
taken from language
prove truth of a proposition

- object oriented
type theories
- operator = $\Sigma \quad \Pi$
taken from set theory
build term of a give type

→ ring like structure Σ = add
 Π - product) algebra of sets

A geography of spaces

Separation scale

	discrete	topology	differentiable geometry	C-space	Alg. geom	combi- natorics	objet for a cat
T_2	Sets	Stone sp. Hausdorff sp.	real Manifolds	complex Manifolds		/	
T_0	poset	Alexanderov sp. Taniishi Spec.			scheme	trees graphs	object in 1-cat (SSet)
T_{-1}	cat/gpd	topos			stacks	...	obj in S.Gpd
$T_{-\infty}$	∞ cat/ ∞ gpd	∞ -topos		∞ -stacks		/	obj in S.Specs

no single notion of space can account for all these notions

The spaces of logic

- spatial semantics
- Space of models
- space of a logical theory

Spatial Semantics

= interpret logical theories in spatial objects -

= logic can talk about space -

plenty of instances of this idea

intuitionist semantics for propositional logic

$$\text{classical } \{ \text{formulae} \} \rightarrow \{ 0,1 \}$$

$$\varphi \mapsto [\varphi]$$

More generally $\{ \text{formula} \} \rightarrow P(E) = \{0,1\}^E$
 (still classical)

intuitionist $\{ \text{formula} \} \rightarrow \mathcal{O}(x)$ (algebra of
 open subspaces
 of a space x)
 (Heyting)
 ($\neg A \neq A \dots$)

example : theory of subsets of a given set E

$$\{\text{atomic formula}\} = \{\text{elt of } E\} = E$$

$$\text{classical model} = E \rightarrow \{\phi, 1\}$$

(\Rightarrow) subset of E

$$\text{intuitionist model} = E \rightarrow \mathcal{O}(x)$$

(\Rightarrow) subspace of $E \times X$

(where E discrete)

Algebraic theory

p. ex theory of groups (but also monoids, rings,)

- one sort G

- one constant $e : G$

- one unary op. $i : G \rightarrow G$

- one binary op. $m : G \times G \rightarrow G$

+ axioms of gp. $x y^{-1} x^+ z$

Vector space ---

Variation : FOL

finite gp

local rings

work the same

models in Set = classical gp.

Top-Sp = topological gp

Man = Lie gp

A geography of mathematical objects

spatial support	ex of obj. structure
set	monoid, group, ring, V-space \mathbb{N} \mathbb{Z} \mathbb{R}, \mathbb{C} $\mathbb{R}^n, \mathbb{C}^n$
poset	lattice, frame $P(E)$ $\mathcal{O}(x)$
n -cat/ n -grpd	monoidal cat, abelian cat, topos $(Bij, +)$ (Ab, \oplus, \otimes) $Sh(X)$
∞ -cat/ ∞ -grpd	E_∞ groups, stable cat. ∞ -topos QS^0 , Spectra $Sh_{\infty}(X)$

λ -calcul

no model in Top-spaces

but models in compact-gen. H-sp.

- SSet

- trees

linear logic

: col spaces

Mellies surfaces

26

: A85 : models in col class of small mags
Set + mags lk fibers

MLTT

Dependent type theory

types can depend on variables

$$\mathbb{N}_n := \{ k : \mathbb{N} \mid k \leq n \}$$

$$\text{list}_n(A) = \{ \text{lists of elt in } A \text{ of length } \leq n \}$$

$$x : X \vdash A(x) \text{ type} \quad \leftrightarrow \quad \begin{array}{c} A(x) \rightarrow A \quad (\text{dep type}) \\ \downarrow \quad \downarrow \text{family of set} \\ \text{fiber} \quad \xrightarrow{x} \quad X \quad (\text{ctx}) \end{array}$$

MLTT : model in Set (type = set = object)

Voevodsky term = fact > arrow

model in Simplicial sets, ∞ -groupoids, ...

	classical models	homotopical models	HoTT
dependent type	map $E \downarrow_{\Gamma} (\text{ctx})$	fibration $E \downarrow_{\Gamma}$	new behavior is univer
identity type	diagonal $E \downarrow_{E \times E}$	fibraut replacement of diagonal $P E \downarrow_{E \times E}$	(= path space)
Σ	$E \downarrow_{\Gamma} \xrightarrow{\dots} \Delta$	composition	composition
\exists	$E \xrightarrow{\exists} \text{image}$		complicated

for a given regime of space, one can hope to find a logical/axiomatic description:

Set $\hookrightarrow \text{ZF}$, type th.

plan, lines, circles \hookrightarrow Euclid/Hilbert axioms

proj space \hookrightarrow incidence theory

,

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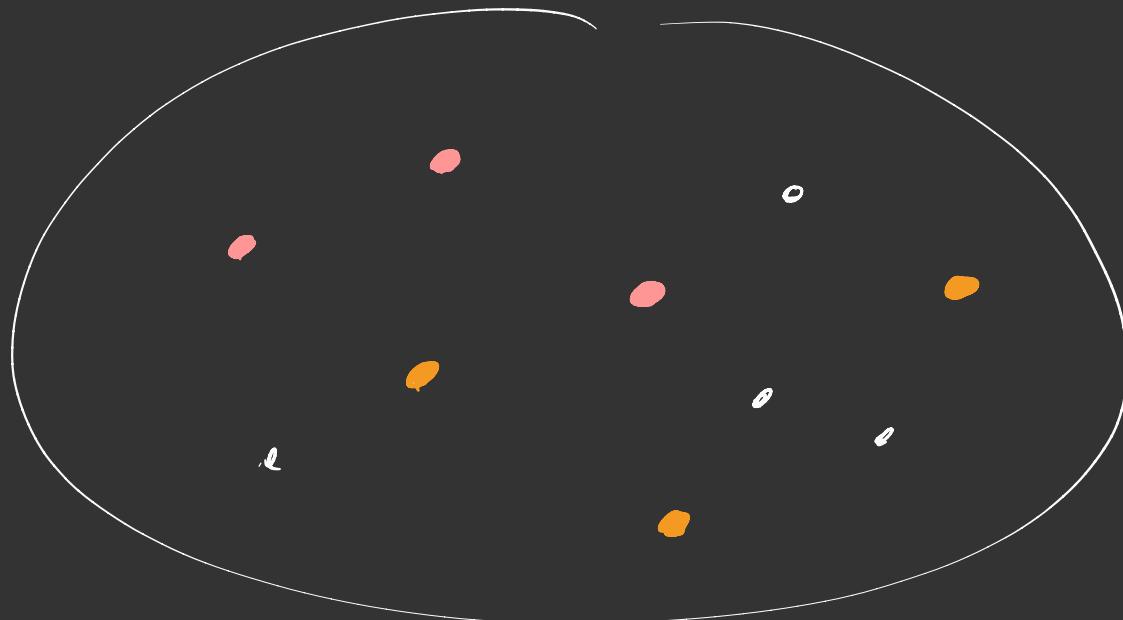
)

for a given logical theory, one can learn
form semantics in other spaces than sets

Homotopical semantics of MLTT :

- explain : why proofs of identity are not unique
 - why fct need not be extensional
- provide univalent universe (not the case of Groth. univ in ZF)
- introduce truncation hierarchy of types

Universe



Observable
universe

Spaces of models

a logical theories has models

for some special theories (called geometric)

the collection of all models in $\text{Set}/\{\emptyset, \{\}\}$

has a canonical "topological structure"

(can be a top-space but more generally
is a Topos)

example : Stone Deafity.

B Boolean alg. (= logical thy) Models = $B \rightarrow \{0, 1\}$ morph of Boolean alg.

Stone duality = the set of models is the set of points of a top. space: a Stone space

(If \mathcal{J} is interpreted as $\text{chess} \rightarrow \text{chess} \times \text{chess}$)

Same with deft lattice and coherent spans ...

example Theory of gp : Models = category of gp.

There exist a space whose collection of points
is the category of groups !

cannot be a top. sp (only have poset of pt at most)

→ Topos.

The types of the theory are interpreted as
(terms)

sheaves over the topos of groups.
(morphisms of sheaves)

Why is there such a topological behaviour of models?

Because

- Book : $\{0, 1\}$ has a canonical topology (discrete)
- coh. $\{0, 1\}$ ————— (Sierpiński)
- gp Set (cat of sets) has a can. top. structure

the idea of enhancing the category set into a
"Space" is quite fantastic ...

cannot define directly this topological structure

definition is done by defining what is a

continuous map $X \rightarrow \underline{\text{Set}}$
sheaf \downarrow étal' wise good old top. space

it is a sheaf (of sets) on X

this
"space"
of
sets
is the
essence of
Topology

formalization of space is often done w/ algebra of fact
 The notion of topos is an instance of this idea

space X	internalizing space	algebra of fact
Stone	$\{0, 1\}$ discrete space	$\text{Chopen}(X)$ (Boole alg)
Gelfand	$[0, 1] \subset \mathbb{R}$ real interval	$C(X, [0, 1])$ (C^* -alg)
top. sp (local)	$\{0, 1\}$ Sierpiński space	$O(X)$ (frame of open)
topos	<u>Set</u>	$\text{Sh}(X)$ (logs of sheaves)

Dualities for logic

logic models space

Boole	Stone sp.	
D. lat	Coh sp	
Alg-theory	varieties	(Lawvere) duality
Ess. alg-theory	loc.fin. pres.cat	(Gabriel-Ulmer) duality
Coh theory	ultracat (coh topos)	(Makkai) duality
geometric theory (kind of FOT)	topos	(measure all)

What to do with these spaces of models?

- use logic to study space

fact that a space classify a structure provide many tools

- every top. sp classifies pres fl.) generates
- — types — FOL) for
 $\odot(x)$, $sh(x)$

- use space to study logic

- universal model

- completeness theorems (Gödel-Deligne)

- reconstruction of syntax from models

the underlying space of a logic

a group can have an underlying { set
ring top-sp
... manifold
:

if logical theory are algebraic structures
what kind of underlying object can they have ?

at minimum :

- set of terms
 - set of types
 - set of formulae
-) multisorted theory

could be replaced by other kinds of space

but better :

lawvere semantics

types \longrightarrow object in a cat

terms \longrightarrow arrow in a cat

formula \longrightarrow monomorphisms (subobjects)

substitution \longrightarrow composition of arrows

a logical theory has an underlying category
operators are structure on this category
deduction rules

↗ Structure ↗ Spatial support

Lawvere : alg th = cartesian categories
 FOL = doctrines
 HOL = elementary topos

Lawbek - Scott : λ -typed
 λ -closed = cat closed
cat

? : Dependent type theory = locally cat.
Closed cat
 or cat of fibrant objects

Girard : linear logic = cat + ...

Lawvere semantics

Theories

models (semantics)

Homogeneous
syntax

translation
(model theorists)

syntax (ZF)

Heterogeneous
syntax

interpretation
(Bourbaki, everyone)

category (Set, Top)

Homogeneous
Category

functor
(cat-theorists)

category

Lawvere made theories homogeneous with semantics again
But now the picture is spatial rather than linguistic

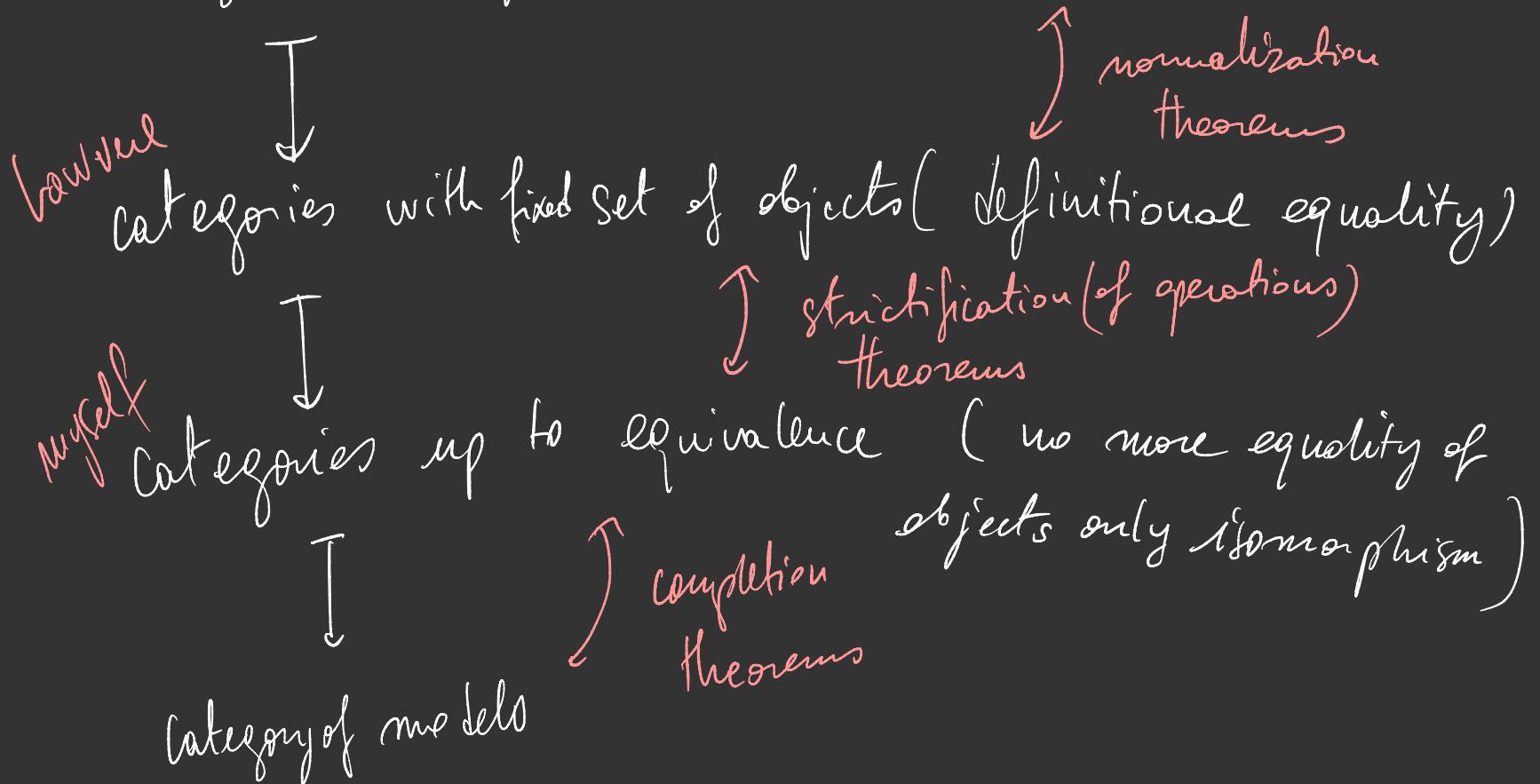
Girard criticism : projecting the logic on
category loses the core of logic :

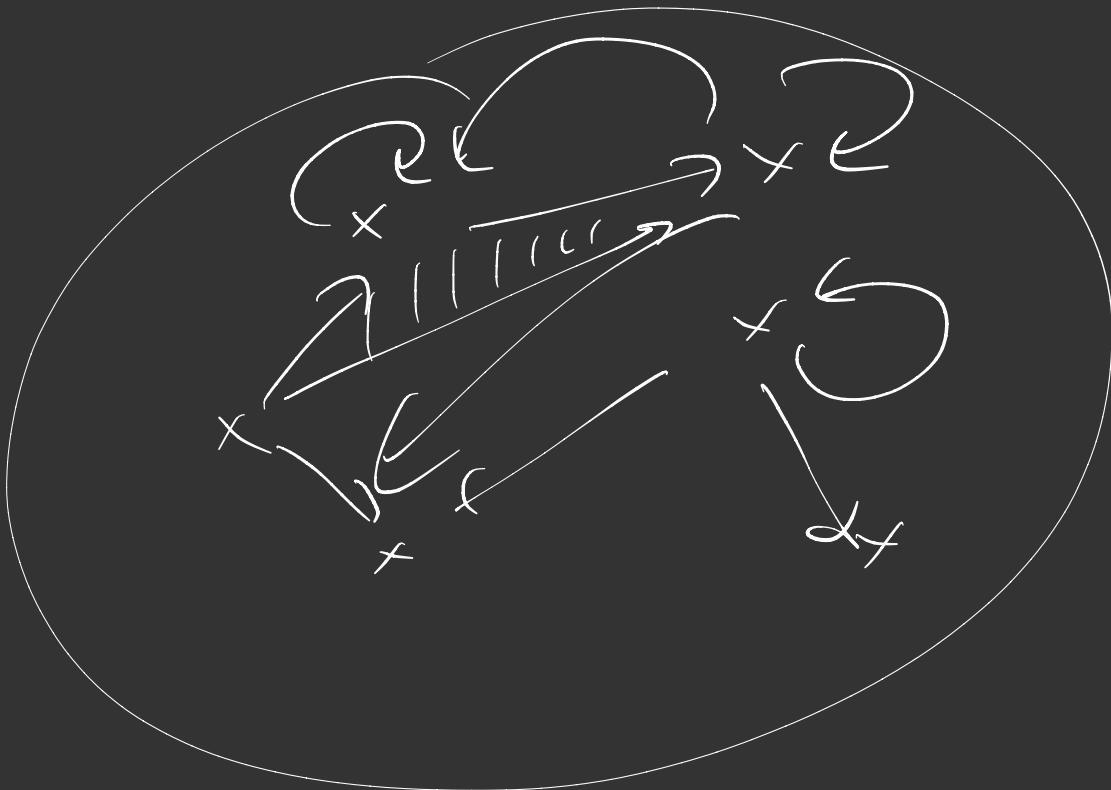
rewriting of terms \Rightarrow

do not see normalization results

can be dealt with : in fact normalization
is classically pres as a geom pls : build a
section of the map $\{ \text{preterms} \} \rightarrow \{ \text{terms} \}$

categories with presentation (preterms)





Thank you

two kinds of intuition for mathematical objects

linguistic : object = formula

schizophrenic : operation on objects

do not fit operations on formulas

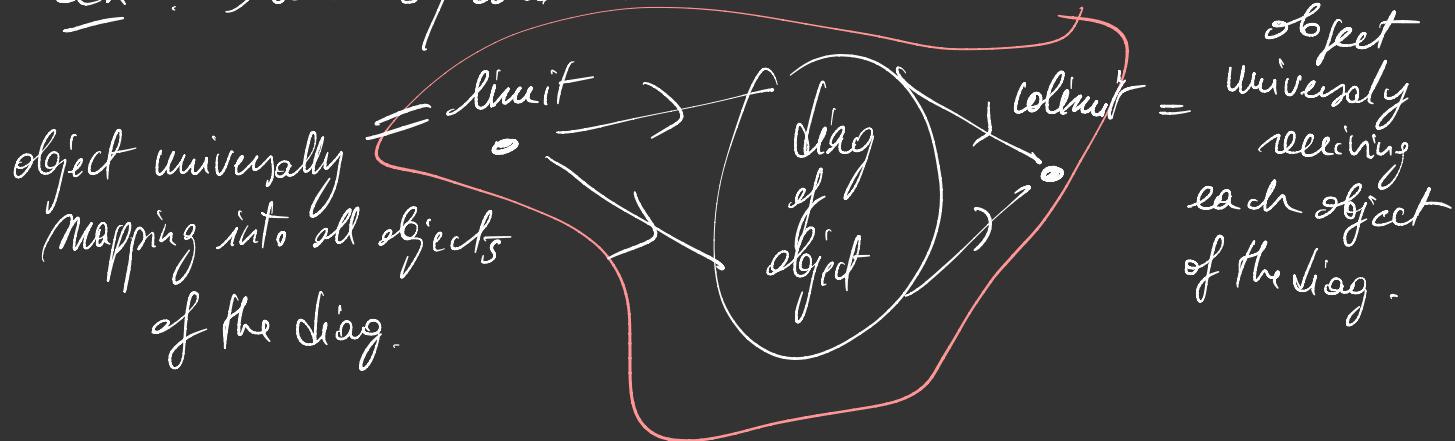
(ZF)

"spatial" : object = spatial extension +
structure

fitting of operations (op on obj = op on sets)

category theory is build around the notion
of universal property.

ex: limits / colimits



the logic in action is spatial (of position)
& category theorist is happy if he can reduce to univ. prop.

the syntax of logic provide another picture

terms = trees

basic operation = grafting