The elementary ∞-topos of truncated coherent spaces

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Abstract

This note shows that the category of truncated spaces with finite homotopy invariants has many of the features expected of an elementary ∞-topos. It should be thought of as the natural higher analogue of the elementary 1-topos of finite sets. We prove several initiality results for this ∞-category.

Contents

1 Introduction

Ever since the notion of ∞-topos (which we shall call Grothendieck ∞-topos in this discussion) started to be studied [Sim99, Rez05, TV05, Lur09], the question of an elementary version of the notion has been around. This would be to ∞-topoi what Lawvere's elementary 1-topoi are to Grothendieck 1-topoi. This question has become less academic with the discovery of the homotopical semantics of Martin-Löf's theory of dependent types and the introduction of the univalence axiom [AW09, GG08, Voe06, KLL21, Uni13]. The interpretation of logical types as homotopy types of spaces and identity types as path spaces has brought a deep and unexpected connection between logic and homotopy theory (aka ∞-category theory [Cis19]). This connection has suggested a more precise content for the notion of elementary ∞-topoi: they should be the ∞-categories which support an interpretation of dependent type theory with identity types and a univalent universe (aka homotopy type theory).

An axiomatization for elementary ∞-topoi has been proposed in [nLa21] and an equivalent axiomatization has been developed in [Ras18]. However examples are still scarce and somehow ad hoc [Ras18, Ras21b]. The

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purpose of this note is to describe an example of an ∞-category having many of the expected features required to interpreted homotopy type theory. Although this example does not verify all the axioms of the definition [nLa21, Ras18], it is nonetheless interesting because it is simple, concrete, and initial in several ways.

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In 1-topos theory, the 1-category Set of (small) sets is a Grothendieck topos and the full subcategory $\mathsf{Set}_{\mathrm{fin}}$ of finite sets is an elementary topos. When Set is generalized into the ∞-category $\mathcal{S}$ of spaces (∞-groupoids), the notion of finite sets has two natural generalizations:

1. *finite spaces*, which are homotopy types of finite CW-complexes.
2. *truncated coherent spaces*, which are truncated homotopy types whose homotopy invariants are all finite (as a set or a group).

Coherent spaces are often called *π-finite spaces* in the literature but, depending on authors, the notion can demand or not the truncation of the space. I have preferred to call them *coherent* because they are the coherent objects of the ∞-category $\mathcal{S}$ [Lur17, Definition A.2.1.6 and Example A.2.1.7].

If $\mathcal{S}$ is the ∞-category of spaces, we denote by $\mathcal{S}_{\mathrm{fin}}$ the full subcategory of finite spaces and by $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ that of truncated coherent spaces. These categories are essentially disjoint: their intersection is reduced to finite sets

$$\begin{tikzcd} \mathsf{Set}_{\mathrm{fin}} \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{S}^{<\infty}_{\mathrm{coh}} \arrow[d, hook] \\ \mathcal{S}_{\mathrm{fin}} \arrow[r, hook] & \mathcal{S} \end{tikzcd} .$$

Their stability properties are also very different (see Table 1). The ∞-category $\mathcal{S}_{\mathrm{fin}}$ is closed under finite colimits but not by fiber products. Conversely, the ∞-category $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ is closed under finite limits, finite sums, but not by pushouts. In particular, the spheres S^n ($n > 0$) are finite but not truncated coherent. These properties make $\mathcal{S}_{\mathrm{fin}}$ into a rather awkward object from the point of view of topos theory, where fiber products are fundamental. The purpose of this note is to show that $\mathcal{S}^{<\infty}_{\mathrm{coh}}$, on the contrary, is very well behaved and could be considered an example of an elementary ∞-topos.

We shall prove the following properties of $\mathcal{S}^{<\infty}_{\mathrm{coh}}$:

(1) it is a lex ∞-category (Proposition 2.2.4),

(2) which is extensive (i.e. finite sums exist, and are universal and disjoint, see Proposition 2.2.7),

(3) and exact (i.e. quotients of Segal groupoids objects exist, and are universal and effective, see Proposition 2.7.3).

(4) $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ has all truncation (Postnikov) modalities (Proposition 2.2.5),

(5) it is locally cartesian closed (Theorem 2.8.6),

(6) its universe U (which lives in $\mathcal{S}$) is a countable coproduct of truncated coherent spaces (Theorem 2.9.2),

(7) $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ has enough univalent maps (Proposition 2.9.7), and they can be chosen closed under diagonals, aka identity types (Proposition 2.9.11).

(8) it has a subobject classifier, which is Boolean (Proposition 2.9.16).

Properties (1) to (3) make $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ into an ∞-pretopos in the sense of [Lur17, Appendix A] where it is mentioned as an example. We shall see that in fact

(9) $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ is the initial ∞-pretopos (Theorem 2.10.1).

We also show a couple of stronger universal properties:

(10) $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ is the initial "locally cartesian closed ∞-pretopos" (see Theorem 2.10.3 for a precise statement).

(11) $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ is the initial "locally cartesian closed ∞-pretopos with a Boolean subobject classifier" (see Corollary 2.10.7 for a precise statement).

Property (4) is a consequence of the ∞-pretopos structure, but it can be checked by hand here. The original contribution of this note seems to be Property (5), that $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ is locally cartesian closed. The main tool to prove it is the folklore result characterizing truncated coherent spaces as realization of Kan complexes with values in finite sets (Proposition 2.6.3). Then the proof of Property (5) is by a descent argument. The rest of the properties are easily derived from there.

In connection to homotopy type theory, $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ does provides a non-trivial univalent family $(U^{<\infty}_{\mathrm{coh}})' \to U^{<\infty}_{\mathrm{coh}}$ (in $\mathcal{S}$) which is closed under dependent sums and dependent products (Theorem 2.9.15). This universe exists in the ∞-category $\mathcal{S}$ of κ-small spaces for *any* inaccessible cardinal $\kappa > \omega$. More generally, the universe $U^{<\infty}_{\mathrm{coh}}$ exists as a subobject of the universe of any Grothendieck ∞-topos $\mathcal{E}$ (see Remark 2.10.4). The universe $U^{<\infty}_{\mathrm{coh}}$ is not the smallest universe of $\mathcal{S}$ closed under dependent sums and products (since the subobject classifier $\Omega = 2$ or the 1-type Fin of finite sets are also examples), but Property (10) can be reformulated by saying that $U^{<\infty}_{\mathrm{coh}}$ is the minimal universe of $\mathcal{S}$ containing Fin and closed under Segal groupoid quotients (see Corollary 2.10.6).

Altogether, this provides the ∞-category $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ with almost all of the properties of the notion of elementary ∞-topos of [nLa21, Ras18], but

(a) $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ does not have all pushouts (e.g. the spheres S^n for $n > 0$, see Proposition 2.3.1),

(b) and it does not have a hierarchy of univalent families closed under dependent sums and/or dependent products Theorem 2.9.17.

In an elementary 1-topos, the existence of finite colimits can be deduced from the existence of finite limits, exponentials and the subobject classifier. Fact (a) shows that this does not generalizes to higher categories, providing a negative answer to a question of Awodey (at least in a context where there are no universe closed under Σ and Π). Nonetheless, it is established in [FR22], that coproducts can be build from finite limits, exponentials and the subobject classifier. It seems reasonable that the pushouts where one leg is a monomorphism can also be produced this way. Fact (b) is essentially due to the fact that there are no inaccessible cardinals between 2 and ω. This is related to the minimality properties of $\mathcal{S}^{<\infty}_{\mathrm{coh}}$.

I would like to finish this introduction with a brief discussion of whether $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ should be considered an elementary ∞-topos. In my opinion, Facts (a) and (b) are not drawbacks with respect to a definition but facts that a definition must accomodate. Examples do not always comply with the a priori expectations of mathematicians, and there are reasons to think that the definition of [nLa21, Ras18] is too strict.

For example, Fact (b) is already a problem for Grothendieck ∞-topoi. Grothendieck ∞-topoi are examples of the definition of [nLa21, Ras18] only if an ad hoc hypothesis is assumed, introducing an infinite hierarchy of inaccessible cardinals whose supremum is the inaccessible cardinal κ bounding the size of small objects. But, from the point of view of the theory of Grothendieck ∞-topoi, such a hierarchy is artificial (and useless). And, from the point of view of ∞-category theory, the demand of such a hierarchy will prevent the existence of free elementary ∞-topoi and the monadicity of their category (unless it is strictified into a structure preserved by morphisms of elementary ∞-topoi, but this is not the idea).

Another issue is that the definition of [nLa21, Ras18] does not connect well with the theory of ∞-pretopoi, that is with *higher coherent logic*. An elementary 1-topos is always a 1-pretopos and this allows one to interpret coherent theories in any elementary 1-topos. It is not known whether the axioms of [nLa21, Ras18] imply that an elementary ∞-topos is an ∞-pretopos in the sense of [Lur17, Appendix A], but this seems unlikely. Lurie has developed a robust theory of ∞-pretopoi and coherent ∞-topoi, with many examples, and with a higher version of Deligne's completude theorem [Lur17, Theorem A.4.0.5]. The corresponding coherent logic is still missing and it is regrettable that elementary ∞-topoi as they are currently defined cannot interact well with this.

The definition of [nLa21, Ras18] also has the problem that the theory of elementary ∞-topoi is not conservative over that of elementary 1-topoi. The 1-category of Grothendieck 1-topoi embeds fully faithfully into the ∞-category of Grothendieck ∞-topoi. But the 1-category of elementary 1-topoi cannot be conservatively embedded into the ∞-category of elementary ∞-topoi. In the best situation, only elementary 1-topoi with a natural number object could be embedded in this way, since it has been proved that the 1-truncation of an elementary ∞-topos is always such an elementary 1-topos [Ras21a].

It would thus seems more satisfying if the definition of elementary ∞-topoi could: encompass the example of $\mathcal{S}_{\mathrm{coh}}^{<\infty}$; be such that any Grothendieck ∞-topos is an example (without assumption on the cardinal bound); be compatible with ∞-pretopoi and higher coherent logic; and recover all elementary 1-topoi via 0-truncated objects. Not having spheres or pushouts around may seem like a problem to do homotopy theory, but the numerous properties of $\mathcal{S}_{\mathrm{coh}}^{<\infty}$ show how far one can go without them. More generally, the whole theory of ∞-pretopoi/coherent Grothendieck ∞-topoi of [Lur17, Appendix A] does not need ∞-pretopoi to contain spheres or have pushouts. The fact that $\mathcal{S}_{\mathrm{coh}}^{<\infty}$ does not have spheres or a hierarchy of universes is analoguous to the fact that $\mathsf{Set}_{\mathrm{fin}}$ does not have a natural number object (some functors are not representable). One can imagine that the core definition of an elementary ∞-topos could not assume spheres, general pushouts, or a hierarchy of universes as part of the structure. Rather these conditions could be demanded as extra properties distinguishing subclasses of important objects.[1]

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Convention This note is written in the language of ∞-categories but we shall drop all "∞-" prefixes and call higher categorical notions by their classical name (category always means ∞-category, topos means ∞-topos, colimit always means ∞-colimit, pullback always means ∞-pullback, etc.) When n-categories and n-categorical notions will be required for $n < \infty$, we shall use an explicit "n-" prefix. We refer to [Lur09, Cis19, RV21] for basics on ∞-category theory. All the arguments of the paper are formulated in terms that make sense in any model of ∞-category theory (limits, colimits, exactness properties, adjunctions...).

2 The ∞-category of truncated coherent spaces

2.1 Definition and characterizations

A space X is called *coherent* if $\pi_0(X)$ is a finite set and all $\pi_n(X,x)$ $(n>0)$ are finite groups, for all choice of base point. Coherent spaces are also called π_**-finite* in the literature, but depending on authors, the notion can demand or not the truncation or the connectedness of the space. We have preferred to follow the terminology of Lurie and call them coherent spaces. This is justified by the fact that they are the coherent objects of the category $\mathcal{S}$ [Lur17, Example A.2.1.7.]. We shall however talk about *truncated coherent* spaces rather than *bounded coherent* spaces when they are truncated. We denote by $\mathcal{S}_{\mathrm{coh}}$ and $\mathcal{S}_{\mathrm{coh}}^{<\infty}$ the subcategory of $\mathcal{S}$ spanned by coherent and truncated coherent spaces.

[1]This is how one proceeds elsewhere. For example, in algebra, the general definition of commutative rings is only useful to have a good category of rings, but in order to work and prove theorems, one needs to assume extra conditions, like being of finite presentation, Nœtherian, excellent, etc. The situation is similar with topological spaces where little can be said about general spaces and one needs quickly to introduce separation conditions, local compactness, countable basis, etc. to prove interesting results.

In the case of spaces, the coherence condition can be understood as a higher analogue of the notion of Kuratowski finite object. We say that a space X is *finitely covered* if there exist a map $E \to X$ where E is a finite set and which is surjective on π_0. We say that a map $X \to Y$ is *finitely covered* if all its fiber are finitely covered. Recall that the diagonal of a map $f : X \to Y$ is the map $\Delta f : X \to X \times_Y X$. The higher diagonals are defined by $\Delta^{n+1} f = \Delta(\Delta^n f)$. When $Y = 1$ is the point, we have $\Delta^{n+1} X := \Delta^{n+1}(X \to 1) = X \to X^{S^n}$.

Proposition 2.1.1 (Kuratowski characterization)**.** *A space X is coherent if all its diagonals $\Delta^{n+1} X$ are finitely covered.*

Proof. The set $\pi_n(X)$ is finite if and only if $\Omega^n X$ is finitely covered. The result follows from the fact that the fibers of $\Delta^{n+1} X : X \to X^{S^n}$ are exactly the loop spaces $\Omega^{n+1} X$. □

Examples of truncated coherent spaces:

- any finite set (including 0 and 1);
- the realization of any finite groupoid ($G_1 \rightrightarrows G_0$ in $\mathsf{Set}_{\mathrm{fin}}$);
- $\mathbb{RP}^\infty = B\mathbb{Z}_2$ (= universe of sets of cardinal 2);
- $\coprod_{k \le n} B\mathfrak{S}_k$ (= universe of sets or cardinal $\le n$);
- the classifying spaces BG, for G a finite group;
- Eilenberg–Mac Lane spaces $K(G, n)$, for G a finite group.

Examples of untruncated coherent spaces:

- $\Omega^{2k+2} S^{2k+1}$, $\Omega^{4k} S^{2k}$;
- The realization of a Kan complex with values in finite sets.

Non-examples of coherent spaces

- the spheres S^n for $n \ge 1$ (since $\pi_n(S^n) = \mathbb{Z}$);
- pushouts of coherent spaces (since $S^1 = 1 \coprod_{S^0} 1$);
- finite CW-complexes (since π_n will only be finitely generated groups).

2.2 Elementary properties of $\mathcal{S}^{<\infty}_{\mathrm{coh}}$

This section proves Properties (1), (2) and (4) of $\mathcal{S}^{<\infty}_{\mathrm{coh}}$.

Lemma 2.2.1. *Any subspace of a truncated coherent space is truncated coherent.*

Proof. A subspace is determined by a subset of connected components. Hence, the π_0 is finite and so are the higher homotopy groups. □

Lemma 2.2.2. *The category $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ has finite sums and the inclusion $\mathcal{S}^{<\infty}_{\mathrm{coh}} \subset \mathcal{S}$ preserves them.*

Proof. The initial object of $\mathcal{S}$ is coherent. Let X and Y be two truncated coherent spaces, then the sum $X + Y$ (computed in $\mathcal{S}$) is truncated coherent and provide a sum for X and Y in $\mathcal{S}^{<\infty}_{\mathrm{coh}}$. □

The following result proves that the category $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ is closed under fibers, extensions, and quotients (see Proposition 2.7.1).

Proposition 2.2.3. *Consider a cartesian square*

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d] \\ 1 \arrow[r] & Y \end{tikzcd}$$

where Y is a connected space. Then, if any two of X, Y, or Z are truncated coherent, so is the third.

Proof. We chose an arbitrary base point z in Z, we denote x its image in X and y its image in Y. We consider the long exact sequence of homotopy invariants:

$$\ldots \pi_2(Z,z) \to \pi_2(X,x) \to \pi_2(Y,y) \to \pi_1(Z,z) \to \pi_1(X,x) \to \pi_1(Y,y) \to \pi_0(Z) \to 1\,.$$

We prove the result in case where X and Y are assumed in $\mathcal{S}^{<\infty}_{\mathrm{coh}}$. The map $\pi_1(Y,y) \to \pi_0(Z,z)$ is surjective, this prove that $\pi_0(Z,z)$ is finite. For $n > 0$, we get a short exact sequence $K \to \pi_n(Z,z) \to Q$ where K is the kernel of $\pi_n(Z,z) \to \pi_n(X,x)$, and Q is the quotient of the map $\pi_{n+1}(Y,y) \to \pi_n(Z,z)$. K is a subgroup of a finite group, Q is a quotient of a finite group, hence they are both finite. Then $\pi_n(Z,z)$ is finite since, as a set, it is in bijection with $K \times Q$. Since the base point of Z was arbitrary, this proves that Z is in $\mathcal{S}^{<\infty}_{\mathrm{coh}}$. The argument is similar in the two other cases. □

Proposition 2.2.4 (Finite limits)**.** *The category $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ has finite limits (in particular loop spaces) and they are preserved by the inclusion $\mathcal{S}^{<\infty}_{\mathrm{coh}} \subset \mathcal{S}$.*

Proof. The point is truncated coherent. The statement for binary products is direct from the formula $\pi_n(X \times Y) = \pi_n(X) \times \pi_n(Y)$. We need only to check fiber products. Given a diagram $X \to Y \leftarrow Y'$ in $\mathcal{S}^{<\infty}_{\mathrm{coh}}$, we want to prove that $X \times_Y Y'$ is in $\mathcal{S}^{<\infty}_{\mathrm{coh}}$. Using Proposition 2.2.3, it is enough to prove that the fibers of the map $X \times_Y Y' \to Y'$ are in $\mathcal{S}^{<\infty}_{\mathrm{coh}}$. But these fibers are fibers of the map $X \to Y$, which are in $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ by Proposition 2.2.3. □

The n-truncation of a truncated coherent space is clearly truncated coherent. The next result proves that the results holds not only for objects but also maps.

Proposition 2.2.5 (Postnikov truncations)**.** *Let $X \to Y$ be a morphism in $\mathcal{S}^{<\infty}_{\mathrm{coh}}$, and $X \to Z \to Y$ its factorization (computed in $\mathcal{S}$) into an n-connected maps followed by an n-truncated map. Then, the space Z is truncated coherent.*

Proof. The fibers of $Z \to Y$ are the n-truncations of the fibers of $X \to Y$, and the result follows from Proposition 2.2.3. □

Let $\mathcal{C}$ be a category with finite limits and all colimits indexed by some small category I. For any diagram $X : I \to \mathcal{C}$, the pullback functor induces a functor $P : \mathcal{C}_{/\operatorname{colim} X_i} \to \lim_i \mathcal{C}_{/X_i}$ and the colimit functor induces a left adjoint $C : \lim_i \mathcal{C}_{/X_i} \to \mathcal{C}_{/\operatorname{colim} X_i}$. We say that an I-colimit is universal (resp. effective) if C (resp. P) is a fully faithful functor. We say that an I-colimit has descent if it is universal and effective [Lur09, 6.1.3, 6.1.8]. If I is a set, effectivity corresponds to the disjunction of sums, and descent to their extensivity. All colimits have descent in $\mathcal{S}$.

Lemma 2.2.6 (Descent)**.** *Let $\mathcal{C} \subset \mathcal{S}$ is a subcategory closed under finite limits such that the colimit of some small diagram $X : I \to \mathcal{C}$ exist in $\mathcal{C}$ and is preserved by $\mathcal{C} \subset \mathcal{S}$, then this colimit has descent in $\mathcal{C}$.*

Proof. Let $X : I \to \mathcal{C}$ be such a diagram. By assumption, the adjunction $\mathcal{C}_{/\operatorname{colim} X_i} \rightleftarrows \lim \mathcal{C}_{/X_i}$ is the restriction to $\mathcal{C}$ of the adjunction $\mathcal{S}_{/\operatorname{colim} X_i} \rightleftarrows \lim \mathcal{S}_{/X_i}$ which is an equivalence of category by descent in $\mathcal{S}$. Hence so is the adjunction $\mathcal{C}_{/\operatorname{colim} X_i} \rightleftarrows \lim \mathcal{C}_{/X_i}$. □

Proposition 2.2.7 (Extensivity)**.** *Finite sums in $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ have descent (are disjoint and universal).*

Proof. We saw that the inclusion $\mathcal{S}^{<\infty}_{\mathrm{coh}} \subset \mathcal{S}$ preserve finite limits and finite sums in Proposition 2.2.4 and Lemma 2.2.2. The result follows from Lemma 2.2.6. ☐

Proposition 2.2.8. *The category $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ is idempotent complete.*

Proof. Let Y be a retract of a truncated coherent space X, then $\pi_n(Y)$ is a retract of $\pi_n(X)$, hence finite and eventually null. ☐

Remark 2.2.9. All the previous results are also true for the category $\mathcal{S}_{\mathrm{coh}}$ of coherent spaces (with the same proofs).

Using the Postnikov towers, any coherent space can be build by successive pullbacks of maps $1 \to K(G,n)$ where G is a finite group and $K(G,n)$ the associated Eilenberg–Mac Lane spaces. Proposition 2.2.5 ensure that all steps of the construction are in $\mathcal{S}^{<\infty}_{\mathrm{coh}}$. This gives the following result.

Proposition 2.2.10. *The category of truncated coherent spaces is the smallest subcategory of $\mathcal{S}$ closed under finite limits and finite coproducts and containing all Eilenberg–Mac Lane spaces $K(G,n)$ for G a finite group.*

2.3 Absence of pushouts

This section proves that $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ does not have all pushouts. We know already that the inclusion $\mathcal{S}^{<\infty}_{\mathrm{coh}} \subset \mathcal{S}$ cannot preserve pushouts since $S^1 = 1 \cup_{S^0} 1$ is not coherent. But that does not prevent pushout to exist in $\mathcal{S}^{<\infty}_{\mathrm{coh}}$. We will see that it is indeed the case by proving that the pushout $1 \leftarrow 2 \to 1$ (where $2 = 1+1$), classically equal to S^1, does not exist in $\mathcal{S}^{<\infty}_{\mathrm{coh}}$.

Proposition 2.3.1. *The pushout of the diagram $1 \leftarrow 2 \to 1$ is not representable in $\mathcal{S}^{<\infty}_{\mathrm{coh}}$.*

Proof. This pushout is by definition the object representing the free loop space functor

$$\begin{aligned} FL : \mathcal{S}^{<\infty}_{\mathrm{coh}} &\longrightarrow \mathcal{S} \\ X &\longmapsto X \times_{X\times X} X = X^{S^1} \end{aligned}$$

Let H be a truncated coherent space representing FL. Then, for any coherent space X, we have a natural equivalence $\mathrm{Hom}(H,X) = \mathrm{Hom}\big(S^1,X\big)$. This is equivalent to the data of a map $S^1 \to H$ presenting H as the reflection of S^1 in the subcategory $\mathcal{S}^{<\infty}_{\mathrm{coh}} \subset \mathcal{S}$ (but we shall need that). First we can deduce that H has to be connected. Indeed, for the discret space $X = 2$, we have

$$\mathrm{Hom}(H,2) = 2^{\pi_0(H)} \qquad \text{and} \qquad \mathrm{Hom}\big(S^1,2\big) = 2$$

hence $\pi_0(H) = 1$.

Recall that for (X,x) and (Y,y) two pointed spaces, the space of pointed morphisms is defined by the fiber product

$$\begin{tikzcd} \mathrm{Hom}_\bullet((Y,x),(X,x)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Hom}(Y,X) \arrow[d, "\mathrm{Hom}(y,X)"] \\ 1 \arrow[r, "x"] & X \end{tikzcd}$$

We fix a base point s in S^1, and consider its image h by the map $S^1 \to H$. The map $S^1 \to H$ induces an equivalence

$$\mathrm{Hom}_\bullet((H,h),(X,x)) = \mathrm{Hom}_\bullet\big((S^1,s),(X,x)\big) = \Omega_x X \tag{1}$$

The space H being connected its 1-truncation is a space BG for some finite group G. We consider the additive group $\mathbb{Z}/p\mathbb{Z}$ for p a prime number prime to the order of G. Then, the only group morphism $G \to \mathbb{Z}/p\mathbb{Z}$ is the constant one. We put $X = B\mathbb{Z}/p\mathbb{Z}$. Using the equivalence between pointed connected 1-type and discrete groups, we get

$$\mathrm{Hom}_\bullet((H,h),(X,x)) = \mathrm{Hom}_\bullet((BG,h),(B\mathbb{Z}/p\mathbb{Z},x)) = \mathrm{Hom}_{Gp}(G,\mathbb{Z}/p\mathbb{Z}) = 1.$$

But, on the other side, we have

$$\Omega_x X = \mathbb{Z}/p\mathbb{Z} \neq 1$$

This contradicts (1) and shows that H cannot exist. □

Remark 2.3.2. A more sophisticated version of the argument proves that S^n does not admit a reflection into $\mathcal{S}_{\text{coh}}^{<\infty}$. It is likely that any connected finite space does not either.

Not all pushouts exists in $\mathcal{S}_{\text{coh}}^{<\infty}$ but some do.

Proposition 2.3.3. *The pushouts of spans where one of leg is a monomorphism exist in $\mathcal{S}_{\text{coh}}^{<\infty}$.*

Proof. All monomorphisms are split in $\mathcal{S}$, hence in $\mathcal{S}_{\text{coh}}^{<\infty}$: if $A \to B$ is a monomorphism in $\mathcal{S}_{\text{coh}}^{<\infty}$ then B is isomorphic to $A \coprod B'$ for some B' in $\mathcal{S}_{\text{coh}}^{<\infty}$. Then, if $B \leftarrow A \to C$ is a span in $\mathcal{S}_{\text{coh}}^{<\infty}$ such that $A \to B$ is a monomorphism, the pushout is $C \coprod B'$ which is in $\mathcal{S}_{\text{coh}}^{<\infty}$. □

2.4 Comparison of coherent and finite spaces

We recall without proof some properties of finite spaces to compare them with coherent ones.

A space is *finite* is it is the homotopy type of a finite CW-complex, or, equivalently, the realization of a simplicial set with only a finite number of non-degenerate simplices. More intrinsically, the category of finite spaces space can be defined as the smallest subcategory of $\mathcal{S}$ containing 0 and 1 (or the whole of $\mathsf{Set}_{\text{fin}}$) and closed under pushouts. (We shall see in Proposition 2.7.3, a similar characterization of coherent spaces.) All spheres S^n ($n \geq -1$) are finite, and any finite space can be build with a finite chain of cell attachments

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & X_n \arrow[d] \\ 1 \arrow[r] & X_{n+1} \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

This is to be contrasted with Proposition 2.2.10. Any subspace of a finite space is finite. Any finite sums or finite product of finite spaces is finite. But $\mathcal{S}_{\text{fin}} \subset \mathcal{S}$ is not closed under finite limits since $\Omega S^1 \simeq \mathbb{Z}$ is not finite. It is also not closed under retracts [Lur09, Remark 5.4.1.6]. Table 1 summarizes the comparison between finite and truncated coherent spaces. A funny fact is that, $\mathcal{S}_{\text{coh}}^{<\infty}$ being closed under finite limits, it is cotensored over $\mathcal{S}_{\text{fin}}$:

Lemma 2.4.1. *The Hom functor $\mathcal{S}^{op} \times \mathcal{S} \to \mathcal{S}$ restricts into a functor $\mathcal{S}_{\text{fin}}^{op} \times \mathcal{S}_{\text{coh}}^{<\infty} \to \mathcal{S}_{\text{coh}}^{<\infty}$.*

Proof. Let K be a finite space and X be a truncated coherent space, then X^K is a finite limit of copies of X, hence in $\mathcal{S}_{\text{coh}}^{<\infty}$ by Proposition 2.2.4. □

Remark 2.4.2 (Generalization to higher cardinals)**.** The notions of finite and coherent spaces (and more generally that of compact and coherent object in a topos) rely implicitely on the notion of finite sets, that is ω-small sets. It can therefore be generalized by replacing ω with a non countable larger regular ordinal κ. If we do so, then the notion of κ-small and κ-coherent spaces do coincide. Only for ω are the two notions different. An explanation is the following: the completion of a simplicial set with a values in finite sets (a fortiori having a finite set of non-degenerate simplices) into a Kan complex has values in countable sets, but for $\kappa > \omega$, Kan completions of complexes with values in κ-small sets stay with values in κ-small sets.

Remark 2.4.3 (Closure of $\mathcal{S}_{\text{coh}}^{<\infty}$ for pushouts)**.** A natural question is to identify the smallest category containing $\mathcal{S}_{\text{coh}}^{<\infty}$ and $\mathsf{Set}_{\text{fin}}$ which is closed under finite limits and colimits. It can be proved that any space in this category will be the realization of a countable simplicial set. In particular, countable sets will be a part of it and therefore this category cannot be cartesian closed (nor even have countable sums, actually). This category has been studied by Berman [Ber20].

Table 1: Comparison between finite and truncated coherent spaces.

	$\mathcal{S}_{\mathrm{fin}}$	$\mathcal{S}_{\mathrm{coh}}^{<\infty}$
finite + and ×	yes	yes
subspaces	yes	yes
pushouts	yes	no
fiber products	no	yes
loop spaces Ω	no	yes
truncations	no	yes
retracts	no	yes
building blocks	$S^n \to 1$ by pushouts	$1 \to K(G,n)$ by fiber products
compactness properties	finite spaces are compact in $\mathcal{S}$	n-truncated coherent spaces are compact in $\mathcal{S}^{\leq n}$ (but not in $\mathcal{S}$)
Euler characteristic	in $\mathbb{Z} = \mathbb{N}[-1]$	in $\mathbb{Q}_{>0} = \mathbb{N}[\frac{1}{2}, \frac{1}{3}, \dots]$ [BD00, Bae03]
Other properties	—	ambidexterity [HL13, Har20] cotensored over $\mathcal{S}_{\mathrm{fin}}$ (Lemma 2.4.1)

2.5 Simplicial spaces

This section introduces definitions and constructions useful in the following sections.

A map $f : X \to Y$ in $\mathcal{S}$ is called *surjective* if the map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is surjective in Set. An object E of $\mathcal{S}$ is called *projective* if for any surjective map $X \to Y$, the map $X^E \to Y^E$ is surjective. Surjective maps being closed under base change, E is projective if and only if any surjective map $X \to E$ splits. Any object Y of $\mathcal{S}$ admits a surjective map $X \to Y$ from a set[2]. Covering a projective object E by a set, we get that it is a set. Reciprocally, any set is projective. We shall say that a map $f : X \to Y$ in $\mathcal{S}$ is *projective* if it is of the type $X \to X + E$ where E is a set.

Two maps $u : A \to B$ and $f : X \to Y$ of an arbitrary category $\mathcal{C}$ are said to be *weakly orthogonal* if the map $\langle u, f\rangle : \mathrm{Hom}(B,Y) \to \mathrm{Hom}(A,X) \times_{\mathrm{Hom}(A,Y)} \mathrm{Hom}(B,Y)$ is surjective in $\mathcal{S}$. The two classes of projective and surjective maps are weakly orthogonal to each other and form a weak factorization system on $\mathcal{S}$. The factorization of $f : X \to Y$ is given by $X \to X + E \to Y$ for $E \to Y$ any surjective map from a set. This factorization cannot be made functorial.

Let Δ be the category of simplices. We shall denote the colimit functor (also called realization) $\mathcal{S}^{\Delta^{op}} \to \mathcal{S}$ by $X_\bullet \mapsto |X_\bullet|$ and its right adjoint (constant diagram) by $X \mapsto \underline{X}$. The latter functor is fully faithful, and $|-|$ presents $\mathcal{S}$ as a reflective subcategory of $\mathcal{S}^{\Delta^{op}}$. We denote W the class of maps of $\mathcal{S}^{\Delta^{op}}$ send to invertible maps by the colimit functor. We call them *colimit equivalences*. For an object X in $\mathcal{S}$, a *resolution* of X is defined as a simplicial diagram $X_\bullet$ equipped with a colimit cocone with apex X. Such a cocone is equivalent to a map $X_\bullet \to \underline{X}$ which is in W.

The projective–surjective weak factorization system induces a Reedy weak factorization system on $\mathcal{S}^{\Delta^{op}}$ (see [MG14, Section 4]). Maps in the right class are sometimes called *hypercoverings* or *trivial fibrations*, we shall call them *hypersurjective*. Their class is defined as the (weak) right orthogonal to the maps $\partial\Delta[n] \to \Delta[n]$ $(n \geq 0)$. The following lemma is a crucial property of hypersurjective maps.

Lemma 2.5.1 ([Lur09, Lemma 6.5.3.11])**.** *All hypersurjective maps are colimit equivalences.*

The maps in the left class are sometimes called *cofibrations*, we shall call them *hyperprojective*. A map $X_\bullet \to Y_\bullet$ is hyperprojective if and only if all relative latching maps $X_n \coprod_{L_n X_\bullet} L_n Y_\bullet \to Y_n$ are projective. Intuitively, this means that $Y_\bullet$ is build from $X_\bullet$ by adding a *set* (rather than an arbitrary space) of non-degenerate simplices in each dimension. In particular, a map $0 \to X_\bullet$ is hyperprojective if and only if $X_\bullet$ is a simplicial *set*. Thus, the hyperprojective–hypersurjective factorization of $\varnothing \to X_\bullet$ always goes through a simplicial set. More generally, a map $X_\bullet \to Y_\bullet$ where $X_\bullet$ is a simplicial set is hyperprojective if and only if it is a monomorphism of simplicial sets. Let $\underline{X}$ be a constant simplicial object. Since sets are the projective object of $\mathcal{S}$, we shall say that a factorization of $\varnothing \to X_\bullet \to \underline{X}$ is a *projective resolution* of X.

We recall some results on how to construct projective resolutions. We shall need this to prove Proposition 2.6.3.

Lemma 2.5.2 (Reedy induction [Lur09, Corollary A.2.9.15 and Remark A.2.9.16])**.** *Let $\mathcal{C}$ be a category with finite limits and colimits. The extension of a functor $X : \Delta_{<n} \to \mathcal{C}$ into a functor $X' : \Delta_{\leq n} \to \mathcal{C}$ is equivalent to the data of a factorization of the map $L_n X \to M_n X$ (where $L_n X$ and $M_n X$ are the latching and matching objects of $X : \Delta_{<n} \to \mathcal{C}$).*

We can apply this to $\mathcal{C} = \mathcal{S}$ with the projective–surjective factorization.

Lemma 2.5.3 (Projective resolution [Lur11, Corollary 1.4.11])**.** *Let X be an object in $\mathcal{S}$. There exists a simplicial object $\Delta \to \mathcal{S}_{/X}$ such that, for every n, the map $L_n X \to X_n$ is a coproduct with some set, and the map $X_n \to M_n X$ is surjective.*

The simplicial object in $\mathcal{C}_{/X}$ of Lemma 2.5.3 provide a map $X_\bullet \to \underline{X}$ which, by construction, is hypersurjective.

[2] In the model of $\mathcal{S}$ with topological spaces, E can be the set of points of Y; in the model with simplicial sets, E can be the set of 0-simplices

2.6 Kan groupoids

This section proves Proposition 2.6.3, which is going to be our main tool to prove that $\mathcal{S}^{<\infty}_{\text{coh}}$ is locally cartesian closed (Theorem 2.8.6).

We say that a simplicial space $Y_\bullet$ is a *Kan groupoid* if it is (weakly) right orthogonal to all horn inclusions $\Lambda^n \to \Delta^n$. This means that all maps of spaces $Y_n \to \mathrm{Hom}(\Lambda^n, Y_\bullet)$ are surjective. A simplicial set is a Kan groupoid if and only if it is a Kan complex. We shall keep the name Kan complex for a Kan groupoid whose values are sets.

Lemma 2.6.1 (Kan resolution). *Given a hypersurjective map $X_\bullet \to \underline{X}$, the simplicial set $X_\bullet$ is a Kan complex.*

Proof. Let $|Y_\bullet|$ the colimit of $Y_\bullet$. By adjunction, we have $\mathrm{Hom}(Y_\bullet, \underline{X}) = \mathrm{Hom}(|Y_\bullet|, X)$. Using that $|\Lambda^n| \to |\Delta^n|$ is an equivalence in $\mathcal{S}$, we get that $\underline{X}$ is a Kan groupoid.

Hence, the result will be proved if the map $c : X_\bullet \to \underline{X}$ is (weakly) right orthogonal to all horn inclusions. By definition of a hypersurjective, the map c is right orthogonal to all maps $\partial\Delta^n \to \Delta^n$. Since the maps $\partial\Delta^n \to \Delta^n$ generate all monomorphisms of simplicial sets by iterated pushouts in $\mathcal{S}^{\Delta^{op}}$, the map c is also right orthogonal to all these maps. This includes all horn inclusions. □

Remark 2.6.2. Following [MG14, Section 4], it is convenient to introduce a second weak factorization system on $\mathcal{S}^{\Delta^{op}}$, generated by the horn inclusions. The maps in the right class are called *Kan fibrations.* A simplicial space $X_\bullet$ is a Kan groupoid if and only if $X_\bullet \to 1$ is a Kan fibration. Hence, if $X_\bullet \to Y_\bullet$ is a Kan fibration and $Y_\bullet$ is a Kan groupoid, then so is $X_\bullet$. The proof of Lemma 2.6.1 follows from the fact that any hypersurjective maps is a Kan fibration.

A simplicial space is n-coskeletal if it is the right Kan extension of its restriction to $\Delta_{\le n} \subset \Delta$. This is equivalent to the condition that the maps $X_k \to \mathrm{Hom}\big(sk_n\Delta^k, X_{\le n}\big)$ be all equivalence for $k > n$. We say that a Kan groupoid (in $\mathcal{S}$) is *truncated* if its colimit is n-truncated for some n. We say that a Kan complex is has *finite values* if is it in $(\mathsf{Set}_{\text{fin}})^{\Delta^{op}} \subset \mathsf{Set}^{\Delta^{op}}$.

Proposition 2.6.3. *A space is coherent if and only if it is the geometric realization of a Kan complex with finite values.*

Proof. Let $X_\bullet$ be a Kan complex. Recall that $\pi_0(|X_\bullet|)$ is a quotient of X_0 and and $\pi_n(|X_\bullet|, x)$ is a subquotient of X_n. Hence they are all finite if the X_n are. This proves that the conditions are sufficient.

To see that they are necessary, we use Reedy induction. Let X be a coherent space, we use Lemma 2.5.2 in $\mathcal{S}_{/X}$ to construct a simplicial object $\Delta \to \mathcal{S}_{/X}$. First, we chose $X_0 \to X$ a surjection from a set X_0. Because X is coherent, X_0 can be chosen finite. At step 1, $L_1(X_{\le 0}) = X_0$ and $M_0(X_{\le 0}) = X_0 \times_X X_0$. The space $X_0 \times_X X_0$ is a finite coproduct of path spaces of X. Since X is coherent, it has a finite number of connected components and we can put $X_1 \coloneqq X_0 + X_1'$ where X_1' is a finite set. At step n, let $\mathrm{Hom}(\partial\Delta^n, X_{<n})$ be the set of maps in $\mathsf{Set}\Delta^{op}_{<n}$. Since all X_k are finite sets, this is a finite set. Then we have $M_n(X_{<n}) = \mathrm{Hom}(\partial\Delta^n, X_{<n}) \times_{X^{|\partial\Delta^n|}} X$ and $M_n(X_{<n})$ is a coproduct of n-fold path spaces of X. Since X is coherent, $M_n(X_{<n})$ has a finite number of connected components and we can put $X_n \coloneqq L_n(X_{<n}) + X_n'$ where X_n' is a finite set. By induction, $L_n(X_{<n})$ is a finite set, hence so is X_n. The resulting simplicial set $X_\bullet$ has finite values. We get a map $X_\bullet \to \underline{X}$ in $\mathcal{S}^{\Delta^{op}}$ which is a hypersurjective, hence a colimit cone by Lemma 2.5.1. The fact that it is Kan is Lemma 2.6.1. □

2.7 Segal groupoids

This section proves Property (3). We prove in fact a stronger result presenting $\mathcal{S}^{<\infty}_{\text{coh}}$ as the closure of finite set under Segal groupoid (Proposition 2.7.3).

Let $X_\bullet$ be a simplicial space and $|X_\bullet|$ its colimit. We shall also call $|X_\bullet|$ the quotient of $X_\bullet$ and refer to the canonical map $q : X_0 \to |X_\bullet|$ as the quotient map. An object $X_\bullet$ in $\mathcal{S}^{\Delta^{op}}$ is *Segal groupoid* if it satisfies the Segal conditions: $X_n = X_1 \times_{X_0} \cdots \times_{X_0} X_1$ $(n > 1)$. Let $f : X \to Y$ be a map in $\mathcal{S}$ and $X_\bullet$ be its nerve $N(f)_\bullet$ $(X_n = X \times_Y \cdots \times_Y X)$. Then, $X_\bullet$ is a Segal groupoid. Intuitively, $N(f)_\bullet$ is the groupoid encoding the equivalence relation "to have same image by f". A Segal groupoid is *effective* if the canonical map $X_\bullet \to N(q)$ is invertible in $\mathcal{S}^{\Delta^{op}}$ (where q is the quotient map). In $\mathcal{S}$, all Segal groupoids are effective, this is part of the Giraud axioms of ∞-topoi [Lur09, Proposition 6.1.3.19]. Moreover, the functor sending a Segal groupoid to its quotient map $q : X_0 \to |X_\bullet|$ induces an equivalence between the full subcategory of $\mathcal{S}^{\Delta^{op}}$ spanned by Segal groupoids and the full subcategory of the arrow category of $\mathcal{S}$ spanned by surjective maps (the inverse equivalence being given by the nerve).

Proposition 2.7.1. *Let $X_\bullet$ be a Segal groupoid in $\mathcal{S}^{<\infty}_{\mathrm{coh}}$, then its quotient $|X_\bullet|$ is in $\mathcal{S}^{<\infty}_{\mathrm{coh}}$.*

Proof. The quotient map $X_0 \to |X_\bullet|$ being surjective, $\pi_0(|X_\bullet|)$ is a finite set. Hence we can restrict to the case where $|X_\bullet|$ is connected. By effectivity of Segal groupoids, we have a cartesian square

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X_0 \arrow[d] \\ X_0 \arrow[r] & {|X_\bullet|} \end{tikzcd}$$

Let x be an element in X_0. The fiber of $X_1 \to X_0$ at x is a coherent space Z by Proposition 2.2.3. Hence we can apply Proposition 2.2.3 again to the cartesian square

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X_0 \arrow[d] \\ 1 \arrow[r] & {|X_\bullet|} \end{tikzcd}$$

to deduce that $|X_\bullet|$ is truncated coherent. □

Lurie proves a similar result for $\mathcal{S}_{\mathrm{coh}}$ and Kan groupoids. We mention it for a comparison.

Proposition 2.7.2 ([Lur17, Theorem A.5.5.1])**.** *Let $X_\bullet$ be a Kan groupoid in $\mathcal{S}_{\mathrm{coh}}$, then its quotient $|X_\bullet|$ is in $\mathcal{S}_{\mathrm{coh}}$.*

The following result gives meaning to the category $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ and $\mathcal{S}_{\mathrm{coh}}$ inside $\mathcal{S}$. We'll use in Theorem 2.10.1 to prove $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ is the initial ∞-pretopos.

Proposition 2.7.3 (Exact completions)**.**

1. *The category $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ is the smallest category of $\mathcal{S}$ containing $\mathsf{Set}_{\mathrm{fin}}$ and closed under quotients of Segal groupoids.*
2. *The category $\mathcal{S}_{\mathrm{coh}}$ is the smallest category of $\mathcal{S}$ containing $\mathsf{Set}_{\mathrm{fin}}$ and closed under quotients of Kan groupoids.*

Proof. (1) Let $\mathcal{C} \subset \mathcal{S}$ be the smallest full subcategory containing $\mathsf{Set}_{\mathrm{fin}}$ and closed under quotients of Segal groupoids. Since $\mathsf{Set}_{\mathrm{fin}} \subset \mathcal{S}^{<\infty}_{\mathrm{coh}}$, Proposition 2.7.1 proves that $\mathcal{C} \subset \mathcal{S}^{<\infty}_{\mathrm{coh}}$. Conversely, we proceed by induction on the truncation level. Let $\mathcal{S}^{\leq n}_{\mathrm{coh}} \subset \mathcal{S}^{<\infty}_{\mathrm{coh}}$ be the full subcategory spanned by n-truncated objects. We have $\mathcal{S}^{\leq 0}_{\mathrm{coh}} = \mathsf{Set}_{\mathrm{fin}}$. Let us prove that any object X of $\mathcal{S}^{\leq n+1}_{\mathrm{coh}}$ can be obtained as the quotient of a Segal groupoid in $\mathcal{S}^{\leq n}_{\mathrm{coh}}$. Let $f : X_0 \to X$ be a surjective map where X_0 is a set, then X_0 is in $\mathcal{S}^{\leq n}_{\mathrm{coh}}$. We consider the nerve $X_\bullet$ of f. It is a Segal groupoid whose quotient is X. The result will be proved if we show that $X_\bullet$ is a simplicial object in $\mathcal{S}^{\leq n}_{\mathrm{coh}}$. The space $X_1 = X_0 \times_X X_0$ is a coproduct of loop spaces of X, hence coherent and n-truncated. More generally, we have $X_n = X_1 \times_{X_0} \cdots \times_{X_0} X_1$, and this shows X_n is also in $\mathcal{S}^{\leq n}_{\mathrm{coh}}$.
(2) By Proposition 2.6.3, $\mathcal{S}_{\mathrm{coh}}$ is included in the smallest full subcategory containing $\mathsf{Set}_{\mathrm{fin}}$ and closed under quotients of Kan groupoids. The converse is given by Proposition 2.7.2. □

Corollary 2.7.4 (Descent properties)**.**

1. *Quotients of Kan groupoids have descent in $\mathcal{S}_{\mathrm{coh}}$.*

2. *Quotients of truncated Kan groupoids have descent in* $\mathcal{S}^{<\infty}_{\mathrm{coh}}$.

3. *Segal groupoids have descent in* $\mathcal{S}^{<\infty}_{\mathrm{coh}}$.

4. *Segal groupoids are universal and effective in* $\mathcal{S}^{<\infty}_{\mathrm{coh}}$.

Proof. The properties (1), (2), and (3) are consequences of Lemma 2.2.6. We are left to prove (4). The universality of Segal groupoids is a consequence of (3), and the effectivity is a consequence of $\mathcal{S}^{<\infty}_{\mathrm{coh}} \subset \mathcal{S}$ preserving finite limits (Proposition 2.2.4). □

Remark 2.7.5. Putting together **??** 2.2.4**??** 2.2.7, Remark 2.2.9, and Corollary 2.7.4, we get that $\mathcal{S}_{\mathrm{coh}}$ and $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ are ∞-pretopoi in the sense of [Lur17, Definition A.6.1.1]. In fact, we shall see in Theorem 2.10.1 that $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ is the initial pretopos. Since $\mathsf{Set}_{\mathrm{fin}}$ is the initial 1-pretopos, $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ is then the ∞-pretopos envelope of $\mathsf{Set}_{\mathrm{fin}}$, and can be thought as its higher exact completion.

2.8 Local cartesian closure

This section proves Property (5) (Theorem 2.8.6). We prove first that $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ is cartesian closed and deduce the statement for the slice categories by a descent argument.

Lemma 2.8.1. *The category* $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ *is cartesian closed, and the embedding* $\mathcal{S}^{<\infty}_{\mathrm{coh}} \subset \mathcal{S}$ *preserves the exponentials.*

Proof. We have seen that $\mathcal{S}^{<\infty}_{\mathrm{coh}} \subset \mathcal{S}$ is closed under finite products (Proposition 2.2.4). We are going to show that for any two spaces X and Y in $\mathcal{S}^{<\infty}_{\mathrm{coh}}$, the space Y^X is in $\mathcal{S}^{<\infty}_{\mathrm{coh}}$. When X is a finite set, this is true because Y^X is a finite product of Y. For a general X, we use Proposition 2.6.3 to present X as the colimit of a simplicial finite set and get

$$Y^X = \lim_m Y^{X_m}.$$

This limit is a priori infinite and $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ is only closed under finite limits. By assumption Y is k-truncated for some k, then so are all the Y^{X_m}. Thus, we can use that the inclusion $\Delta_{\leq k+1} \subset \Delta$ is coinitial for diagrams of k-truncated objects (see Lemma 2.8.4 below), and replace the limit by an equivalent one which is finite. □

We say that a functor $f : C \to D$ between n-categories is n-cofinal if for any cocomplete n-category $\mathcal{C}$, the colimit of any diagram $X : D \to C$ coincide with the colimit of $X \circ f : C \to \mathcal{C}$. For C a small n-category, its free cocompletion (as an n-category) is $\mathcal{P}_n(C) := \left[C^{op}, \mathcal{S}^{\leq n-1}\right]$.

Lemma 2.8.2. *The following condition are equivalent:*

1. *the functor* $f : C \to D$ *is* n*-cofinal;*

2. *the functor* $\mathcal{P}_n(f) : \mathcal{P}_n(C) \to \mathcal{P}_n(D)$ *preserves the terminal object;*

3. *for any* d *in* D*, the realization of the category* $C_{d/} := C \times_D D_{d/}$ *is an* $(n-1)$*-connected space.*

Proof. (2) ⇔ (1). The colimit of the Yoneda embedding $C \to \mathcal{P}_n(C)$ is the terminal object. If $\mathcal{C}$ is a cocomplete n-category the colimit of a diagram $C \to \mathcal{C}$ is the image by $\mathcal{P}_n(C) \to \mathcal{C}$ of the terminal object. This proves that (1) ⇒ (2). Reciprocally, given a diagram $D \to \mathcal{C}$, the commutative diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d, "f"'] & \mathcal{P}_n(C) \arrow[d, "\mathcal{P}_n(f)"] \arrow[dr, dashed, bend left] & \\ D \arrow[r] & \mathcal{P}_n(D) \arrow[r, dashed] & \mathcal{C} \end{tikzcd}$$

(where the dashed arrows are cocontinuous) proves that (2) ⇒ (1).

(2) $\Leftrightarrow$ (3). We have $\mathcal{P}_n(f)(1) = 1$ if and only if, for any d in D, the space $\mathrm{Hom}(d, \mathcal{P}_n(f)(1))$ is contractible. But we have (in $\mathcal{S}^{\leq n-1}$)

$$\mathrm{Hom}(d, \mathcal{P}_n(f)(1)) = \mathrm{Hom}\left(\widehat{d}, \operatorname*{colim}_C \widehat{f(c)}\right) = \operatorname*{colim}_C \mathrm{Hom}(d, f(c)) = \operatorname*{colim}_{C_{d/}} 1 = |C_{d/}|^{\leq n-1}$$

where $|C_{d/}|^{\leq n-1}$ is the $(n-1)$-truncation of the realization of $C_{d/}$. This space is contractible if and only if the realization of $C_{d/}$ is $(n-1)$-connected. This proves (2) $\Leftrightarrow$ (3). ☐

Lemma 2.8.3. *The inclusion $\Delta_{\leq n} \to \Delta$ is coinitial for diagrams in n-categories.*

Proof. If $k \leq n$, $(\Delta_{\leq n})_{/[k]}$ has a terminal object and is weakly contractible. If $k > n$, the realization of $(\Delta_{\leq n})_{/[k]}$ is $sk_n(\Delta[k])$ which is a bouquet of n-spheres, hence $(n-1)$-connected. The result follows from Lemma 2.8.2. ☐

Recall that an object X in category $\mathcal{C}$ is called n-truncated if the functor $\mathrm{Hom}(-, X) : \mathcal{C}^{op} \to \mathcal{S}$ takes values in n-truncated spaces.

Lemma 2.8.4. *The inclusion $\Delta_{\leq n} \to \Delta$ is coinitial for diagrams of $(n-1)$-truncated objects.*

Proof. Lemma 2.8.3 Let $\mathcal{C}$ be a category and $\mathcal{C}^{\leq n-1} \subset \mathcal{C}$ be the full subcategory of n-truncated objects. Let us see that $D : I \to \mathcal{C}^{\leq n-1}$ be a diagram having a limit in $\mathcal{C}$, then its limit is in $\mathcal{C}^{\leq n}$. The result is true in $\mathcal{S}$ because the subcategory $\mathcal{S}^{\leq n-1} \subset \mathcal{S}$ of n-truncated spaces is reflective, hence closed under arbitrary limits. For a general $\mathcal{C}$, the limit of D is the object representing the functor

$$\begin{array}{ccc} \mathcal{C}^{op} & \longrightarrow & \mathcal{S} \\ X & \longmapsto & \lim\limits_i \mathrm{Hom}(X, D_i) \end{array}$$

If all the D_i are $(n-1)$-truncated, this functor takes values in $(n-1)$-truncated spaces, so any representative will be an $(n-1)$-truncated object. This reduces the problem to prove the coinitiality of $\Delta_{\leq n} \to \Delta$ to diagrams in the n-category $\mathcal{C}^{\leq n-1}$, but then it follows from Lemma 2.8.3. ☐

Lemma 2.8.5. *The limit of a diagram of cartesian closed categories and cartesian closed functors is cartesian closed.*

Proof. Several arguments can be given: the most conceptual is that the category of cartesian closed categories is monadic over that of categories, hence the forgetful functor creates limits. More down to earth, the proof is straightforward for products, so we need only give an argument for fiber products. Let $\mathcal{C}_1 \xrightarrow{p} \mathcal{C}_0 \xleftarrow{q} \mathcal{C}_2$ be a diagram of cartesian closed categories and cartesian closed functors. Objects in the limits are families $X = (X_1, X_0, X_2, x_1 : p(X_1) \simeq X_0, x_2 : q(X_2) \simeq X_0)$. We leave the reader to check that the internal hom between two such families X and Y are computed termwise as

$$\left(X_1^{Y_1},\ X_0^{Y_0},\ X_2^{Y_2},\ x_1^{y_1^{-1}} : p(X_1^{Y_1}) \simeq X_0^{Y_0},\ x_2^{y_2^{-1}} : q(X_2^{Y_2}) \simeq X_0^{Y_0}\right).$$

☐

Theorem 2.8.6. *The category $\mathcal{S}_{\mathrm{coh}}^{<\infty}$ is locally cartesian closed.*

Proof. We need to prove that for any X in $\mathcal{S}_{\mathrm{coh}}^{<\infty}$, the category $(\mathcal{S}_{\mathrm{coh}}^{<\infty})_{/X}$ is cartesian closed. If X is a finite set, then $(\mathcal{S}_{\mathrm{coh}}^{<\infty})_{/X} = (\mathcal{S}_{\mathrm{coh}}^{<\infty})^X$ is cartesian closed as a product of cartesian closed categories. For a general X, we use Proposition 2.6.3 to get a Kan complex with finite values $X_\bullet$ with colimit X. Corollary 2.7.4 gives that $(\mathcal{S}_{\mathrm{coh}}^{<\infty})_{/X} = \lim_\Delta (\mathcal{S}_{\mathrm{coh}}^{<\infty})_{/X_n}$ and the result follow from Lemma 2.8.5. ☐

Remark 2.8.7. This result is not true for the category $\mathcal{S}_{\mathrm{coh}}$. Let $X = \prod_n K(\mathbb{Z}_2, n)$. Any sequence of groups morphisms $\phi_n : \mathbb{Z}_2 \to \mathbb{Z}_2$ defines an endomorphisms $\phi := \prod_n K(\phi_n, n)$ of X. Acting differently on the π_n, these ϕ are non-homotopic in X^X. Any group morphism $\mathbb{Z}_2 \to \mathbb{Z}_2$ is either the identity or constant. Hence, the set of such sequences is $2^{\mathbb{N}}$. This proves $\pi_0(X^X)$ is not finite (it's not even countable).

2.9 The universe of truncated coherent spaces

This section proves Properties (6) to (8). We do so by constructing a universe in $\mathcal{S}$ for truncated coherent spaces (Theorem 2.9.2).

Let $\mathcal{S}^{\to}$ be the arrow category of $\mathcal{S}$. Consider the codomain cartesian fibration $\mathrm{cod} : \mathcal{S}^{\to} \to \mathcal{S}$. Its fibers are the slices categories $\mathcal{S}_{/X}$. We denote by $\mathcal{S}^{\to}_{\mathrm{cart}} \subset \mathcal{S}^{\to}$ the subcategory with the same objects but only cartesian morphisms. The restriction $\mathrm{cod} : \mathcal{S}^{\to}_{\mathrm{cart}} \to \mathcal{S}$ is still a fibration, whose fibers at X is the *interior groupoid* (maximal subgroupoid) $\mathcal{S}^{\mathrm{int}}_{/X}$ of $\mathcal{S}_{/X}$. Let $\widehat{\mathcal{S}}$ be the category of spaces in a larger universe. We shall implicitly embed $\mathcal{S}$ in $\widehat{\mathcal{S}}$. This fibration has a classifying functor

$$\begin{aligned} U : \mathcal{S}^{op} &\longrightarrow \widehat{\mathcal{S}} \\ X &\longmapsto \mathcal{S}^{\mathrm{int}}_{/X} . \end{aligned}$$

For a space X, let $Aut(X) \subset X^X$ be its group of automorphisms. This group acts on X and we shall denote the quotient by $X/Aut(X)$. It also acts trivially on the point 1 and the quotient $1/Aut(X)$ is the gerbe $BAut(X)$ classifying spaces isomorphic to X. Precisely, the space of maps $Z \to BAut(X)$ is equivalent to the full subgroupoid of $\mathcal{S}^{\mathrm{int}}_{/Z}$ spanned by X-*bundles* (maps $Z' \to Z$ whose fiber are all isomorphic to X). The canonical map $X \to 1$ induces a map $X/Aut(X) \to BAut(X)$ which is the universal X-bundle. For any X-bundle $Z' \to Z$, there exists a unique cartesian square

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X/Aut(X) \arrow[d] \\ Z \arrow[r] & BAut(X) . \end{tikzcd}$$

Let S be a set (in $\widehat{\mathcal{S}}$) of representative for each isomorphism class of objects in $\mathcal{S}$. We define

$$U_S := \coprod_{X \in S} BAut(X) \qquad \text{and} \qquad U'_S := \coprod_{X \in S} X/Aut(X).$$

Then, the functor U is representable (in $\widehat{\mathcal{S}}$) by the space U_S, and the map $U'_S \to U_S$ is the corresponding universal family.

Notice that for any small set $S' \subset S$, the object $U_{S'} = \coprod_{X \in S'} BAut(X)$ is in $\mathcal{S}$. For a space X, let $S'(X) \subset \mathcal{S}^{\mathrm{int}}_{/X}$ be the full subcategory of maps $X' \to X$ whose fibers are isomorphic to some element of S'. Then, the space $U_{S'}$ represents the functor $X \mapsto S'(X)$. When $S' = S^{<\infty}_{\mathrm{coh}} \subset S$ is the subset of elements of X that are truncated coherent, we denote $U_{S'}$ and $U'_{S'}$ by $U^{<\infty}_{\mathrm{coh}}$ and $(U^{<\infty}_{\mathrm{coh}})'$. The next result proves that $S^{<\infty}_{\mathrm{coh}}$ is a countable set.

Lemma 2.9.1. *The set of isomorphism classes of objects of $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ is countable.*

Proof. Let $X_\bullet$ be a Kan complex whose realization $|X|$ is n-truncated. Then the map $X_\bullet \to cosk_{n+1}(X_\bullet)$ is a colimit equivalence. Moreover, if the values X_n are finite, then so are the values of $cosk_{n+1}(X_\bullet)$. Then, Proposition 2.6.3 proves that all truncated coherent spaces can be described as colimits of diagrams $\Delta_{\le n+1} \to \mathsf{Set}_{\mathrm{fin}}$ for some n. The conclusion follows from the set of isomorphism classes of diagrams $\Delta_{\le n+1} \to \mathsf{Set}_{\mathrm{fin}}$ being countable. □

We say that a map $X \to Y$ in $\mathcal{S}$ is *truncated coherent* if all its fibers are truncated coherent (or, equivalently, isomorphic to some element in $S^{<\infty}_{\mathrm{coh}}$). Let $\mathcal{S}^{<\infty}_{\mathrm{coh}}(X) \subset \mathcal{S}^{\mathrm{int}}_{/X}$ the full subgroupoid spanned by truncated coherent maps.

Theorem 2.9.2. *The space $U^{<\infty}_{\mathrm{coh}}$ represents the functor $X \mapsto \mathcal{S}^{<\infty}_{\mathrm{coh}}(X)$. For any truncated coherent map $X \to Y$, there exists a unique cartesian square*

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & (U^{<\infty}_{\mathrm{coh}})' \arrow[d] \\ Y \arrow[r] & U^{<\infty}_{\mathrm{coh}} \end{tikzcd} \tag{2}$$

Moreover, $U^{<\infty}_{\mathrm{coh}}$ and $(U^{<\infty}_{\mathrm{coh}})'$ are countable coproducts of truncated coherent spaces.

Proof. The first part is a consequence of the remarks preceding the theorem. We need only to prove the last assertion. For a coherent space X, we've seen in Lemma 2.8.1 that X^X is coherent. Hence, so is $Aut(X) \subset X^X$ by Lemma 2.2.1. The group $Aut(X)$ defines a Segal groupoid whose quotient is the classifying space $BAut(X)$. The action of $Aut(X)$ on X also defines a Segal groupoid whose quotient is $X/Aut(X)$. By Proposition 2.7.1, these quotients are truncated coherent. Using Lemma 2.9.1, this shows that $U^{<\infty}_{\mathrm{coh}}$ is a countable coproduct of truncated coherent spaces. □

Remark 2.9.3. In other words, the class of truncated coherent maps is a local class in the sense of [Lur09, Definition 6.1.3.8].

Corollary 2.9.4. *The space $U^{<\infty}_{\mathrm{coh}}$ is the realization of a countable simplicial set.*

Proof. By Proposition 2.6.3 and Theorem 2.9.2, $U^{<\infty}_{\mathrm{coh}}$ is the realization of a countable coproduct of Kan complexes with values in finite sets. □

Definition 2.9.5. A map $V' \to V$ in $\mathcal{S}$ is called *univalent* if its classifying map $V \to U$ (in $\widehat{\mathcal{S}}$) is a monomorphism. In other words, the univalent maps in $\mathcal{S}$ are exactly the maps $U'_{S'} \to U_{S'}$ (for some small set $S' \subset S$), introduced above. A map $f : X \to Y$ is said to be *classified* by a univalent map $p : V' \to V$ if f is a base change of p. A space X is said to be *classified* by a univalent map $p : V' \to V$ if $X \to 1$ is a base change of p.

Lemma 2.9.6. *The map $t : \{1\} \to \{0, 1\}$ is univalent in $\mathcal{S}$.*

Proof. For $S' := \{\varnothing, 1\}$, we have $U_{S'} = BAut(\varnothing) + BAut(\{*\}) = 1 + 1$ and $U'_{S'} = 1$. Thus, the map $U'_{S'} \to U_{S'}$ is isomorphic to the map t. This proves that t is univalent. □

Proposition 2.9.7 (Enough univalent maps)**.** *For any map $f : X \to Y$ in $\mathcal{S}^{<\infty}_{\mathrm{coh}}$, there exists a univalent map u_f in $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ such that f is a pullback of u_f.*

Proof. Because Y has a finite number of connected components, the classifying map $U \to U^{<\infty}_{\mathrm{coh}}$ of (2) factors through some space $U_{S'} = \coprod_{X \in S'} BAut(X)$ for some finite set $S' \subset S^{<\infty}_{\mathrm{coh}}$. Both U_S and U'_S are truncated coherent as finite coproduct of truncated coherent spaces, and $X \to Y$ is the pullback in $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ of the $U'_{S'} \to U_{S'}$. □

Lemma 2.9.8. *If $f' \to f$ is a cartesian square in $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ there exists a canonical monomorphism $u_{f'} \to u_f$. Moreover, if the components of the map $f' \to f$ are surjections, then the map $u_{f'} \to u_f$ is invertible.*

Proof. By construction of the maps $u_{f'}$ and u_f in Proposition 2.9.7, $u_{f'}$ is isomorphic to a summand of u_f. Then, the assumption of surjectivity of $f' \to f$, ensures that all summands of u_f are in the image of $u_{f'} \to u_f$. □

Lemma 2.9.9. *In a category with pullbacks, if a map f' is a pullback of a map f, then $\Delta f'$ is a pullback of Δf. Moreover, if f' is a pullback of f along a surjection, then $\Delta f'$ is a pullback of Δf along a surjection.*

Proof. Given a cartesian square

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\ulcorner", very near start] & A \arrow[d, "f"] \\ B' \arrow[r] & B \end{tikzcd}$$

The map $\Delta f' \to \Delta f$ is the square

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[d] \\ A' \times_{B'} A' \arrow[r] & A \times_B A\,. \end{tikzcd}$$

Using that $A' = A \times_B B'$, the previous square becomes

$$\begin{tikzcd} A \times_B B' \arrow[r] \arrow[d] & A \arrow[d] \\ A \times_B A \times_B B' \arrow[r] & A \times_B A \end{tikzcd}$$

which is clearly cartesian.

If the map $B' \to B$ is a surjection, then so is the map $A \times_B A \times_B B' \to A \times_B A$ in the previous diagram. This proves the second assertion. □

The following definition is meant to capture the type theoretical idea of a universe closed under the construction of identity types.

Definition 2.9.10 (Universe with diagonals)**.** A univalent map u is said to be *closed under diagonals* if Δu is a pullback of u. If f is a pullback of u, then Δf is a pullback of Δu (Lemma 2.9.9 below), and thus a a pullback of u. In particular, all iterated diagonals of u are pullbacks of u.

Proposition 2.9.11 (Enough universes with diagonals)**.** *For any map $f : X \to Y$ in $\mathcal{S}^{<\infty}_{\mathrm{coh}}$, there exists a univalent map u in $\mathcal{S}^{<\infty}_{\mathrm{coh}}$, which is closed under diagonals and such that f (and therefore any of its diagonals) is a pullback of u.*

Proof. The map f being truncated, there exists an n such that for all $k \geq n$ the iterated diagonal $\Delta^k f$ are invertible. In particular, for such a k, $\Delta^k f$ will always be a pullback of $\Delta^n f$. We put $g := \coprod_{0 \leq k \leq n} \Delta^k f$. Using the effectivity of coproducts and the previous remark, we get that the map Δg is a pullback of g. We consider the map u_g of Proposition 2.9.7. Let us see that it is closed under diagonals. By construction of u_g, there exists a cartesian square $g \to u_g$ whose component are surjections. The result will follows from the diagram in the arrow category of $\mathcal{S}^{<\infty}_{\mathrm{coh}}$

$$\begin{tikzcd} g \arrow[d, "\mathrm{cart}"'] & \Delta g \arrow[l, "\mathrm{cart}"'] \arrow[r, "\mathrm{cart}"] \arrow[d, "\mathrm{cart}"] & \Delta u_g \arrow[d, "\mathrm{cart}"] \\ u_g & u_{\Delta g} \arrow[l, "\mathrm{mono}"] \arrow[r, equal] & u_{\Delta u_g}\,. \end{tikzcd}$$

The vertical maps are obtained from Proposition 2.9.7. The map $\Delta g \to \Delta u_g$ is obtained by Lemma 2.9.9 applied to $g \to u_g$. The components of $\Delta g \to \Delta u_g$ are surjections and we can apply Lemma 2.9.8 to get the bottom map $u_{\Delta g} = u_{\Delta u_g}$. Another application of Lemma 2.9.8 to $\Delta g \to g$ gives the monomorphism $u_{\Delta g} \to u_g$. All maps in the diagram are cartesian. Then so is the map $\Delta_{u_g} \to u_g$ by composition. This proves that g is closed under diagonals. □

If $f : X \to Y$ is a map in $\mathcal{S}$, recall that there exists a triple adjunction $f_! \dashv f^* \dashv f_*$ where $f^* : \mathcal{S}_{/Y} \to \mathcal{S}_{/X}$ is the pullback along f. The functor $f_!$ is given by the composition with f. Let $p : V' \to V$ be a univalent family in $\mathcal{S}$ and, for any X in $\mathcal{S}$, let $V(X)$ be the full subcategory of $\mathcal{S}_{/X}$ spanned by maps $X' \to X$ which are base change of p. If $f : X \to Y$ is a map in $\mathcal{S}$, the pullback along f restricts to a functor $f^* : V(Y) \to V(X)$.

Definition 2.9.12 (Dependent sums and products)**.** Let $p : V' \to V$ be a univalent family in $\mathcal{S}$.

1. The map p is *closed under dependent sums* if, for any map $f : A \to B$ which is a base change of p, the left adjoint $f_!$ sends $V(X)$ into $V(Y)$.

2. The map p is *closed under dependent products* if, for any map f which is a base change of p, the right adjoint f_* sends $V(X)$ into $V(Y)$.

Remark 2.9.13. The general definition of having dependent sums or products involves Beck–Chevalley conditions, but these conditions are automatic in $\mathcal{S}$.

Lemma 2.9.14. *Let $p : V' \to V$ be a univalent family in $\mathcal{S}$ and let X be a space classified by p.*

1. *If the map p is closed under dependent sums, then all finite powers X^n are classified by p.*

2. *If the map p is closed under dependent products, then all iterated exponential X^X, $X^{(X^X)}$, ... are classified by p.*

Proof. (1) The map $q : X \to 1$ is a base change of p by assumption. The map $p_1 : X \times X \to X$ is a base change of q and thus of p. The composition $qp_1 = q_!(p_1)$ is then a base change of p since p is closed under dependent sums. The higher powers are obtained from there by an induction left to the reader.
(2) We proceed as in (1), the map $q_*(p_1) = X^X \to 1$ is then a base change of p since p is closed under dependent products. The higher iterated exponential are obtained by induction. □

Theorem 2.9.15. *The univalent family (in $\mathcal{S}$) $p : (U^{<\infty}_{\mathrm{coh}})' \to U^{<\infty}_{\mathrm{coh}}$ has dependent sums and dependent products.*

Proof. Let $\mathcal{M}$ be the class of maps in $\mathcal{S}$ which are base change of p. Definition 2.9.12.(1) says that for any two maps $f : A \to B$ and $g : A' \to A$ in $\mathcal{M}$, the composite $fg : A' \to B$ is in $\mathcal{M}$. This is a consequence of Proposition 2.2.3. Definition 2.9.12.(2) says that for any two maps $f : A \to B$ and $g : A' \to A$ in $\mathcal{M}$, the map $f_*(g)$ is in $\mathcal{M}$. This is a consequence of Theorem 2.8.6. □

Proposition 2.9.16. *The set $2 := \{0, 1\}$ is a (Boolean) subobject classifier in $\mathcal{S}_{\mathrm{coh}}$.*

Proof. The set $2 = \{0, 1\}$ is a subobject classifier in $\mathsf{Set}_{\mathrm{fin}}$. A map $X \to Y$ in $\mathcal{S}$ is a monomorphism if and only if the map $\pi_0(X) \to \pi_0(Y)$ is injective. If $Sub(X)$ is the set of subobjects of a space X, we have natural bijections

$$Sub(X) = 2^{\pi_0(X)} = 2^X.$$

This proves that 2 is a subobject classifier in $\mathcal{S}$. The result follows from the fact that any subobject of a truncated coherent space X is truncated coherent (Lemma 2.2.1). □

Theorem 2.9.17. *The map $1 \to 2$ is the largest univalent map in $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ with dependent sums or dependent products.*

Proof. The map $1 \to 2$ is univalent by Lemma 2.9.6. We leave the proof that it is closed under dependent sums and dependent products to the reader. Let $U'_{S'} \to U_{S'}$ be a univalent map in $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ such that one of the component of $U_{S'}$ is $BAut(X)$ for a coherent space X which is not subterminal (not 0 or 1). Let us see now that $U'_{S'} \to U_{S'}$ cannot be closed under dependent sums. If this was the case, by Lemma 2.9.14(1), all finite powers X^n, would be classified by $U_{S'}$. But when X is not subterminal, all finite powers X^n have non isomorphic π_0. This proves that all X^n are non equivalent and must belong to different connected components of $U_{S'}$. This is contradictory with the fact that $U_{S'}$ has only a finite number of connected components. This proves that any object X classified by $U_{S'}$ must be subterminal. The argument is similar if we consider a univalent family with dependent products (using Lemma 2.9.14(2)). □

2.10 Initiality properties

This section proves the initiality results of Properties (9) to (11).

2.10.1 Initial pretopos

Recall from [Lur17, Definition A.6.1.1], that an ∞-pretopos (we shall say simply a *pretopos*) is a category $\mathcal{E}$ with finite limits, with extensive finite coproducts, and with universal and effective quotients of Segal groupoids. A morphism of pretopoi is a functor preserving finite limits, finite sums and quotients of Segal groupoids (or equivalently surjective maps).

Theorem 2.10.1. *The category $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ is the initial pretopos.*

Sketch of the proof. We need to prove that, for any pretopos $\mathcal{E}$, the category of morphisms of pretopoi $\mathcal{S}^{<\infty}_{\mathrm{coh}} \to \mathcal{E}$ is contractible. The proof of Proposition 2.7.3 shows that all objects of $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ can be build from finite sets by successive quotients of Segal groupoids. Hence any pretopos morphism $\mathcal{S}^{<\infty}_{\mathrm{coh}} \to \mathcal{E}$ is completely determined by its restriction to finite sets. Since the morphism preserve sums, it is in fact determined by the image of a singleton. But this must be the terminal object of $\mathcal{E}$ by left-exactness. This proves that the category of morphisms pretopoi $\mathcal{S}^{<\infty}_{\mathrm{coh}} \to \mathcal{E}$ is either empty of contractible.

We're only going to sketch the proof of the existence of a pretopos a morphism $i : \mathcal{S}^{<\infty}_{\mathrm{coh}} \to \mathcal{E}$. A first argument is to define i as the left Kan extension

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & \mathcal{E} \\ \mathcal{S}^{<\infty}_{\mathrm{coh}} \arrow[ur, dashed, "i"'] & \end{tikzcd}$$

where both maps $1 \to \mathcal{E}$ and $1 \to \mathcal{S}^{<\infty}_{\mathrm{coh}}$ are pointing the terminal objects. The existence of this Kan extension is clear enough, but the fact that is it a left-exact functor require an argument. The argument is similar to the one proving that, for a topos $\mathcal{E}$, the canonical cocontinuous functor $\mathcal{S} \to \mathcal{E}$ is left-exact (see [Lur09, Proposition 6.1.5.2] or [AL19, Theorem 2.1.4]) and would be too long to reproduce here.

Another argument would be to cocomplete both $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ and $\mathcal{E}$ into topoi $\overline{\mathcal{S}^{<\infty}_{\mathrm{coh}}}$ and $\overline{\mathcal{E}}$, to precisely use the analoguous result in this context. The topos $\overline{\mathcal{E}}$ is the cocompletion of $\mathcal{E}$ preserving finite sums and quotients of Segal groupoid. Explicitely, $\overline{\mathcal{E}}$ it the category of sheaves $\mathcal{E}^{op} \to \mathcal{S}$ for the effective epimorphism topology on $\mathcal{E}$, see [Lur17, A.6.4]. By definition, the embedding $\mathcal{E} \to \overline{\mathcal{E}}$ preserve finite limits, finite sums and quotient of Segal groupoids. The $\overline{\mathcal{S}^{<\infty}_{\mathrm{coh}}}$ can be defined similary, but by Proposition 2.7.3.(1), it is simply $\mathcal{S}$. The constant sheaf functor $\mathcal{S} \to \overline{\mathcal{E}}$ is a cocontinuous and left-exact functor. Composing with $\mathcal{S}^{<\infty}_{\mathrm{coh}} \to \overline{\mathcal{S}^{<\infty}_{\mathrm{coh}}}$, we get a functor $\mathcal{S}^{<\infty}_{\mathrm{coh}} \to \overline{\mathcal{E}}$ preserving finite limits and quotients of Segal groupoids. Since this functor sends the terminal object of $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ in $\mathcal{E}$, the whole image $\mathcal{S}^{<\infty}_{\mathrm{coh}} \to \overline{\mathcal{E}}$ is in $\mathcal{E}$. This proves the existence of a pretopos morphism $\mathcal{S}^{<\infty}_{\mathrm{coh}} \to \mathcal{E}$ (but relying on the material of [Lur17, A.6.4]). ☐

2.10.2 Initial Π-pretopos

Let $\mathcal{E}$ be a pretopos and $i : \mathcal{S}^{<\infty}_{\mathrm{coh}} \to \mathcal{E}$ the morphism of Theorem 2.10.1. For X a space, we denote by $\mathcal{E}^X$ the category of X-diagrams in $\mathcal{E}$. We show it is equivalent to $\mathcal{E}_{/iX}$.

Lemma 2.10.2. *For X a truncated coherent space, there exists a canonical equivalence $\mathcal{E}_{/iX} \simeq \mathcal{E}^X$.*

Proof. We prove it by descent. When X is a finite set this is true by extensionality of sums in $\mathcal{E}$ and because $i : \mathcal{S}^{<\infty}_{\mathrm{coh}} \to \mathcal{E}$ preserve finite sums. For a general X, we use a resolution $X_\bullet$ by a truncated Kan complex (Proposition 2.6.3). By Theorem 2.10.1 we have $iX = \operatorname{colim} i(X_n)$ in $\mathcal{E}$ (we shall simply write X_n for $i(X_n)$ henceforth). By the descent property of Corollary 2.7.4 we get $\mathcal{E}_{/iX} = \lim_n \mathcal{E}_{/X_n} = \lim_n \mathcal{E}^{X_n}$. By extensivity we get $\lim_n \mathcal{E}_{/X_n} = \lim_n \mathcal{E}^{X_n}$. Recall that the embedding $\mathcal{S} \subset \mathsf{Cat}$ of groupoids in categories preserves all limits and colimits (since it has both left and a right adjoint). This gives $\lim_n \mathcal{E}^{X_n} = \mathcal{E}^{\operatorname{colim} X_n} = \mathcal{E}^X$. Altogether, this provides the expected equivalence. ☐

Let $f : \mathcal{E} \to \mathcal{F}$ be a morphism of pretopoi. We denote by $i : \mathcal{S}^{<\infty}_{\mathrm{coh}} \to \mathcal{E}$ and $j : \mathcal{S}^{<\infty}_{\mathrm{coh}} \to \mathcal{E}$ the canonical morphisms of Theorem 2.10.1. Then f induces a functor between diagram categories $f^X : \mathcal{E}^X \to \mathcal{F}^X$ and a functor between slice categories $f_X : \mathcal{E}_{/iX} \to \mathcal{F}_{/jX}$ (sending $Y \to iX$ to $f(Y) \to f(iX) = jX$). We leave to the reader the proof that these two functors correspond to each other under the equivalences $\mathcal{E}^X = \mathcal{E}_{/iX}$ and $\mathcal{F}^X = \mathcal{F}_{/jX}$ of Lemma 2.10.2.

We define a Π-*pretopos* as a pretopos which is locally cartesian closed. A morphism of Π-pretopoi is a morphism of pretopoi which is also a morphism of locally cartesian closed categories.

Theorem 2.10.3. *The category $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ is the initial Π-pretopos.*

Proof. Let $\mathcal{E}$ be a Boolean $\Pi\Omega$-pretopos. Then, $\mathcal{E}$ is in particular a pretopos and we get a unique pretopos morphism $i : \mathcal{S}^{<\infty}_{\mathrm{coh}} \to \mathcal{E}$ from Theorem 2.10.1. The result will be proved if we show that i is a morphism of Boolean $\Pi\Omega$-pretopos. Since i preserves finite sums, it does preserve the subobject classifier. We are left to prove that the functor is a morphism of locally cartesian closed categories. For any X in $\mathcal{S}^{<\infty}_{\mathrm{coh}}$, we need to prove that the pretopos morphism $i_X : (\mathcal{S}^{<\infty}_{\mathrm{coh}})_{/X} \to \mathcal{E}_{/iX}$ preserves exponentials. Using Lemma 2.10.2, we can use the same descent strategy as in Theorem 2.8.6, to present i_X as a limit of morphisms of cartesian closed categories. This reduces the problem to proving that $i : \mathcal{S}^{<\infty}_{\mathrm{coh}} \to \mathcal{E}$ preserves exponentials, for which we use the same strategy as in Lemma 2.8.1. Let X and Y be two truncated coherent spaces and $X_\bullet$ a truncated Kan complexes with colimit X (Proposition 2.6.3). Then we have $Y^X = \lim_n Y^{X_n}$ in $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ and $(iY)^{iX} = \lim_n (iY)^{X_n}$ in $\mathcal{E}$. Since Y is N-truncated for some N and i is left-exact, then iY is also N-truncated and we can use Lemma 2.8.4 to reduce both cosimplicial limits to finite limits. Then we can use that $i : \mathcal{S}^{<\infty}_{\mathrm{coh}} \to \mathcal{E}$ preserves finite products and finite limits, and therefore sends $Y^X = \lim_n Y^{X_n}$ to $\lim_n (iY)^{C_n} = (iY)^{iX}$. □

Remark 2.10.4. To appreciate the strength of the initiality condition of Theorem 2.10.3, it is useful to compare it with the initial property of the topos $\mathcal{S}$. Recall that the category of spaces $\mathcal{S}$ is initial in the category of topoi and cocontinuous and left-exact functors [Lur09, Proposition 6.3.4.1]. However, $\mathcal{S}$ is no longer initial in the (non-full) subcategory of topoi and cocontinuous and left-exact functors which are also morphisms of locally cartesian closed categories. If this was true, this would imply that for any topos $\mathcal{E}$, the canonical cocontinuous and left-exact functor $i : \mathcal{S} \to \mathcal{E}$ always preserves exponentials ($i(Y^X) = (iY)^{iX}$) which is false if $\mathcal{E}$ is not locally contractible (for example, if $\mathcal{E}$ is the category of sheaves over the Cantor space). However, it is always true that the restriction $\mathcal{S}^{<\infty}_{\mathrm{coh}} \hookrightarrow \mathcal{S} \xrightarrow{i} \mathcal{E}$ does preserve exponentials. In fact, any topos $\mathcal{E}$ being a Π-pretopos, Theorem 2.10.3 says that the canonical functor $\mathcal{S}^{<\infty}_{\mathrm{coh}} \to \mathcal{E}$ is even a morphism of locally cartesian closed categories.

Definition 2.10.5. A univalent map $p : V' \to V$ in $\mathcal{S}$ is said to be *closed under quotient of Segal groupoids* if, for any X and any Segal groupoid object $Y_\bullet \to X$ in $\mathcal{S}_{/X}$ such that all maps $Y_n \to X$ are base change of p, the map $\operatorname{colim} Y_\bullet \to X$ is also a base change of p.

In $\mathcal{S}$, this condition can be tested pointwise in X. Then the following result follows from Proposition 2.7.3(1).

Corollary 2.10.6. *The universe $U^{<\infty}_{\mathrm{coh}}$ is the smallest universe of $\mathcal{S}$ containing finite sets and closed under quotients of Segal groupoids.*

2.10.3 Initial Boolean $\Pi\Omega$-pretopos

The category of finite sets is known to be the universal Boolean elementary 1-topos [Awo97, pp. 71–73]. We can deduce from Theorem 2.10.3 a similar result for $\mathcal{S}^{<\infty}_{\mathrm{coh}}$. We define a $\Pi\Omega$-*pretopos* as a Π-pretopos which admits a subobject classifier. A morphism of $\Pi\Omega$-pretopoi is a morphism of Π-pretopoi which preserves the subobject classifier. A $\Pi\Omega$-pretopos is said to be Boolean if its subobject classifier is isomorphic to $2 = 1 + 1$. The category of Boolean $\Pi\Omega$-pretopos is defined as a full subcategory of that of $\Pi\Omega$-pretopoi.

Corollary 2.10.7. *The category $\mathcal{S}^{<\infty}_{\mathrm{coh}}$ is the initial Boolean $\Pi\Omega$-pretopos.*

Proof. Let $\mathcal{E}$ be a Boolean $\Pi\Omega$-pretopos. It is sufficient to prove that the morphism $i : \mathcal{S}^{<\infty}_{\mathrm{coh}} \to \mathcal{E}$ of Π-pretopoi given by Theorem 2.10.3 is in fact a morphism of Boolean $\Pi\Omega$-pretopoi, that is that i preserves the subobject classifiers ($i(\Omega_{\mathcal{S}^{<\infty}_{\mathrm{coh}}}) = \Omega_{\mathcal{E}}$). By assumption the subobject classifier of $\mathcal{E}$ is $\Omega_{\mathcal{E}} = 2 = 1 + 1$. Using the fact that i preserves sums and Proposition 2.9.16, we get that $\Omega_{\mathcal{E}} = 2 = i(2) = i(\Omega_{\mathcal{S}^{<\infty}_{\mathrm{coh}}})$. $\square$

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