The elementary $\infty$-topos  
of truncated coherent spaces

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Abstract

This note shows that the category of truncated spaces with finite homotopy invariants (truncated coherent spaces, or truncated $\pi$-finite spaces) has many of the expected features of what should be an elementary $\infty$-topos. It should be thought as the natural higher analogue of the elementary topos of finite sets.

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1 Introduction

In 1-topos theory, the 1-category Set of (small) sets is a Grothendieck topos and the full subcategory $\text{Set}_{\text{fin}}$ of finite sets is an elementary topos. When Set is generalized into the $\infty$-category $S$ of spaces ($\infty$-groupoids), the notion of finite sets has two natural generalizations:

1. finite spaces which are homotopy types of finite CW-complexes.

2. truncated coherent spaces which are truncated homotopy types whose homotopy invariants are all finite (as a set or a group).

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Coherent spaces are often called $\pi$-finite spaces in the literature but depending on authors, the notion can demand or not the truncation of the space. We have preferred to call them coherent because they are the coherent objects of the category $\mathcal{S}$.

If $\mathcal{S}$ is the $\infty$-category of spaces, we denote by $\mathcal{S}_{\text{fin}}$ the full subcategory of finite spaces and by $\mathcal{S}_{\text{coh}}^{\infty}$ that of truncated coherent spaces. These categories are essentially disjoint: their intersection is reduced to finite sets

$$\begin{array}{ccc}
\text{Set}_{\text{fin}} & \xhookrightarrow{} & \mathcal{S}_{\text{coh}}^{\infty} \\
\downarrow & & \downarrow \\
\mathcal{S}_{\text{fin}} & \hookrightarrow & \mathcal{S}
\end{array}$$

Their stability properties are also very different (see Table 1). The category $\mathcal{S}_{\text{fin}}$ is stable by finite colimits but not by fiber products. Conversely, the category $\mathcal{S}_{\text{coh}}^{\infty}$ is stable by finite limits, finite sums, but not by pushouts. In particular, the spheres $S^n$ ($n > 0$) are finite but not truncated coherent. These properties makes $\mathcal{S}_{\text{fin}}$ into a rather awkward object from the point of view of topos theory, where fiber products are fundamental. The purpose of this note is to prove that $\mathcal{S}_{\text{coh}}^{\infty}$, on the contrary, is very well behaved and should be considered an example of an elementary $\infty$-topos.

We shall prove the following list of property of $\mathcal{S}_{\text{coh}}^{\infty}$:

(1) it is a lex category (Proposition 2.2.4),
(2) which is extensive (i.e. finite sums exist, and are universal and disjoint, see Proposition 2.2.7),
(3) and exact (i.e. quotients of Segal groupoids objects exist, and are universal and effective, see Proposition 2.7.3).
(4) $\mathcal{S}_{\text{coh}}^{\infty}$ has all truncation (Postnikov) modalities (Proposition 2.2.5),
(5) it is locally cartesian closed (Theorem 2.8.6),
(6) its universe $U$ (which lives in $\mathcal{S}$) is a countable coproduct of truncated coherent spaces (Theorem 2.9.2),
(7) $\mathcal{S}_{\text{coh}}^{\infty}$ has enough univalent maps (Corollary 2.9.5),
(8) it has a subobject classifier which is Boolean (Proposition 2.9.8).

Properties (1) to (3) make $\mathcal{S}_{\text{coh}}^{\infty}$ into an $\infty$-pretopos in the sense of [Lur17, Appendix A] where it is mentioned as an example. We shall see that

(9) $\mathcal{S}_{\text{coh}}^{\infty}$ is the initial $\infty$-pretopos (Theorem 2.10.1).

We shall also see a stronger universal property:

(10) $\mathcal{S}_{\text{coh}}^{\infty}$ is the initial “Boolean locally cartesian closed $\infty$-pretopos” (see Theorem 2.10.3 for a precise statement).

Property (4) is a consequence of the $\infty$-pretopos structure, but it can be checked by hand here. The original contribution of this note seems to be Property (5), that is $\mathcal{S}_{\text{coh}}^{\infty}$ is locally cartesian closed. The main tool to prove it is a characterization of truncated coherent spaces as realization of Kan complexes with values in finite sets (Proposition 2.6.3). The rest of the properties are easily derived from there.

Altogether, this provides the category $\mathcal{S}_{\text{coh}}^{\infty}$ with many properties of the notion of elementary $\infty$-topos proposed by Shulman [nLa21] and Rasekh [Ras18], but

(a) $\mathcal{S}_{\text{coh}}^{\infty}$ does not have all pushouts (e.g. the spheres $S^n$ for $n > 0$, see Proposition 2.3.1),
(b) and it does not (seemingly) have univalent families stable by dependent sums and/or dependent products.

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In an elementary 1-topos, the existence of finite colimits can be deduced from the existence of finite limits, exponential and the subobject classifier. Fact (a) shows that this does not generalize to higher categories. This provide a negative answer to a question of Awodey (at least in a context where there are no universe closed under $\Sigma$ and $\Pi$).

I do not have a formal proof of impossibility for Fact (b), only a couple of remarks (see Remark 2.9.7). The first one is that there cannot be a univalent family $E \to B$ stable by finite sums since this would force $\pi_0(B)$ to be infinite. Another argument is that universes closed under $\Sigma$ (or $\Sigma$ and $\Pi$) are essentially equivalent to regular (or inaccessible) cardinals and the only (non-trivial) regular or inaccessible cardinals lower than $\omega$ is $2$. This suggests that the best one can do in $S^\infty_{coh}$ is to have a classifier for types whose homotopy invariants are all in $2$, that is a subobject classifier.

Acknowledgments I thank Steve Awodey, Jonas Frey, André Joyal, Nima Rasekh, Mike Shulman, and Andrew Swan for many discussions about pretopoi, elementary topoi and comments about earlier drafts. Theorem 2.10.3 was suggested by Awodey. I learn the theory of $\infty$-pretopos in the great Appendix A of [Lur17]. Many techniques and results are taken from there. I gratefully acknowledge the support of the Air Force Office of Scientific Research through grant FA9550-20-1-0305.

Convention This note is written in the language of $\infty$-categories but we shall drop all “$\infty$-” prefixes and call higher categorical notions by their classical name (category always means $\infty$-category, topos means $\infty$-topos, colimit always means $\infty$-colimit, pullback always means $\infty$-pullback, etc.) When $n$-categories and $n$-categorical notions will be required for $n < \infty$, we shall use an explicit “$n$”-prefix.

2 The $\infty$-category of truncated coherent spaces

2.1 Definition and characterizations

A space $X$ is called coherent if $\pi_0(X)$ is a finite set and all $\pi_n(X, x)$ ($n > 0$) are finite groups, for all choice of base point. Coherent spaces are also called $\pi_n$-finite in the literature, but depending on authors, the notion can demand or not the truncation or the connectedness of the space. We have preferred to follow the terminology of Lurie and call them coherent spaces. This is justified by the fact that they are the coherent objects of the category $\mathbb{S}$ [Lur17, Example A.2.1.7]. We shall however talk about truncated coherent spaces rather than bounded coherent spaces when they are truncated. We denote by $S_{coh}$ and $S^\infty_{coh}$ the subcategory of $\mathbb{S}$ spanned by coherent and truncated coherent spaces.

In the case of spaces, the coherence condition can be understood as a higher analogue of the notion of Kuratowski finite object. We say that a space $X$ is finitely covered if there exist a map $E \to X$ where $E$ is a finite set and which is surjective on $\pi_0$. We say that a map $X \to Y$ is finitely covered if all its fiber are finitely covered. Recall that the diagonal of a map $f : X \to Y$ is the map $\Delta f : X \to X \times_Y X$. The higher diagonals are defined by $\Delta^{n+1} f = \Delta(\Delta^n f)$. When $Y = 1$ is the point, we have $\Delta^{n+1} X := \Delta^{n+1}(X \to 1) = X \to X^{S^n}$.

Proposition 2.1.1 (Kuratowski characterization). A space $X$ is coherent if all its diagonals $\Delta^{n+1} X$ are finitely covered.

Proof. The set $\pi_n(X)$ is finite if and only if $\Omega^n X$ is finitely covered. The result follows from the fact that the fibers of $\Delta^{n+1} X : X \to X^{S^n}$ are exactly the loop spaces $\Omega^{n+1} X$. \qed

Examples of truncated coherent spaces:

• any finite set (including 0 and 1);
• the realization of any finite groupoid ($G_1 \rightrightarrows G_0$ in $\text{Set}_{\text{fin}}$);
• $\mathbb{RP}^\infty = B\mathbb{Z}_2$ (= universe of sets of cardinal 2);
• $\bigsqcup_{k\leq n} B\Sigma_k$ (= universe of sets or cardinal $\leq n$);
• the classifying spaces $BG$, for $G$ a finite group;
• Eilenberg–Mac Lane spaces $K(G,n)$, for $G$ a finite group.

Examples of untruncated coherent spaces:
• $\Omega^{2k+2}S^{2k+1}$, $\Omega^{4k}S^{2k}$;
• The realization of a Kan complex with values in finite sets.

Non-examples of coherent spaces
• the spheres $S^n$ for $n \geq 1$ (since $\pi_n(S^n) = \mathbb{Z}$);
• pushouts of coherent spaces (since $\mathbb{Z}^1 = 1$);
• finite CW-complex (since $\pi_n$ will only be finitely generated groups).

2.2 Elementary properties of $S_{\text{coh}}^{<\infty}$

This section proves Properties (1), (2) and (4) of $S_{\text{coh}}^{<\infty}$.

Lemma 2.2.1. Any subspace of truncated coherent space is truncated coherent.

Proof. A subspace is determined by a subset of connected components. Hence, the $\pi_0$ is finite and so are the higher homotopy groups.

Lemma 2.2.2. The category $S_{\text{coh}}^{<\infty}$ has finite sums and the inclusion $S_{\text{coh}}^{<\infty} \subseteq S$ preserves them.

Proof. The initial object of $S$ is coherent. Let $X$ and $Y$ be two truncated coherent spaces, then the sum $X + Y$ (computed in $S$) is truncated coherent and provide a sum for $X$ and $Y$ in $S_{\text{coh}}^{<\infty}$.

The following result proves that the category $S_{\text{coh}}^{<\infty}$ is stable by fibers, extensions, and quotients (see Proposition 2.7.1).

Proposition 2.2.3. Consider a cartesian square

$$
\begin{array}{ccc}
Z & \longrightarrow & X \\
\downarrow & & \downarrow \\
1 & \longrightarrow & Y
\end{array}
$$

where $Y$ is a connected space. Then, if any two of $X$, $Y$, or $Z$ are truncated coherent, so is the third.

Proof. We chose an arbitrary base point $z$ in $Z$, we denote $x$ its image in $X$ and $y$ its image in $Y$. We consider the long exact sequence of homotopy invariants:

$$
\cdots \pi_2(Z,z) \to \pi_2(X,x) \to \pi_2(Y,y) \to \pi_1(Z,z) \to \pi_1(X,x) \to \pi_1(Y,y) \to \pi_0(Z) \to 1.
$$

We prove the result in case where $X$ and $Y$ are assumed in $S_{\text{coh}}^{<\infty}$. The map $\pi_1(Y,y) \to \pi_0(Z,z)$ is surjective, this prove that $\pi_0(Z,z)$ is finite. For $n > 0$, we get a short exact sequence $K \to \pi_n(Z,z) \to Q$ where $K$ is the kernel of $\pi_n(Z,z) \to \pi_n(X,x)$, and $Q$ is the quotient of the map $\pi_{n+1}(Y,y) \to \pi_n(Z,z)$. $K$ is a subgroup of a finite group, $Q$ is a quotient of a finite group, hence they are both finite. Then $\pi_n(Z,z)$ is finite since, as a set, it is in bijection with $K \times Q$. Since the base point of $Z$ was arbitrary, this proves that $Z$ is in $S_{\text{coh}}^{<\infty}$.

The argument is similar in the two other cases. \[\square\]
Proposition 2.2.3. The fibers of is truncated coherent.

Proof. The point is truncated coherent. The statement for binary products is direct from the formula \( \pi_n(X \times Y) = \pi_n(X) \times \pi_n(Y) \). We need only to check fiber products. Given a diagram \( X \to Y \leftarrow Y' \) in \( S_{\infty}^{\text{coh}} \), we want to prove that \( X \times_Y Y' \) is in \( S_{\infty}^{\text{coh}} \). Using Proposition 2.2.3, it is enough to prove that the fibers of the map \( X \times_Y Y' \to Y' \) are in \( S_{\infty}^{\text{coh}} \). But these fibers are fibers of the map \( X \to Y \), which are in \( S_{\infty}^{\text{coh}} \) by Proposition 2.2.3.

The \( n \)-truncation of a truncated coherent space is clearly truncated coherent. The next result proves that the results holds not only for objects but also maps.

Proposition 2.2.5 (Postnikov truncations). Let \( X \to Y \) be a morphism in \( S_{\infty}^{\text{coh}} \), and \( X \to Z \to Y \) its factorization (computed in \( S \)) into an \( n \)-connected maps followed by an \( n \)-truncated map. Then, the space \( Z \) is truncated coherent.

Proof. The fibers of \( Z \to Y \) are the \( n \)-truncations of the fibers of \( X \to Y \), and the result follows from Proposition 2.2.3.

Let \( \mathcal{C} \) be a category with finite limits and all colimits indexed by some small category \( I \). For any diagram \( X : I \to \mathcal{C} \), the pullback functor induces a functor \( P : \mathcal{C}_{/ \text{colim } X_i} \to \lim_i \mathcal{C}_{/X_i} \), and the colimit functor induces a left adjoint \( C : \lim_i \mathcal{C}_{/X_i} \to \mathcal{C}_{/ \text{colim } X_i} \). We say that an \( I \)-colimit is universal (resp. effective) if \( C \) (resp. \( P \)) is a fully faithful functor. We say that an \( I \)-colimit has descent if it is universal and effective [Lur09, 6.1.3, 6.1.8]. If \( I \) is a set, effectivity corresponds to the disjunction of sums, and descent to their extensivity. All colimits have descent in \( S \).

Lemma 2.2.6 (Descent). Let \( \mathcal{C} \subset S \) is a subcategory stable by finite limits such that the colimit of some small diagram \( X : I \to \mathcal{C} \) exist in \( \mathcal{C} \) and is preserved by \( \mathcal{C} \subset S \), then this colimit has descent in \( \mathcal{C} \).

Proof. Let \( X : I \to \mathcal{C} \) be such a diagram. By assumption, the adjunction \( \mathcal{C}_{/ \text{colim } X_i} \rightleftarrows \lim_i \mathcal{C}_{/X_i} \) is the restriction to \( \mathcal{C} \) of the adjunction \( S_{/ \text{colim } X_i} \rightleftarrows \lim S_{/X_i} \), which is an equivalence of category by descent in \( S \). Hence so is the adjunction \( \mathcal{C}_{/ \text{colim } X_i} \rightleftarrows \lim \mathcal{C}_{/X_i} \).

Proposition 2.2.7 (Extensivity). Finite sums in \( S_{\infty}^{\text{coh}} \) have descent (are disjoint and universal).

Proof. We saw that the inclusion \( S_{\infty}^{\text{coh}} \subset S \) preserve finite limits and finite sums in Proposition 2.2.4 and Lemma 2.2.2. The result follows from Lemma 2.2.6.

Proposition 2.2.8. The category \( S_{\infty}^{\text{coh}} \) is idempotent complete.

Proof. Let \( Y \) be a retract of a truncated coherent space \( X \), then \( \pi_n(Y) \) is a retract of \( \pi_n(X) \), hence finite and eventually null.

Remark 2.2.9. All the previous results are also true for the category \( S_{\text{coh}} \) of coherent spaces (with the same proofs).

Using the Postnikov towers, any coherent space can be build by successive pullbacks of maps \( 1 \to K(G,n) \) where \( G \) is a finite group and \( K(G,n) \) the associated Eilenberg–Mac Lane spaces. Proposition 2.2.5 ensure that all steps of the construction are in \( S_{\infty}^{\text{coh}} \). This gives the following result.

Proposition 2.2.10. The category of truncated coherent spaces is the smallest subcategory of \( S \) stable by finite limits and finite coproducts and containing all Eilenberg–Mac Lane spaces \( K(G,n) \) for \( G \) a finite group.
2.3 Absence of pushouts

This section proves that $S_{\text{coh}}^{<\infty}$ does not have all pushouts. We know already that the inclusion $S_{\text{coh}}^{<\infty} \subset S$ cannot preserve pushouts since $S^1 = 1 \cup S^0$ is not coherent. But that does not prevent pushout to exist in $S_{\text{coh}}^{<\infty}$. We will see that it is indeed the case by proving that the pushout $1 \leftarrow 2 \rightarrow 1$ (where $2 = 1 + 1$), classically equal to $S^1$, does not exist in $S_{\text{coh}}^{<\infty}$.

**Proposition 2.3.1.** The pushout of the diagram $1 \leftarrow 2 \rightarrow 1$ is not representable in $S_{\text{coh}}^{<\infty}$.

**Proof.** This pushout is by definition the object representing the free loop space functor

$$FL : S_{\text{coh}}^{<\infty} \longrightarrow S$$

$X \longleftarrow X \times_{X \times X} X = X S^1$

Let $H$ be a truncated coherent space representing $FL$. Then, for any coherent space $X$, we have a natural equivalence $\text{Hom}(H, X) = \text{Hom}(S^1, X)$. This is equivalent to the data of a map $S^1 \rightarrow H$ presenting $H$ as the reflection of $S^1$ in the subcategory $S_{\text{coh}}^{<\infty} \subset S$ (but we shall need that). First we can deduce that $H$ has to be connected. Indeed, for the discret space $X = 2$, we have

$$\text{Hom}(H, 2) = 2^{\pi_0(H)} \quad \text{and} \quad \text{Hom}(S^1, 2) = 2$$

hence $\pi_0(H) = 1$.

Recall that for $(X, x)$ and $(Y, y)$ two pointed spaces, the space of pointed morphisms is defined by the fiber product

$$\text{Hom}_*(((Y, y), (X, x))) \longrightarrow \text{Hom}(Y, X)$$

$$\begin{array}{ccc}
1 & \longleftarrow & X \\
\downarrow & & \downarrow \\text{Hom}(y, X) \\
r & & x
\end{array}$$

We fix a base point $s$ in $S^1$, and consider its image $h$ by the map $S^1 \rightarrow H$. The map $S^1 \rightarrow H$ induces an equivalence

$$\text{Hom}_*((H, h), (X, x)) = \text{Hom}_*((S^1, s), (X, x)) = \Omega_x X$$

(1)

The space $H$ being connected its 1-truncation is a space $BG$ for some finite group $G$. We consider the additive group $\mathbb{Z}/p\mathbb{Z}$ for $p$ a prime number prime to the order of $G$. Then, the only group morphism $G \rightarrow \mathbb{Z}/p\mathbb{Z}$ is the constant one. We put $X = B\mathbb{Z}/p\mathbb{Z}$. Using the equivalence between pointed connected 1-type and discrete groups, we get

$$\text{Hom}_*((H, h), (X, x)) = \text{Hom}_*((BG, h), (B\mathbb{Z}/p\mathbb{Z}, x)) = \text{Hom}_{Gp}(G, \mathbb{Z}/p\mathbb{Z}) = 1.$$ 

But, on the other side, we have

$$\Omega_x X = \mathbb{Z}/p\mathbb{Z} \neq 1$$

This contradicts (1) and shows that $H$ cannot exist. 

**Remark 2.3.2.** A more sophisticated version of the argument proves that $S^n$ does not admit a reflection into $S_{\text{coh}}^{<\infty}$. It is likely that any connected finite space does not either.

2.4 Comparison of coherent and finite spaces

We recall without proof some properties of finite spaces to compare them with coherent ones.

A space is finite if it is the homotopy type of a finite CW-complex, or, equivalently, the realization of a simplicial set with only a finite number of non-degenerate simplices. More intrinsically, the category of finite spaces space can be defined as the smallest subcategory of $S$ containing 0 and 1 (or the whole of $\text{Set}_{\text{fin}}$) and...
stable by pushouts. (We shall see in Proposition 2.7.3, a similar characterization of coherent spaces.) All spheres $S^n (n \geq -1)$ are finite, and any finite space can be built with a finite chain of cell attachments

$$
\begin{array}{ccc}
S^n & \to & X_n \\
\downarrow & & \downarrow \\
1 & \to & X_{n+1}
\end{array}
$$

This is to be contrasted with Proposition 2.2.10. Any subspace of a finite space is finite. Any finite sums or finite product of finite spaces is finite. But $S_{\text{fin}} \subset S$ is not stable by finite limits since $\Omega S^1 \approx \mathbb{Z}$ is not finite. It is also not stable by retracts [Lur09, Remark 5.4.1.6]. Table 1 summarizes the comparison between finite and truncated coherent spaces. A funny fact is that, $S_{\text{coh}}^{\infty}$ being stable by finite limits, it is cotensored over $S_{\text{fin}}$:

**Lemma 2.4.1.** The Hom functor $S^{op} \times S \to S$ restricts into a functor $S_{\text{fin}}^{op} \times S_{\text{coh}}^{\infty} \to S_{\text{coh}}^{\infty}$.

**Proof.** Let $K$ be a finite space and $X$ be a truncated coherent space, then $X^K$ is a finite limit of copies of $X$, hence in $S_{\text{coh}}^{\infty}$ by Proposition 2.2.4.

**Remark 2.4.2** (Generalization to higher cardinals). The notions of finite and coherent spaces (and more generally that of compact and coherent object in a topos) rely implicitly on the notion of finite sets, that is $\omega$-small sets. It can therefore be generalized by replacing $\omega$ with a non countable larger regular ordinal $\kappa$. If we do so, then the notion of $\kappa$-small and $\kappa$-coherent spaces do coincide. Only for $\omega$ are the two notions different. An explanation is the following: the completion of a simplicial set with a values in finite sets (a fortiori having a finite set of non-degenerate simplices) into a Kan complex has values in countable sets, but for $\kappa > \omega$, Kan completions of complexes with values in $\kappa$-small sets stay with values in $\kappa$-small sets.

**Remark 2.4.3** (Closure of $S_{\text{coh}}^{\infty}$ for pushouts). A natural question is to identify the smallest category containing $S_{\text{coh}}^{\infty}$ and $S_{\text{fin}}$ which is stable by finite limits and colimits. It can be proved that any space in this category will be the realization of a countable simplicial set. In particular, countable sets will be a part of it and therefore this category cannot be cartesian closed (nor even have countable sums, actually). This category has been studied by Berman [Ber20].

### 2.5 Simplicial spaces

This section introduces some definitions and constructions needed in the following sections.

A map $f : X \to Y$ in $S$ is called surjective if the map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is surjective in $\text{Set}$. An object $E$ of $S$ is called projective if for any surjective map $X \to Y$, the map $X^E \to Y^E$ is surjective. Surjective maps being stable by base change, $E$ is projective if and only if any surjective map $X \to E$ splits. Any object $Y$ of $S$ admits a surjective map $X \to Y$ from a set$^1$. Covering a projective object $E$ by a set, we get that it is a set. Reciprocally, any set is projective. We shall say that a map $f : X \to Y$ in $S$ is projective if it is of the type $X \to X + E$ where $E$ is a set.

Two maps $u : A \to B$ and $f : X \to Y$ of an arbitrary category $\mathcal{C}$ are said to be weakly orthogonal if the map $(u,f) : \text{Hom}(B,Y) \to \text{Hom}(A,X) \times \text{Hom}(A,Y)$ is surjective in $S$. The two classes of projective and surjective maps are weakly orthogonal to each other and form a weak factorization system on $S$. The factorization of $f : X \to Y$ is given by $X \to X + E \to Y$ for $E \to Y$ any surjective map from a set. This factorization cannot be made functorial.

Let $\Delta$ be the category of simplices. We shall denote the colimit functor (also called realization) $S^{\Delta^{op}} \to S$ by $X \mapsto |X|$ and its right adjoint (constant diagram) by $X \mapsto X$. The latter functor is fully faithful, and

$^1$In the model of $S$ with topological spaces, $E$ can be the set of points of $Y$; in the model with simplicial sets, $E$ can be the set of $0$-simplices.
Table 1: Comparison between finite and truncated coherent spaces.

<table>
<thead>
<tr>
<th></th>
<th>$\mathcal{S}_{\text{fin}}$</th>
<th>$\mathcal{S}_{\text{coh}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>finite + and $\times$</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>subspaces</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>pushouts</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>fiber products</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>loop spaces $\Omega$</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>truncations</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>retracts</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>building blocks</td>
<td>$S^n \rightarrow 1$</td>
<td>$1 \rightarrow K(G,n)$</td>
</tr>
<tr>
<td>compactness properties</td>
<td>finite spaces are compact in $\mathcal{S}$</td>
<td>$n$-truncated coherent spaces are compact in $S^{\leq n}$ (but not in $\mathcal{S}$)</td>
</tr>
<tr>
<td>Euler characteristic</td>
<td>in $\mathbb{Z} = \mathbb{N}[-1]$</td>
<td>in $\mathbb{Q}_{&gt;0} = \mathbb{N}[\frac{1}{2}, \frac{1}{3}, \ldots]$ [Bae03, BD00]</td>
</tr>
<tr>
<td>Other properties</td>
<td>—</td>
<td>ambidexterity [Har20, HL13]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>cotensored over $\mathcal{S}_{\text{fin}}$ (??)</td>
</tr>
</tbody>
</table>
|−| presents S as a reflective subcategory of $S^{\Delta^{op}}$. We denote W the class of maps of $S^{\Delta^{op}}$ send to invertible maps by the colimit functor. We call them colimit equivalences. For an object $X$ in S, a resolution of $X$ is defined as a simplicial diagram $X_\bullet$ equipped with a colimit cocone with apex $X$. Such a cocone is equivalent to a map $X_\bullet \to \underline{X}$ which is in W.

The projective-surjective weak factorization system induces a Reedy weak factorization system on $S^{\Delta^{op}}$ (see [MG14, Section 4]). Maps in the right class are sometimes called hypercoverings or trivial fibrations, we shall call them hypersurjective. Their class is defined as the (weak) right orthogonal to the maps $\partial \Delta[n] \to \Delta[n]$ $(n \geq 0)$. The following lemma is a crucial property of hypersurjective maps.

**Lemma 2.5.1** ([Lur09, Lemma 6.5.3.11]). All hypersurjective maps are colimit equivalences.

The maps in the left class are sometimes called cofibrations, we shall call them hyperprojective. A map $X_\bullet \to Y_\bullet$ is hyperprojective if and only if all relative latching maps $X_n \coprod_{L_n X_\bullet} L_n Y_\bullet \to Y_n$ are projective. Intuitively, this means that $Y_\bullet$ is build from $X_\bullet$ by adding a set (rather than an arbitrary space) of non-degenerate simplices in each dimension. In particular, a map $0 \to X_\bullet$ is hyperprojective if and only if $X_\bullet$ is a simplicial set. Thus, the hyperprojective–hypersurjective factorization of $\emptyset \to X_\bullet$ always goes through a simplicial set. More generally, a map $X_\bullet \to Y_\bullet$ where $X_\bullet$ is a simplicial set is hyperprojective if and only if it is a monomorphism of simplicial sets. Let $\underline{X}$ be a constant simplicial object. Since sets are the projective object of S, we shall say that a factorization of $\emptyset \to X_\bullet \to \underline{X}$ is a projective resolution of $X$.

We recall some results on how to construct projective resolutions. We shall need this to prove Proposition 2.6.3.

**Lemma 2.5.2** (Reedy induction [Lur09, Corollary A.2.9.15 and Remark A.2.9.16]). Let $\mathcal{C}$ be a category with finite limits and colimits. The extension of a functor $X: \Delta^{op} \to \mathcal{C}$ into a functor $X': \Delta^{op} \to \mathcal{C}$ is equivalent to the data of a factorization of the map $L_n X \to M_n X$ (where $L_n X$ and $M_n X$ are the latching and matching objects of $X: \Delta^{op} \to \mathcal{C}$).

We can apply this to $\mathcal{C} = S$ with the projective–surjective factorization.

**Lemma 2.5.3** (Projective resolution [Lur11, Corollary 1.4.11]). Let $X$ be an object in S. There exists a simplicial object $\Delta \to S_{/X}$ such that, for every $n$, the map $L_n X \to X_n$ is a coproduct with some set, and the map $X_n \to M_n X$ is surjective.

The simplicial object in $\mathcal{C}_{/X}$ of Lemma 2.5.3 provide a map $X_\bullet \to \underline{X}$ which, by construction, is hypersurjective.

### 2.6 Kan groupoids

This section proves Proposition 2.6.3, which is going to be our main tool to prove that $S^{\Delta^{op}}_{\text{col}}$ is locally cartesian closed (Theorem 2.8.6).

We say that a simplicial space $Y_\bullet$ is a Kan groupoid if it is (weakly) right orthogonal to all horn inclusions $\Lambda^n \to \Delta^n$. This means that all maps of spaces $Y_n \to \text{Hom}(\Lambda^n, Y_\bullet)$ are surjective. A simplicial set is a Kan groupoid if and only if it is a Kan complex. We shall keep the name Kan complex for a Kan groupoid whose values are sets.

**Lemma 2.6.1** (Kan resolution). Given a hypersurjective map $X_\bullet \to \underline{X}$, the simplicial set $X_\bullet$ is a Kan complex.

**Proof.** Let $|Y_\bullet|$ the colimit of $Y_\bullet$. By adjunction, we have $\text{Hom}(Y_\bullet, \underline{X}) = \text{Hom}(|Y_\bullet|, X)$. Using that $|\Lambda^n| \to |\Delta^n|$ is an equivalence in S, we get that $X$ is a Kan groupoid.

Hence, the result will be proved if the map $c: X_\bullet \to \underline{X}$ is (weakly) right orthogonal to all horn inclusions. By definition of a hypersurjective, the map $c$ is right orthogonal to all maps $\partial \Delta^n \to \Delta^n$. Since the maps $\partial \Delta^n \to \Delta^n$ generate all monomorphisms of simplicial sets by iterated pushouts in $S^{\Delta^{op}}$, the map $c$ is also right orthogonal to all these maps. This includes all horn inclusions. □
Remark 2.6.2. Following [MG14, Section 4], it is convenient to introduce a second weak factorization system on $S^\Delta^{op}$, generated by the horn inclusions. The maps in the right class are called Kan fibrations. A simplicial space $X_*$ is a Kan groupoid if and only if $X_* \to 1$ is a Kan fibration. Hence, if $X_* \to Y_*$ is a Kan fibration and $Y_*$ is a Kan groupoid, then so is $X_*$. The proof of Lemma 2.6.1 follows from the fact that any hypersurjective maps is a Kan fibration.

A simplicial space is $n$-coskeletal if it is the right Kan extension of its restriction to $\Delta_{n}\subset \Delta$. This is equivalent to the condition that the maps $X_k \to \text{Hom}(sk_n \Delta^k, X_{<n})$ be all equivalence for $k > n$. We say that a Kan groupoid (in S) is truncated if its colimit is $n$-truncated for some $n$. We say that a Kan complex is has finite values if is in $(\text{Set}_{\mathbb{N}})^{\Delta^{op}} \subset \text{Set}^{\Delta^{op}}$.

Proposition 2.6.3. A space is coherent if and only if it is the geometric realization of a Kan complex with finite values.

Proof. Let $X_*$ be a Kan complex. Recall that $\pi_0(\cdot|X_*\cdot)$ is a quotient of $X_0$ and and $\pi_n(\cdot|X_*\cdot, x)$ is a subquotient of $X_n$. Hence they are all finite if the $X_n$ are. This proves that the conditions are sufficient.

To see that they are necessary, we use Reedy induction. Let $X$ be a coherent space, we use Lemma 2.5.2 in $S/X$ to construct a simplicial object $\Delta \to S/X$. First, we chose $X_0 \to X$ a surjection from a set $X_0$. Because $X$ is coherent, $X_0$ can be chosen finite. At step 1, $L_1(X_{<0}) = X_0$ and $M_0(X_{<0}) = X_0 \times X_0$. The space $X_0 \times X_0$ is a finite coproduct of path spaces of $X$. Since $X$ is coherent, it has a finite number of connected components and we can put $X_1 := X_0 + X_1'$ where $X_1'$ is a finite set. At step $n$, let $\text{Hom}(\partial \Delta^n, X_{<n})$ be the set of maps in $\text{Set}_{\mathbb{N}}^{\Delta^{op}}$. Since all $X_k$ are finite sets, this is a finite set. Then we have $M_n(X_{<n}) = \text{Hom}(\partial \Delta^n, X_{<n}) \times X_{<\partial \Delta^n}$. And $M_n(X_{<n})$ is a coproduct of $n$-fold path spaces of $X$. Since $X$ is coherent, $M_n(X_{<n})$ has a finite number of connected components and we can put $X_n := L_n(X_{<n}) + X_n'$ where $X_n'$ is a finite set. By induction, $L_n(X_{<n})$ is a finite set, hence so is $X_n$. The resulting simplicial set $X_*$ has finite values. We get a map $X_* \to X$ in $S^{\Delta^{op}}$ which is a hypersurjective, hence a colimit cone by Lemma 2.5.1. The fact that it is Kan is Lemma 2.6.1.

2.7 Segal groupoids

This section proves Property (3). We prove in fact a stronger result presenting $S_{\text{coh}}^{<\infty}$ as the closure of finite set under Segal groupoid (Proposition 2.7.3).

Let $X_*$ be a simplicial space and $|X_*|$ its colimit. We shall also call $|X_*|$ the quotient of $X_*$ and refer to the canonical map $q : X_0 \to |X_*|$ as the quotient map of $X_*$. An object $X_*$ in $S^{\Delta^{op}}$ is Segal groupoid if it satisfies the Segal conditions: $X_n = X_1 \times X_0 \cdots \times X_0 X_1$ ($n > 1$). Let $f : X \to Y$ be a map in S and $X_*$ be its nerve $N(f)_*$. Then, $X_*$ is a Segal groupoid. Intuitively, $N(f)_*$ is the groupoid encoding the equivalence relation “to have same image by $f$". A Segal groupoid is effective if the canonical map $X_* \to N(q)_*$ is invertible in $S^{\Delta^{op}}$ (where $q$ is the quotient map). In $S$, all Segal groupoids are effective, this is part of the Giraud axioms of $\infty$-topoi [Lur09, Proposition 6.1.3.19]. Moreover, the functor sending a Segal groupoid to its quotient map $q : X_0 \to |X_*|$ induces an equivalence between the full subcategory of $S^{\Delta^{op}}$ spanned by Segal groupoids and the full subcategory of the arrow category of $S$ spanned by surjective maps (the inverse equivalence being given by the nerve).

Proposition 2.7.1. Let $X_*$ be a Segal groupoid in $S_{\text{coh}}^{<\infty}$ then its quotient $|X_*|$ is in $S_{\text{coh}}^{<\infty}$.

Proof. The quotient map $X_0 \to |X_*|$ is surjective. Hence we can restrict to the case where $|X_*|$ is connected. By effectivity of Segal groupoids, we have a cartesian square

$$
\begin{array}{ccc}
X_1 & \longrightarrow & X_0 \\
\downarrow & & \downarrow \\
X_0 & \longrightarrow & |X_*|
\end{array}
$$
Let $x$ be an element in $X_0$. The fiber of $X_1 \to X_0$ at $x$ is a coherent space $Z$ by Proposition 2.2.3. Hence we can apply Proposition 2.2.3 again to the cartesian square

$$
\begin{array}{ccc}
Z & \longrightarrow & X_0 \\
\downarrow & & \downarrow \\
1 & \longrightarrow & |X_*|
\end{array}
$$

to deduce that $|X_*|$ is truncated coherent.

Lurie proves a similar result for $\mathcal{S}_{\text{coh}}$ and Kan groupoids. We mention it for a

**Proposition 2.7.2** ([Lur17, Theorem A.5.5.1]). Let $X_*$ be a Kan groupoid in $\mathcal{S}_{\text{coh}}$, then its quotient $|X_*|$ is in $\mathcal{S}_{\text{coh}}$.

The following result gives meaning to the category $\mathcal{S}_{\text{coh}}^{\leq n}$ and $\mathcal{S}_{\text{coh}}$ inside $\mathcal{S}$. We’ll use in Theorem 2.10.1 to prove $\mathcal{S}_{\text{coh}}^{\leq \infty}$ is the initial $\infty$-pretopos.

**Proposition 2.7.3** (Exact completions). 1. The category $\mathcal{S}_{\text{coh}}^{\leq n}$ is the smallest category of $\mathcal{S}$ containing $\mathcal{S}_{\text{fin}}$ and closed under quotients of Segal groupoids.

2. The category $\mathcal{S}_{\text{coh}}$ is the smallest category of $\mathcal{S}$ containing $\mathcal{S}_{\text{fin}}$ and closed under quotients of Kan groupoids.

**Proof.** (1) Let $\mathcal{C} \subset \mathcal{S}$ be the smallest full subcategory containing $\mathcal{S}_{\text{fin}}$ and closed under quotients of Segal groupoids. Since $\mathcal{S}_{\text{fin}} \subset \mathcal{S}_{\text{coh}}^{\leq n}$, Proposition 2.7.1 proves that $\mathcal{C} \subset \mathcal{S}_{\text{coh}}^{\leq n}$. Conversely, proceed by induction on the truncation level. Let $\mathcal{S}_{\text{coh}}^{\leq n} \subset \mathcal{S}_{\text{coh}}^{\leq n}$ be the full subcategory spanned by $n$-truncated objects. We have $\mathcal{S}_{\text{coh}}^{\leq 0} = \mathcal{S}_{\text{fin}}$. Let us prove that any object $X$ of $\mathcal{S}_{\text{coh}}^{\leq n+1}$ can be obtained as the quotient of a Segal groupoid in $\mathcal{S}_{\text{coh}}^{\leq n}$. Let $f : X_0 \to X$ be a surjective map where $X_0$ is a set, then $X_0$ is in $\mathcal{S}_{\text{coh}}^{\leq n}$. We consider the nerve $X_*$ of $f$. It is a Segal groupoid whose quotient is $X$. The result will be proved if we show that $X_*$ is a simplicial object in $\mathcal{S}_{\text{coh}}^{\leq n}$. The space $X_1 = X_0 \times_X X_0$ is a coproduct of loop spaces of $X$, hence coherent and $n$-truncated. More generally, we have $X_n = X_1 \times_{X_0} \cdots \times_{X_0} X_1$, and this shows $X_n$ is in $\mathcal{S}_{\text{coh}}^{\leq n}$.

(2) By Proposition 2.6.3, $\mathcal{S}_{\text{coh}}$ is included in the smallest full subcategory containing $\mathcal{S}_{\text{fin}}$ and closed under quotients of Kan groupoids. The converse is given by Proposition 2.7.2.

**Corollary 2.7.4** (Descent properties). 1. Quotients of Kan groupoids have descent in $\mathcal{S}_{\text{coh}}$.

2. Quotients of truncated Kan groupoids have descent in $\mathcal{S}_{\text{coh}}^{\leq n}$.

3. Segal groupoids have descent in $\mathcal{S}_{\text{coh}}^{\leq \infty}$.

4. Segal groupoids are universal and effective in $\mathcal{S}_{\text{coh}}^{\leq \infty}$.

**Proof.** The properties (1), (2), and (3) are consequences of Lemma 2.2.6. We are left to prove (4). The universality of Segal groupoids is a consequence of (3), and the effectivity is a consequence of $\mathcal{S}_{\text{coh}}^{\leq \infty} \subset \mathcal{S}$ preserving finite limits (Proposition 2.2.4).

**Remark 2.7.5.** Putting together Propositions 2.2.4 and 2.2.7, Remark 2.2.9, and Corollary 2.7.4, we get that $\mathcal{S}_{\text{coh}}$ and $\mathcal{S}_{\text{coh}}^{\leq \infty}$ are $\infty$-pretopoi in the sense of [Lur17, Definition A.6.1.1]. In fact, we shall see in Theorem 2.10.1 that $\mathcal{S}_{\text{coh}}^{\leq \infty}$ is the initial pretopos. Since $\mathcal{S}_{\text{fin}}$ is the initial 1-pretopos, $\mathcal{S}_{\text{coh}}^{\leq \infty}$ is then the $\infty$-pretopos envelope of $\mathcal{S}_{\text{fin}}$, and can be thought as its higher exact completion.
2.8 Local cartesian closure

This section proves Property (5) (Theorem 2.8.6). We prove first that $S_{\text{coh}}^{\infty}$ is cartesian closed and deduce the statement for the slice categories by a descent argument.

**Lemma 2.8.1.** The category $S_{\text{coh}}^{\infty}$ is cartesian closed, and the embedding $S_{\text{coh}}^{\infty} \subset S$ preserves the exponentials.

**Proof.** We have seen that $S_{\text{coh}}^{\infty} \subset S$ is stable by finite products (Proposition 2.2.4). We are going to show that for any two spaces $X$ and $Y$ in $S_{\text{coh}}^{\infty}$, the space $Y^X$ is in $S_{\text{coh}}^{\infty}$. When $X$ is a finite set, this is true because $Y^X$ is a finite product of $Y$. For a general $X$, we use Proposition 2.6.3 to present $X$ as the colimit of a simplicial finite set and get

$$Y^X = \lim_{m} Y^{X_m}.$$ 

This limit is a priori infinite and $S_{\text{coh}}^{\infty}$ is only stable by finite limits. By assumption $Y$ is $k$-truncated for some $k$, then so are all the $Y^{X_m}$. Thus, we can use that the inclusion $\Delta_{\leq k+1} \subset \Delta$ is coinitial for diagrams of $k$-truncated objects (see Lemma 2.8.4 below), and replace the limit by an equivalent one which is finite. □

We say that a functor $f : C \to D$ between $n$-categories is $n$-cofinal if for any cocomplete $n$-category $\mathcal{C}$, the colimit of any diagram $X : D \to C$ coincides with the colimit of $X \circ f : C \to \mathcal{C}$.

**Lemma 2.8.2.** The following conditions are equivalent:

1. the functor $f : C \to D$ is $n$-cofinal;
2. the functor $\mathcal{P}_n(f) : \mathcal{P}_n(C) \to \mathcal{P}_n(D)$ preserves the terminal object;
3. for any $d$ in $D$, the realization of the category $C_{dl} := C \times_D D_{dl}$ is an $(n-1)$-connected space.

**Proof.** (2) $\iff$ (1). Let $C$ a small $n$-category, its free cocompletion as an $n$-category is $\mathcal{P}_n(C) := [C^{op}, S_{\text{coh}}^{\infty}]$. The colimit of the Yoneda embedding $C \to \mathcal{P}_n(C)$ is the terminal object. If $\mathcal{C}$ is a cocomplete $n$-category the colimit of a diagram $C \to \mathcal{C}$ is the image by $\mathcal{P}_n(C) \to \mathcal{C}$ of the terminal object. This proves that (1) $\Rightarrow$ (2). Reciprocally, given a diagram $D \to \mathcal{C}$, the commutative diagram

$$
\begin{array}{ccc}
C & \longrightarrow & \mathcal{P}_n(C) \\
\downarrow f & & \downarrow \mathcal{P}_n(f) \\
D & \longrightarrow & \mathcal{P}_n(D) \\
& \nearrow & \\
& & \mathcal{C}
\end{array}
$$

(where the dashed arrows are cocontinuous) proves that (2) $\Rightarrow$ (1).

(2) $\iff$ (3). We have $\mathcal{P}_n(f)(1) = 1$ if and only if, for any $d$ in $D$, the space $\text{Hom}(d, \mathcal{P}_n(f)(1))$ is contractible. But we have (in $S_{\leq n-1}$)

$$\text{Hom}(d, \mathcal{P}_n(f)(1)) = \text{Hom}(\tilde{d}, \underset{\mathcal{C}}{\text{colim}} f(c)) = \underset{\mathcal{C}}{\text{colim}} \text{Hom}(d, f(c)) = \underset{C_{dl}}{\text{colim}} 1 = |C_{dl}|_{\leq n-1}$$

where $|C_{dl}|_{\leq n-1}$ is the $(n-1)$-truncation of the realization of $C_{dl}$. This space is contractible if and only if the realization of $C_{dl}$ is $(n-1)$-connected. This proves (2) $\iff$ (3). □

**Lemma 2.8.3.** The inclusion $\Delta_{\leq n} \to \Delta$ is coinitial for diagrams in $n$-categories.

**Proof.** If $k \leq n$, $(\Delta_{\leq n})_k$ has a terminal object and is weakly contractible. If $k > n$, the realization of $(\Delta_{\leq n})_k$ is $sk_n(\Delta[k])$ which is a bouquet of $n$-spheres, hence $(n-1)$-connected. The result follows from Lemma 2.8.2. □

Recall that an object $X$ in category $\mathcal{C}$ is called $n$-truncated if the functor $\text{Hom}(-,X) : \mathcal{C}^{op} \to S$ takes values in $n$-truncated spaces.
Lemma 2.8.4. The inclusion $\Delta_{\leq n} \to \Delta$ is coinitial for diagrams of $(n-1)$-truncated objects.

Proof. Lemma 2.8.3 Let $\mathcal{E}$ be a category and $\mathcal{E}^{\leq n-1} \subseteq \mathcal{E}$ be the full subcategory of $n$-truncated objects. Let us see that $D : I \to \mathcal{E}^{\leq n-1}$ be a diagram having a limit in $\mathcal{E}$, then its limit is in $\mathcal{E}^{\leq n}$. The result is true in $\mathcal{S}$ because the subcategory $\mathcal{S}^{\leq n-1} \subseteq \mathcal{S}$ of $n$-truncated spaces is reflective, hence stable by arbitrary limits. For a general $\mathcal{E}$, the limit of $D$ is the object representing the functor

$$\mathcal{E}^{op} \longrightarrow \mathcal{S}$$

$$X \longmapsto \lim_{i} \text{Hom}(X, D_i)$$

If all the $D_i$ are $(n-1)$-truncated, this functor takes values in $(n-1)$-truncated spaces, so any representative will be an $(n-1)$-truncated object. This reduces the problem to prove the coinitiality of $\Delta_{\leq n} \to \Delta$ to diagrams in the $n$-category $\mathcal{E}^{\leq n-1}$, but then it follows from Lemma 2.8.3.

Lemma 2.8.5. The limit of a diagram of cartesian closed categories and cartesian closed functors is cartesian closed.

Proof. Several arguments can be given: the most conceptual is that the category of cartesian closed categories is straightforward for products, so we need only give an argument for fiber products. Let $\mathcal{E}_1 \overset{p}{\to} \mathcal{E}_0 \overset{q}{\leftarrow} \mathcal{E}_2$ be a diagram of cartesian closed categories and cartesian closed functors. Objects in the limits are families $X = (X_1, X_0, X_2, x_1 : p(X_1) \cong X_0, x_2 : q(X_2) \cong X_0)$. We leave the reader to check that the internal hom between two such families $X$ and $Y$ are computed termwise as

$$\big(X_{Y_1}, X_{Y_0}, X_{Y_2}, x_{Y_1}^1 : p(X_{Y_1}) \cong X_{Y_0}, x_{Y_2}^2 : q(X_{Y_2}) \cong X_{Y_0}\big).$$

Theorem 2.8.6. The category $\mathcal{S}_{\text{coh}}^{\leq \infty}$ is locally cartesian closed.

Proof. We need to prove that for any $X$ in $\mathcal{S}_{\text{coh}}^{\leq \infty}$, the category $(\mathcal{S}_{\text{coh}}^{\leq \infty})/X$ is cartesian closed. If $X$ is a finite set, then $(\mathcal{S}_{\text{coh}}^{\leq \infty})/X = (\mathcal{S}_{\text{coh}}^{\leq \infty})^X$ is cartesian closed as a product of cartesian closed categories. For a general $X$, we use Proposition 2.6.3 to get a Kan complex with finite values $X_\bullet$ with colimit $X$. Corollary 2.7.4 gives that $(\mathcal{S}_{\text{coh}}^{\leq \infty})/X = \lim_{\Delta} (\mathcal{S}_{\text{coh}}^{\leq \infty})/(X_n)$ and the result follow from Lemma 2.8.5.

Remark 2.8.7. This result is not true for the category $\mathcal{S}_{\text{coh}}$. Let $X = \prod_n K(\mathbb{Z}_2, n)$. Any sequence of groups morphisms $\phi_n : \mathbb{Z}_2 \to \mathbb{Z}_2$ defines an endomorphisms $\phi := \prod_n K(\phi_n, n)$ of $X$. Acting differently on the $\pi_n$, these $\phi$ are non-homotopic in $X^X$. Any group morphism $\mathbb{Z}_2 \to \mathbb{Z}_2$ is either the identity or constant. Hence, the set of such sequences is $2^{\mathbb{N}}$. This proves $\pi_0(X^X)$ is not finite (it’s not even countable).

2.9 The universe of truncated coherent spaces

This section proves Properties (6) to (8). We do so by constructing a universe in $\mathcal{S}$ for truncated coherent spaces (Theorem 2.9.2).

Let $\mathcal{S}^{\geq}$ be the arrow category of $\mathcal{S}$. Consider the codomain cartesian fibration $\text{cod} : \mathcal{S}^{\geq} \to \mathcal{S}$. Its fibers are the slices categories $\mathcal{S}_{/X}$. We denote by $\mathcal{S}^{\geq}_{\text{cart}} \subseteq \mathcal{S}^{\geq}$ the subcategory with the same objects but only cartesian morphisms. The restriction $\text{cod} : \mathcal{S}^{\geq}_{\text{cart}} \to \mathcal{S}$ is still a fibration, whose fibers at $X$ is the interior groupoid (maximal subgroupoid) $\mathcal{S}^{\text{int}}_{/X}$ of $\mathcal{S}_{/X}$. Let $\mathcal{S}$ be the category of spaces in a larger universe. We shall implicitly embed $\mathcal{S}$ in $\widetilde{\mathcal{S}}$. This fibration has a classifying functor

$$U : \mathcal{S}^{\geq} \longrightarrow \mathcal{S}^{\geq}_{/X}.$$
For a space $X$, let $Aut(X) \subset X^X$ be its group of automorphisms. This group acts on $X$ and we shall denote the quotient by $X/Aut(X)$. It also acts trivially on the point $1$ and the quotient $1/Aut(X)$ is the gerbe $BAut(X)$ classifying spaces isomorphic to $X$. Precisely, the space of maps $Z \to BAut(X)$ is equivalent to the full subgroupoid of $S^\int_{\sslash Z}$ spanned by $X$-bundles (maps $Z' \to Z$ whose fiber are all isomorphic to $X$). The canonical map $X \to 1$ induces a map $X/Aut(X) \to BAut(X)$ which is the universal $X$-bundle. For any $X$-bundle $Z' \to Z$, there exists a unique cartesian square

$$
\begin{array}{ccc}
Z' & \longrightarrow & X/Aut(X) \\
\downarrow & & \downarrow \\
Z & \longrightarrow & BAut(X)
\end{array}
$$

Let $S$ be a set (in $\overline{S}$) of representative for each isomorphism class of objects in $S$. We define

$$U_S := \bigsqcup_{X \in S} BAut(X) \quad \text{and} \quad U'_S := \bigsqcup_{X \in S} X/Aut(X).$$

Then, the functor $U$ is representable (in $\overline{S}$) by the space $U_S$, and the map $U'_S \to U_S$ is the corresponding universal family.

Notice that for any small set $S' \subset S$, the object $U_{S'} = \bigsqcup_{X \in S'} BAut(X)$ is in $\overline{S}$. For a space $X$, let $S'(X) \subset S^\int_{\sslash X}$ be the full subcategory of maps $X' \to X$ whose fibers are isomorphic to some element of $S'$. Then, the space $U_{S'}$ represents the functor $X \mapsto S'(X)$. When $S' = S^\infty_{coh} \subset S$ is the subset of elements of $X$ that are truncated coherent, we denote $U_{S'}$ and $U'_{S'}$ by $U'_{coh}$ and $(U'_{coh})'$. The next result proves that $S^\infty_{coh}$ is a countable set.

**Lemma 2.9.1.** The set of isomorphism classes of objects of $S^\infty_{coh}$ is countable.

**Proof.** The set of diagrams $\Delta_{\infty} \to \Set_{\fin}$ is countable, and a fortiori the subset $Kan_n$ of Kan complexes. Then Proposition 2.6.3 proves that the set $\bigsqcup_n Kan_n$ has a surjective map to the set of isomorphism classes of objects of $S^\infty_{coh}$. \qed

We say that a map $X \to Y$ in $S$ is truncated coherent if all its fibers are truncated coherent (or, equivalently, isomorphic to some element in $S^\infty_{coh}$). Let $S^\infty_{coh}(X) \subset S^\int_{\sslash X}$ the full subgroupoid spanned by truncated coherent maps.

**Theorem 2.9.2.** The space $U'_{coh}$ represents the functor $X \mapsto S^\infty_{coh}(X)$. For any truncated coherent map $X \to Y$, there exists a unique cartesian square

$$
\begin{array}{ccc}
X & \longrightarrow & (U'_{coh})' \\
\downarrow & & \downarrow \\
Y & \longrightarrow & U'_{coh}
\end{array}
$$

Moreover, $U'_{coh}$ and $(U'_{coh})'$ are countable coproducts of coherent spaces.

**Proof.** The first part is a consequence of the remarks preceding the theorem. We need only to prove the last assertion. For a coherent space $X$, we’ve seen in Lemma 2.8.1 that $X^X$ is coherent. Hence, so is $Aut(X) \subset X^X$ by Lemma 2.2.1. The group $Aut(X)$ defines a Segal groupoid whose quotient is the classifying space $BAut(X)$. The action of $Aut(X)$ on $X$ also defines a Segal groupoid whose quotient is $X/Aut(X)$. By Proposition 2.7.1, these quotients are truncated coherent. Using Lemma 2.9.1, this shows that $U'_{coh}$ is a countable coproduct of truncated coherent spaces. \qed

**Remark 2.9.3.** In other words, the class of truncated coherent maps is a local class in the sense of [Lur09, Definition 6.1.3.8].
Corollary 2.9.4. The space $U_{\text{coh}}^{<\infty}$ is the realization of a countable simplicial set.

Proof. By Proposition 2.6.3 and Theorem 2.9.2, $U_{\text{coh}}^{<\infty}$ is the realization of a countable coproduct of Kan complexes with values in finite sets. □

A map $X \to Y$ in $\mathcal{S}$ is called univalent if its classifying map $Y \to U$ (in $\hat{\mathcal{S}}$) is a monomorphism. In other words, the univalent maps in $\mathcal{S}$ are exactly the maps $U_{S'} \to U_S$ (for some small set $S' \subset S$), introduced above.

Corollary 2.9.5 (Enough univalent maps). Any map $X \to Y$ in $\mathcal{S}_{\text{coh}}^{<\infty}$ is the pullback of a univalent map in $\mathcal{S}_{\text{coh}}^{<\infty}$.

Proof. Because $Y$ has a finite number of connected components, the classifying map $U \to U_{\text{coh}}^{<\infty}$ of (4) factors through some space $U_{S'} = \prod_{X \in S'} BAut(X)$ for some finite set $S' \subset S_{\text{coh}}^{<\infty}$. Both $U_S$ and $U_{S'}$ are truncated coherent as finite coproduct of truncated coherent spaces, and $X \to Y$ is the pullback in $\mathcal{S}_{\text{coh}}^{<\infty}$ of the $U_{S'} \to U_S$. □

Theorem 2.9.6. The univalent family (in $\mathcal{S}$) $(U_{\text{coh}}^{<\infty})^I \to U_{\text{coh}}^{<\infty}$ has dependent sums and dependent products.

Sketch of the proof. Since I don’t want to enter the precise definition of having dependent sums and products here, I’ll just say that the stability by dependent sums (i.e. by composition) is a consequence of Proposition 2.2.3 and that by dependent products is a consequence of Theorem 2.8.6. □

Remark 2.9.7. It is not clear whether any univalent map $U_{S'} \to U_S$ in $\mathcal{S}_{\text{coh}}^{<\infty}$ can have dependent sums and dependent product. It is clear however that there cannot be a univalent family stable by finite sums, since this would require $\pi_0(B)$ to be infinite. This seems to forbid also univalent families containing non-connected truncated coherent spaces. But this does not forbid the existence of a class of connected truncated coherent spaces which could be stable under $\Sigma$ or even $\Pi$.

Finally we conclude by proving that $\mathcal{S}_{\text{coh}}^{<\infty}$ has a subobject classifier.

Proposition 2.9.8. The set $2 = \{0, 1\}$ is a (Boolean) subobject classifier in $\mathcal{S}_{\text{coh}}$.

Proof. The set $2 = \{0, 1\}$ is a subobject classifier in $\text{Set}_{\text{fin}}$. A map $X \to Y$ in $\mathcal{S}$ is a monomorphism if and only if the map $\pi_0(X) \to \pi_0(Y)$ is injective. If $\text{Sub}(X)$ is the set of subobjects of a space $X$, we have natural bijections

$$\text{Sub}(X) = 2^{\pi_0(X)} = 2^X.$$  

This proves that $2$ is a subobject classifier in $\mathcal{S}$. The result follows from the fact that any subobject of a truncated coherent space $X$ is truncated coherent (Lemma 2.2.1). □

2.10 Initiality properties

This section proves the initiality results of Properties (9) and (10).

Recall from [Lur17, Definition A.6.1.1], that an $\infty$-pretopos (we shall say simply a pretopos) is a category $\mathcal{E}$ with finite limits, with extensive finite coproducts, and with universal and effective quotients of Segal groupoids. A morphism of pretopoi is a functor preserving finite limits, finite sums and quotients of Segal groupoids (or equivalently surjective maps).

Theorem 2.10.1. The category $\mathcal{S}_{\text{coh}}^{<\infty}$ is the initial pretopos.

Sketch of the proof. We need to prove that, for any pretopos $\mathcal{E}$, the category of morphisms of pretopoi $\mathcal{S}_{\text{coh}}^{<\infty} \to \mathcal{E}$ is contractible. The proof of Proposition 2.7.3 shows that all objects of $\mathcal{S}_{\text{coh}}^{<\infty}$ can be build from finite sets by successive quotients of Segal groupoids. Hence any pretopos morphism $\mathcal{S}_{\text{coh}}^{<\infty} \to \mathcal{E}$ is completely determined by its restriction to finite sets. Since the morphism preserve sums, it is in fact determined by the image of a singleton. But this must be the terminal object of $\mathcal{E}$ by left-exactness. This proves that the category of morphisms pretopoi $\mathcal{S}_{\text{coh}}^{<\infty} \to \mathcal{E}$ is either empty of contractible.
We’re only going to sketch the proof of the existence of a pretopos a morphism \( i : S_{\text{coh}}^{\infty} \to \mathcal{E} \). A first argument is to define \( i \) as the left Kan extension

\[
\begin{array}{ccc}
1 & \rightarrow & \mathcal{E} \\
\downarrow & & \downarrow i \\
S_{\text{coh}}^{\infty} & \rightarrow & \mathcal{E} \\
\end{array}
\]

where both maps \( 1 \to \mathcal{E} \) and \( 1 \to S_{\text{coh}}^{\infty} \) are pointing the terminal objects. The existence of this Kan extension is clear enough, but the fact that is it a left-exact functor require an argument. The argument is similar to the one proving that, for a topos \( \mathcal{E} \), the canonical cocontinuous functor \( S \to \mathcal{E} \) is left-exact [AL19, Theorem 2.1.4] and is too long to reproduce here.

Another argument would be to cocomplete both \( S_{\text{coh}}^{\infty} \) and \( \mathcal{E} \) into topos \( \overline{S}_{\text{coh}}^{\infty} \) and \( \overline{\mathcal{E}} \), to precisely use the analogous result in this context. The topos \( \overline{\mathcal{E}} \) is the cocompletion of \( \mathcal{E} \) preserving finite sums and quotients of Segal groupoid. Explicitly, \( \overline{\mathcal{E}} \) it the category of sheaves \( \mathcal{E}^{\text{op}} \to S \) for the effective epimorphism topology on \( \mathcal{E} \), see [Lur17, A.6.4]. By definition, the embedding \( \mathcal{E} \to \overline{\mathcal{E}} \) preserve finite limits, finite sums and quotient of Segal groupoids. The \( S_{\text{coh}}^{\infty} \) can be defined similarly, but by Proposition 2.7.3.(1), it is simply \( S \). The constant sheaf functor \( S \to \overline{\mathcal{E}} \) is a cocontinuous and left-exact functor. Composing with \( S_{\text{coh}}^{\infty} \to S_{\text{coh}}^{\infty} \), we get a functor \( S_{\text{coh}}^{\infty} \to \overline{\mathcal{E}} \) preserving finite limits and quotients of Segal groupoids. Since this functor sends the terminal object of \( S_{\text{coh}}^{\infty} \) in \( \mathcal{E} \), the whole image \( S_{\text{coh}}^{\infty} \to \overline{\mathcal{E}} \) is in \( \mathcal{E} \). This proves the existence of a pretopos morphism \( S_{\text{coh}}^{\infty} \to \mathcal{E} \) (but relying on the material of [Lur17, A.6.4]).

Let \( \mathcal{E} \) be a pretopos and \( i : S_{\text{coh}}^{\infty} \to \mathcal{E} \) the morphism of Theorem 2.10.1. For \( X \) a space, we denote by \( \mathcal{E}^X \) the category of \( X \)-diagrams in \( \mathcal{E} \). We show it is equivalent to \( \mathcal{E}_{j/1X} \).

**Lemma 2.10.2.** For \( X \) a truncated coherent space, there exists a canonical equivalence \( \mathcal{E}_{j/1X} \simeq \mathcal{E}^X \).

**Proof.** We prove it by descent. When \( X \) is a finite set this is true by extensionality of sums in \( \mathcal{E} \) and because \( i : S_{\text{coh}}^{\infty} \to \mathcal{E} \) preserve finite sums. For a general \( X \), we use a resolution \( X_n \) by a truncated Kan complex (Proposition 2.6.3). By Theorem 2.10.1 we have \( iX = \text{colim } i(X_n) \) in \( \mathcal{E} \) (we shall simply write \( X_n \) for \( i(X_n) \) henceforth). By the descent property of Corollary 2.7.4 we get \( \mathcal{E}_{j/1X} = \text{lim}_n \mathcal{E}_{j/X_n} = \text{lim}_n \mathcal{E}^{X_n} \). By extensivity we get \( \text{lim}_n \mathcal{E}_{j/X_n} = \text{lim}_n \mathcal{E}^{X_n} \). Recall that the embedding \( S \in \text{Cat} \) of groupoids in categories preserves all limits and colimits (since it has both left and a right adjoint). This gives \( \text{lim}_n \mathcal{E}^{X_n} = \mathcal{E}^{\text{colim } X_n} = \mathcal{E}^X \). Altogether, this provides the expected equivalence.

Let \( f : \mathcal{E} \to \mathcal{F} \) be a morphism of pretopoi. We denote \( i : S_{\text{coh}}^{\infty} \to \mathcal{E} \) and \( j : S_{\text{coh}}^{\infty} \to \mathcal{E} \) the canonical morphism of Theorem 2.10.1. Then \( f \) induces a functor between diagram categories \( f^X : \mathcal{E}^X \to \mathcal{F}^X \) and a functor between slice categories \( f_X : \mathcal{E}_{j/1X} \to \mathcal{F}_{f/1X} \) (sending \( Y \to iX \) to \( f(Y) \to f(iX) = jX \)). We leave to the reader the proof that these two functors correspond to each other under the equivalences \( \mathcal{E}^X = \mathcal{E}_{j/1X} \) and \( \mathcal{F}^X = \mathcal{F}_{f/1X} \).

The category of finite sets is known to be the universal Boolean elementary 1-topos [Awo97, pp. 71–73]. We will prove now a similar result for \( S_{\text{coh}}^{\infty} \). Since the category \( S_{\text{coh}}^{\infty} \) is not an elementary topos in the sense of Shulman–Rasekh [Ras18, nLa21], we introduce a new notion. We define a \( \Pi \Omega \)-pretopos as a pretopos which is locally cartesian closed and admits a subobject classifier. A morphism of \( \Pi \Omega \)-pretopoi is a morphism of pretopoi which is also a morphism of locally cartesian closed categories and which preserves the subobject classifier. A \( \Pi \Omega \)-pretopos is Boolean if its subobject classifier is isomorphic to \( 2 = 1 + 1 \). The category of Boolean \( \Pi \Omega \)-pretopoi is defined as a full subcategory of that of \( \Pi \Omega \)-pretopoi.

**Theorem 2.10.3.** The category \( S_{\text{coh}}^{\infty} \) is the initial Boolean \( \Pi \Omega \)-pretopos.

**Proof.** Let \( \mathcal{E} \) be a Boolean \( \Pi \Omega \)-pretopos. Then, \( \mathcal{E} \) is in particular a pretopos and we get a unique pretopos morphism \( i : S_{\text{coh}}^{\infty} \to \mathcal{E} \) from Theorem 2.10.1. The result will be proved if we show that \( i \) is a morphism of Boolean \( \Pi \Omega \)-pretopoi. Since \( i \) preserves finite sums, it does preserve the subobject classifier. We are left to prove that for any \( X \) in \( S_{\text{coh}}^{\infty} \), the morphism \( i_X : (S_{\text{coh}}^{\infty})_{j/1X} \to \mathcal{E}_{j/1X} \) preserves exponentials. We use the same
descent strategy as in Theorem 2.8.6, to present \( i \) as a limit of morphisms of cartesian closed categories. This reduces the problem to proving that \( i : S^\infty_{\text{coh}} \to \mathcal{E} \) preserves exponentials. We use the same strategy as in Lemma 2.8.1. Let \( X \) and \( Y \) be two truncated coherent spaces and \( X_n \) a truncated Kan complexes with colimit \( X \) (Proposition 2.6.3). Then we have \( Y^X = \lim_n Y^{X_n} \) in \( S^\infty_{\text{coh}} \) and \((iY)^iX = \lim_n (iY)^{X_n} \) in \( \mathcal{E} \). Since \( X \) is \( N \)-truncated for some \( N \) and \( i \) is left-exact, then \( iX \) is also \( N \)-truncated and we can use Lemma 2.8.4 to reduce both cosimplicial limits to finite limits. Then we can use that \( i : S^\infty_{\text{coh}} \to \mathcal{E} \) preserves finite products and finite limits, and therefore sends \( Y^X = \lim_n Y^{X_n} \) to \( \lim_n (iY)^{X_n} = (iY)^iX \).

\[ \square \]

References


