Well-pointed ∞-endofunctors

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General framework

Given an endofunctor $T : C \to C$,

and a natural transformation $t_X : X \to TX$,

a fixed point is an X such that $t_X : X \simeq TX$.

We get a (full) subcategory of fixed points $Fix(T) \subset C$.

When is the colimit of

$$X \xrightarrow{t_x} TX \xrightarrow{t_{TX}} T^2X \xrightarrow{t_{T^2X}} T^3X \xrightarrow{t_{T^3X}} \dots$$

the reflection of X into $Fix(T, t)$?

Example 1

$$\mathbb{Z} = \mathbb{N}[-1]$$

$$\mathbb{Z} = \mathbb{N} \times \mathbb{N} / \sim$$

(p, q) must be thought as $p - q$

$$(p, q) \sim (p', q') \iff \exists k,\ p + q' + k = p' + q + k$$

$$\mathbb{Z} \; = \; \operatorname{colim} \; \mathbb{N} \xrightarrow{+1} \mathbb{N} \xrightarrow{+1} \mathbb{N} \xrightarrow{+1} \dots$$

at the limit

$$\mathbb{Z} \xrightarrow{+1} \mathbb{Z} \quad \text{is an isomorphism}$$

Example 1

$C = \mathbb{N}$-modules

$T : C \to C =$ identity

$1 \to T = id \xrightarrow{+1} id$

$Fix(T, t) = \mathbb{Z}$-modules

The reflector $\mathbb{N}$-module $\to \mathbb{Z}$-module is

$$\operatorname*{colim}_{\mathbb{N}} \; id \xrightarrow{+1} id \xrightarrow{+1} id \xrightarrow{+1} id \xrightarrow{+1} \dots$$

Example 2

S_n symmetric group

BS_n its classifying space

$M = \coprod_{n \in \mathbb{N}} BS_n$ = topological monoid

multiplication comes from the canonical inclusions

$$S_m \times S_n \to S_{m+n}$$

(M = free E_∞-monoid on 1 generator)

Example 2

fix $m \in BS_1$

$$M[-m] = ?$$

A topological monoid is a group iff $\pi_0(M)$ is a group.

$$\pi_0(M) = \mathbb{N} \qquad \pi_0(M[-m]) = \mathbb{N}[-1] = \mathbb{Z}$$

Hence $M[-m]$ is a group (= group completion of M)

$$\operatorname{colim} \; M \xrightarrow{+1} M \xrightarrow{+1} M \xrightarrow{+1} M \xrightarrow{+1} \ldots \stackrel{?}{=} M[-m]$$

Example 2

$$\operatorname{colim}\ M \xrightarrow{+1} M \xrightarrow{+1} M \xrightarrow{+1} M \xrightarrow{+1} \dots \ = \ \coprod_{n\in\mathbb{Z}} BS_\infty$$

where

$$S_\infty \ = \ \operatorname{colim}\ S_1 \hookrightarrow S_2 \hookrightarrow S_3 \hookrightarrow S_4 \hookrightarrow \dots$$

$=$ permutations of $\mathbb{N}$ with finite support

Problem

$$\coprod_{n\in\mathbb{Z}} BS_\infty \quad \text{is not a group! (not even a monoid!)}$$

It cannot be $M[-m]$ (it is know that $M[-m] = \Omega^\infty\Sigma^\infty S^0$ instead)

Example 2

If M is a topological monoid

then $\pi_1(M, 0)$ is an abelian group (Eckmann–Hilton)

($= \Omega_0 M$ is a E_2-group)

$$\pi_1\left(\coprod_{n\in\mathbb{Z}} BS_\infty, 0\right) = S_\infty \quad \text{is not abelian!}$$

Example 2

$C = M$-modules

$T : C \to C =$ identity

$1 \to T = id \xrightarrow{+1} id$

$Fix(T, t) = M[-m]$-modules

The reflector M-mod $\to M[-m]$-mod is not

$$\text{colim } id \xrightarrow{+1} id \xrightarrow{+1} id \xrightarrow{+1} id \xrightarrow{+1} \ldots$$

Sometimes this works sometimes not...

We must understand why.

Example 3

$O(n)$ the orthogonal group

$BO(n)$ its classifying space

$M = \coprod_{n \in \mathbb{N}} BO(n)$ = topological monoid

multiplication comes from the canonical maps

$$O(m) \times O(n) \to O(m+n)$$

fix $m \in BO(1)$

$$M[-m] = ?$$

$$\text{colim } M \xrightarrow{+1} M \xrightarrow{+1} M \xrightarrow{+1} M \xrightarrow{+1} \dots \overset{?}{=} M[-m]$$

Example 3

$$\text{colim} \ M \xrightarrow{+1} M \xrightarrow{+1} M \xrightarrow{+1} M \xrightarrow{+1} \dots \ = \ \coprod_{n \in \mathbb{Z}} BO(\infty)$$

where

$$O(\infty) \ = \ \text{colim} \ O(1) \hookrightarrow O(2) \hookrightarrow O(3) \hookrightarrow O(4) \hookrightarrow \dots$$

= orthogonal $\mathbb{N} \times \mathbb{N}$ matrices with finite support

group law = product of matrices

Is $\coprod_{n \in \mathbb{Z}} BO(\infty)$ a group ?

YES ! Astonishing fact : $O(\infty)$ is an E_∞-group!

$$M[-m] = \coprod_{n \in \mathbb{Z}} BO(\infty)$$

Example 4

$(M, \otimes)$ a monoidal category

X an object of M

$$M[X^{-1}] = ?$$

When is

$$M[X^{-1}] \;=\; \mathrm{colim}\;\; M \xrightarrow{X\otimes -} M \xrightarrow{X\otimes -} M \xrightarrow{X\otimes -} \ldots\ ?$$

When is

$$\mathrm{colim}\;\; id \xrightarrow{X\otimes -} id \xrightarrow{X\otimes -} id \xrightarrow{X\otimes -} \ldots\ ?$$

the reflection $\{\text{Cat with } M\text{-action}\} \to \{\text{Cat with } M[-1]\text{-action}\} \to$

Known necessary and sufficient condition (due to Smith?) [Rob15, Voe98]

the action of cyclic permutations on $X \otimes X \otimes X$ must be trivial

quite a fantastic condition... Where is it coming from ?

Example 4

$(S^\bullet, \wedge)$ monoidal category of pointed spaces and smash product

S^1 is an object of $S^\bullet$

$$S^1 \wedge X = \Sigma X \qquad = \text{suspension of } X$$

$$S^\bullet[S^{-1}] \;=\; \text{colim}\;\; S^\bullet \xrightarrow{\Sigma} S^\bullet \xrightarrow{\Sigma} S^\bullet \xrightarrow{\Sigma} \cdots = \text{Spectra}$$

Voevodsky condition: cyclic permutations on $S^1 \wedge S^1 \wedge S^1 = S^3$ are trivial ?

yes = rotation of S^3 = homotopic to identity ($SO(4)$ is connected)

$$R = \text{colim}\;\; id \xrightarrow{\Sigma} id \xrightarrow{\Sigma} id \xrightarrow{\Sigma} id \xrightarrow{\Sigma} \dots$$

is the reflector

$$\text{Pointed cocomp. Cat.} \quad \longrightarrow \quad \text{Stable categories}$$

General framework

Given an pointed endofunctor $1 \xrightarrow{t} T : C \to C$,

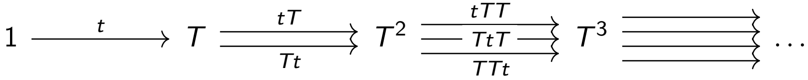
When is the colimit of

$$X \xrightarrow{t_X} TX \xrightarrow{t_{TX}} T^2X \xrightarrow{t_{T^2X}} T^3X \xrightarrow{t_{T^3X}} \dots$$

the reflection of X into $Fix(T, t)$

Free pointed objet

Given $t : 1 \to T$, the powers of T define a Δ_{inj} diagram



This is the free monoidal category on a pointed object $t : 1 \to T$

The colimit of iterating t is the top row of this diagram.

Why not looking at the bottom row ? or any other path ? or the whole diagram ?

Free pointed objet

Theorem (Dubuc)

If T is finitary, $\operatorname{colim}_{\Delta_{inj}} T^n$ *is the free monoid on* $1 \xrightarrow{t} T$.

This is not what we're looking for.

We want a reflector, which is an idempotent monad.

We are missing relations.

Kelly's solution within 1-categories

$1 \xrightarrow{t} T$ is well-pointed if there exists an equality

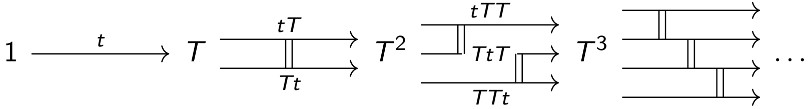
$$1 \xrightarrow{\quad t \quad} T \underset{Tt}{\overset{tT}{\rightrightarrows}} T^2 \qquad (\tau \,\|\,)$$

Theorem (Kelly)

If T is well-pointed, $\mathrm{colim}_{\mathbb{N}} T^n$ is the free idempotent monoid on $1 \xrightarrow{t} T$.

Kelly's solution within 1-categories

This equality $\tau : tT = Tt$ propagate to the higher powers



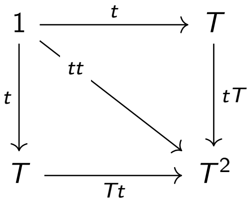
The resulting category collapse to $(\mathbb{N}, \leq)$.

Or does it ?...

Not in ∞-categories !

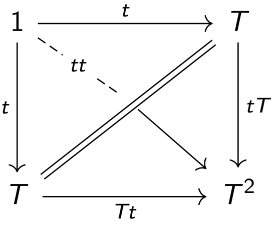
Higher Homs

Let's look at $Hom(1, T^2)$



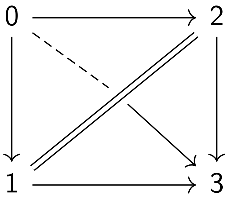
Higher Homs

Let's look at $Hom(1, T^2)$



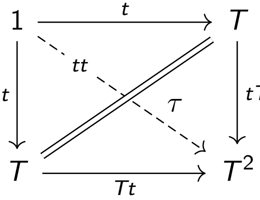
Higher Homs

This is the 3-horn $\Lambda^0[3]$



Higher Homs

The identification $\tau : tT = Tt$ provide the missing face



but the inside is still empty!

This says that

$$Hom(1, T^2) = S^1$$

Higher Homs

The "unique" element of $Hom(1, T^2) = S^1$ is

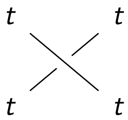
$$tt : 1 \to T^2$$

The generator for $\pi_1\left(Hom(1, T^2), tt\right) = \mathbb{Z}$ is

$$\begin{array}{ccc} 1 & \xrightarrow{\;t\;} & T \\ {\scriptstyle t}\downarrow & \theta = t \star \tau & \downarrow{\scriptstyle tT} \\ T & \xrightarrow[\;tT\;]{} & T^2 \end{array} \quad = \quad \begin{array}{ccc} 1 & \xrightarrow{\;t\;} & T \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle tT} \\ T & \xrightarrow{\;Tt\;} & T^2 \\ \| & \tau & \| \\ T & \xrightarrow[\;tT\;]{} & T^2 \end{array}$$

Higher Homs

It is a good idea to think of θ as a braiding for tt



θ propagates to $t^n : 1 \to T^n$ into a action of the braid group Br_n

$$Br_0 = Br_1 = 1 \qquad Br_2 = \mathbb{Z}$$

Higher Homs

We call a triplet (T, t, τ) a braided-pointed object

Let

$$\Theta = \langle T, t : 1 \to T, \tau : tT = Tt \rangle^{\otimes}$$

be the free monoidal ∞-category generated on a braided-pointed object.

Theorem (A–Henry)

$$Hom_{\Theta}\left(T^n, T^{n+k}\right) = B(Br_k)$$

Remark that the 1-truncation is $(\mathbb{N}, \leq)$.

Well-pointed ∞-endofunctors

Now we understand better the structure of braided-pointed object, we can come back to our original question.

When is

$$\text{colim}\; X \xrightarrow{t_x} TX \xrightarrow{t_{TX}} T^2X \xrightarrow{t_{T^2X}} T^3X \xrightarrow{t_{T^3X}} \ldots$$

the reflection of X into $Fix(T, t)$

Well-pointed ∞-endofunctors

We can specialize the question:

Given a braided-pointed ∞-endofunctor (T, t, τ) of some ∞-category C

What is a condition on θ for

$$\text{colim } X \xrightarrow{t_x} TX \xrightarrow{t_{TX}} T^2X \xrightarrow{t_{T^2X}} T^3X \xrightarrow{t_{T^3X}} \dots$$

to be the reflection of X into $Fix(T, t)$

Well-pointed ∞-endofunctors

What is the action of a braided-pointed ∞-endofunctor on the fixed points ?

The answer is given by the localization

$$\begin{aligned}
\Theta[t^{-1}] &= \langle T, t : 1 \simeq T, \tau : tT = Tt \rangle^{\otimes} \\
&= \langle \tau : id_1 = id_1 \rangle^{\otimes} \\
&= \text{free monoid on a 2-cell} \\
&= \text{free group on a 2-cell} \\
&= \Omega \Sigma S^2 = \Omega S^3
\end{aligned}$$

Well-pointed ∞-endofunctors

The functor

$$\Theta \quad \longrightarrow \quad \Theta[t^{-1}] = \Omega S^3$$

induces a map on 2-cells

$$Br_k = \pi_1\left(Hom_\Theta\left(T^n, T^{n+k}\right)\right) \quad \longrightarrow \quad \pi_2(\Omega S^3) = \mathbb{Z}$$

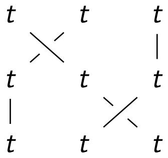
It is the degree map sending a braid to its winding number.

T acts as the identity on $Fix(T)$ (by definition),

but the braiding τ acts on $T^n_{|Fix(T)} = id_{Fix(T)}$ by -1

Well-pointed ∞-endofunctors

$$\begin{array}{ccc} 1 & \xrightarrow{t} & T \\ {\scriptstyle t}\downarrow & \color{blue}{\theta_{12}} & \downarrow{\scriptstyle tT} \\ T & \xrightarrow{tT} & T^2 \\ {\scriptstyle tT}\downarrow & \color{blue}{\theta_{23}^{-1}} & \downarrow{\scriptstyle tTT} \\ T^2 & \xrightarrow[tTT]{} & T^3 \end{array} \qquad \Longleftrightarrow$$



braid for the cycle $(123 \to 231)$

$\theta_{12}\theta_{23}^{-1}$ = generator of the kernel of degree map $Br_3 \to \mathbb{Z}$

must be send to 0 by $\Theta \quad \longrightarrow \quad \Omega S^3$

This is Voevodsky's condition.

Well-pointed ∞-endofunctors

Theorem (A–Henry)

For a braided-pointed ∞-endofunctor (T, t, τ)

$$T^\infty := \operatorname{colim}\ 1 \xrightarrow{t} T \xrightarrow{tT} T^2 \xrightarrow{tT^2} T^3 \xrightarrow{tT^3} \dots$$

is the reflection into fixed points
iff
$\forall n,\ \theta_{n,n+1}\theta^{-1}_{n+1,n+2}$ *eventually become the identity in the sequence.*
iff
the action of Br_∞ *on* $t^\infty = ttt\dots\ :\ 1 \to T^\infty$ *factors through the degree map* $Br_\infty \to \mathbb{Z}$

This our definition of a well-pointed ∞-endofunctor.

Examples

For $M = \mathbb{N}$

$$\operatorname{colim} \ \mathbb{N} \xrightarrow{+1} \mathbb{N} \xrightarrow{+1} \mathbb{N} \xrightarrow{+1} \mathbb{N} \xrightarrow{+1} \dots \ = \ \mathbb{Z}$$

$\theta_{n,n+1}\theta_{n+1,n+2} \in \pi_1(\mathbb{Z}, 0) = 1$

is the identity (we're in a contractible group)

Examples

For $M = \coprod_n BS_n$

$$\operatorname{colim}\ M \xrightarrow{+1} M \xrightarrow{+1} M \xrightarrow{+1} M \xrightarrow{+1} \dots \ = \ \coprod_{n \in \mathbb{Z}} BS_\infty$$

$\theta_{n,n+1}\theta_{n+1,n+2} = (n+1, n+2, n) \in S_\infty$

cannot be connected to the identity (we're in a discrete group)

Examples

For $M = \coprod_n BO(n)$

$$\text{colim } M \xrightarrow{+1} M \xrightarrow{+1} M \xrightarrow{+1} M \xrightarrow{+1} \dots = \coprod_{n \in \mathbb{Z}} BO(\infty)$$

$\theta_{n,n+1}\theta_{n+1,n+2} \in SO(\infty)$

can be connected to the identity (we're in a connected group)

Swell-pointed ∞-endofunctors

A stronger condition on a well-pointed ∞-endofunctor (T, t, τ) is that is it classified by the monoidal poset $\mathbb{N} = \{0 < 1 < 2 \ldots\}$.

We call such functors swell-pointed.

Theorem (Conjecture)

A well-pointed ∞-endofunctor (T, t, τ) is swell-pointed
iff
$\forall n, \ \theta_{n,n+1}$ eventually become the identity in the sequence.
iff
the action of Br_∞ on $t^\infty = ttt \ldots : 1 \to T^\infty$ is trivial.

Open questions

1. Examples ?!

   1.1 Sheafification in ∞-topos theory?

   1.2 Thierry Coquand's construction ?

   1.3 Can adapt Kelly's construction of well-pointed by transfer along adjunction to ∞-categories ?

2. What is free $\otimes$-cat on a well-pointed ∞-functor ?

M. Robalo.
K-theory and the bridge from motives to noncommutative motives. Advances in Mathematics, 269:399 – 550, 2015.

V. Voevodsky.
$\mathbf{A}^1$-*Homotopy Theory*, in Proceedings of the International Congress of Mathematicians, Volume I, pages 579–604. Documenta Mathematica, 1998.

Thanks!