An example of elementary ∞-topos

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	1-topoi	**∞-topoi**
Grothendieck	Set	$\mathcal{S}$
Elementary	Set_{fin}	???

What is the higher analogue of finite sets?

	1-topoi	**∞-topoi**
Grothendieck	Set	S
Elementary	Set_{fin}	$S^{<\infty}_{coh}$

What is the higher analogue of finite sets?

Bounded coherent spaces.

I will prove that the category $S_{coh}^{<\infty}$ of bounded coherent spaces

- has finite limits;
- has finite sums, and they are extensive;
- has quotients of Segal groupoids, and they satisfy descent;
- is idempotent complete;
- has all truncation modalities;
- is locally cartesian closed;
- has enough univalent families;
- and has a subobject classifier.

Moreover, all these features are preserved by the enbedding

$$S_{coh}^{<\infty} \subset S.$$

We will see also that

- $S_{coh}^{<\infty}$ has a universe in S stable under Σ and Π.

PLAN

1. Finite spaces S_{fin}
2. Bounded coherent spaces $S_{coh}^{<\infty}$
3. Comparison S_{fin} v. $S_{coh}^{<\infty}$
4. Proofs of the properties of $S_{coh}^{<\infty}$

What are the finite spaces?

Finite sets have two natural generalizations to spaces:

- finite spaces S_{fin}
- bounded coherent spaces $S_{coh}^{<\infty}$

Finite spaces

A space is finite is it is the homotopy type of a finite CW-complex.

or, equivalently, the realization of a simplicial set with only a finite number of non-degenerate simplices
(or of a finite semi-simplicial set).

All spheres S^n $(n \geq -1)$ are finite and any finite space is build with a finite chain of cell attachments

$$\begin{array}{ccc} S^n & \longrightarrow & X_n \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & X_{n+1} \end{array}$$

Finite spaces

The subcategory $S_{fin} \subset S$ of finite spaces is stable by

- finite sums and products
- pushouts

but

- NOT by fiber products/loop spaces ($\Omega S^1 = \mathbb{Z}$)
- NOT by truncation
- NOT by retracts

Theorem (Univ. prop of S_{fin})

S_{fin} is the smallest subcategory of S containing Set_{fin} and stable under pushouts.

Coherent spaces

A space X is coherent if all its homotopy invariants ($\pi_0(X)$ and all $\pi_n(X, x)$) are finite sets.

A coherent space X is bounded if it is truncated.

Bounded coherent spaces are also called π_*-finite, because the set

$$\pi_*(X) := \coprod_{x \in \pi_0(X)} \bigvee_{n \geq 1} \pi_n(X, x)$$

is finite.

Coherent spaces

Examples of bounded coherent spaces

- any finite set (including 0 and 1)
- any finite groupoid ($G_1 \rightrightarrows G_0$ in Set_{fin})
- $\mathbb{RP}^\infty = B\mathbb{Z}_2$ (= universe of sets of cardinal 2)
- $\coprod_{k \leq n} BAut(k)$ (= universe of sets or cardinal $\leq n$)
- classifying spaces BG, for G a finite group
- Eilenberg–Mac Lane spaces $K(G, n)$, for G a finite group

Examples of unbounded coherent spaces

- $\Omega^{2n+2} S^{2n+1}$, $\Omega^{4n} S^{2n}$
- The realization of a simplicial Kan complex of finite sets

Non-examples of coherent spaces

- the spheres S^n ($n \geq 1$)

Kan complexes

Theorem

- *A space is coherent iff it is the realization of a Kan complex of finite sets.*
- *A space is bounded coherent iff it is the realization of a coskeletal Kan complex of finite sets.*

Proof.

The conditions are clearly sufficient.
For the necessity, we proceed à la Reedy:
cover X by a finite set $X_0 \to X$,
cover $X_0 \times_X X_0$ by a finite set X_1 to get $X_1 \rightrightarrows X_0$,
etc.
At each step, we can cover by a finite set because the loop spaces of X are assumed to have a finite π_0. □

Kan complexes

Corollary

The set of isomorphism classes of objects of $S^{<\infty}_{coh}$ is countable.

Proof.

The set of diagrams $\Delta_{\leq n} \to Set_{fin}$ is countable, a fortiori the subset Kan_n of Kan complexes.
Then, the set $\coprod_n Kan_n$ has a surjective map to the set of isomorphism classes of objects of $S^{<\infty}_{coh}$. □

Coherent spaces à la Kuratowski

Coherent spaces are a natural generalization of Kuratowski finite objects:

A space X is 0-finite if there exists a surjective map $n \twoheadrightarrow X$. This is equivalent to $\pi_0(X)$ being finite.

A map $f : X \to Y$ is 0-finite if all its fibers are 0-finite spaces.

A map $f : X \to Y$ is n-finite if all its diagonals $\Delta^n f$ are 0-finite. This is equivalent to all $\pi_k(X, x)$ ($k \leq n$) being finite.

A space is coherent iff $X \to 1$ is ∞-finite.

Coherent spaces à la Eilenberg–Mac Lane

The EM spaces $K(G,n)$ (G finite) are the basic building blocks of $S_{coh}^{<\infty}$.

They are analogous to the spheres for finite spaces.

Using Postnikov tower, any $X \in S_{coh}^{<\infty}$ is build with a finite chain of "co-cell" attachments

$$\begin{array}{ccc} X_{n+1} & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow \\ X_n & \longrightarrow & K(G,n) \end{array}$$

First properties

The definition of $S^{<\infty}_{coh} \subset S$ implies immediately

- stability by finite sums
- stability by finite products
- truncation

but

- NOT by pushouts:

  Recall that $S^1 = 1 \coprod_{1+1} 1$ in S. If $1 \coprod_{1+1} 1$ existed in $S^{<\infty}_{coh}$, it would exist also in the category $(S^{<\infty}_{coh})^{\bullet}$ of pointed objects in $S^{<\infty}_{coh}$. This would imply that the group $\mathbb{Z}$ has a reflection in finite groups, which it does not (it has only a profinite reflection).

An easy computation shows also that

- $S^{<\infty}_{coh}$ is stable by fiber products.

Finite limits

Let $E \to B$ be a map in $S^{<\infty}_{coh}$, let us see that the fibers are in $S^{<\infty}_{coh}$:

$$\begin{array}{ccc} F & \longrightarrow & E \\ \big\downarrow & \ulcorner & \big\downarrow \\ 1 & \longrightarrow & B \end{array}$$

We use the the long exact sequence in homotopy

$$\dots \to \pi_{n+1}(B) \to \pi_n(F) \to \pi_n(E) \to \dots$$

Introducing the image of each map

$$\pi_{n+1}(B) \twoheadrightarrow K \rightarrowtail \pi_n(F) \twoheadrightarrow I \rightarrowtail \pi_n(E)$$

K is finite because $\pi_{n+1}(B)$ is, and I is finite because $\pi_n(E)$ is. Then, $\pi_n(F)$ is finite because it is in bijection with $K \times I$.

Extensions

$S^{<\infty}_{coh}$ is also stable by extension (Σ-types):

If F and B are in $S^{<\infty}_{coh}$ (with B connected),

$$\begin{array}{ccc} F & \longrightarrow & E \\ \downarrow & \ulcorner & \downarrow \\ 1 & \longrightarrow & B \end{array}$$

then a long exact sequence argument as before proves that E is in $S^{<\infty}_{coh}$.

Retracts

Theorem

$S_{coh}^{<\infty}$ is idempotent complete.

Proof.

If Y is a retract of a bounded coherent space X, $\pi_n(Y)$ is a retract of $\pi_n(X)$, hence finite. □

Coherent v. finite spaces so far

	S_{fin}	$S_{coh}^{<\infty}$
finite + and ×	yes	yes
pushouts	yes	no
fiber products	no	yes
loop spaces Ω	no	yes
truncations	no	yes
retracts	no	yes
building blocks	$S^n \to 1$	$1 \to K(G, n)$
compactness	compact in S	n-truncated bounded coh. spaces are compact in $S^{\leq n}$ (but not in S)

Coherent v. finite spaces

Theorem (?)

$$S_{fin} \bigcap S_{coh}^{<\infty} = Set_{fin}$$

The example of

$$B\mathbb{Z}_2 = \mathbb{RP}^\infty = 1 \cup \mathbb{R} \cup \mathbb{R}^2 \cup \mathbb{R}^3 \ldots$$

shows that bounded coherent spaces have an infinite number of non-degenerate cells.

Coherent v. finite spaces

Theorem (Miller)

For X a finite space and G a finite group

$$\mathrm{Map}(BG, X) = X$$

$$X \xrightarrow{\simeq} X^{BG}$$

$$X \to X^Y \text{ invertible} \quad \Leftrightarrow \quad \begin{matrix} Y \\ \downarrow \\ 1 \end{matrix} \quad \perp \quad \begin{matrix} X \\ \downarrow \\ 1 \end{matrix}$$

Do we have

$$(S_{coh}^{<\infty})_{\text{connected}} \quad \perp \quad S_{fin} \qquad ?$$

Coherent v. finite spaces

What is the modality generated by $(S^{<\infty}_{coh})_{\text{connected}}$?

Coherent spaces are killed by rationalization $- \otimes \mathbb{Q}$ but not all spaces.

This proves there are plenty of objects in $(S^{<\infty}_{coh})^{\perp}_{\text{connected}}$.

Euler characteristics

Any finite space has a Euler characteristic in $\mathbb{Z}$

$$\begin{aligned}\chi(X) &= \sum |n\text{-cells}|(-1)^n \\ &= b_0 - b_1 + b_2 \dots\end{aligned}$$

Any bounded coherent space has a Euler characteristic in $\mathbb{Q}_{\geq 0}$ (also called homotopy cardinality)

$$\begin{aligned}\chi(X) &= \sum_{x \in \pi_0(X)} \prod |\pi_n(X, x)|^{(-1)^n} \\ &= \sum_x \frac{|\pi_2(X, x)||\pi_4(X, x)| \dots}{|\pi_1(X, x)||\pi_3(X, x)| \dots}\end{aligned}$$

Both characteristics are compatible with $+$ and $\times$ (and more)

$$\chi(X + Y) = \chi(X) + \chi(Y) \qquad \chi(X \times Y) = \chi(X)\chi(Y)$$

Euler characteristics – Analogies

$$\begin{array}{ccc} Set_{fin} & \longrightarrow & S_{coh}^{<\infty} \\ \downarrow & \ulcorner & \downarrow \\ S_{fin} & \longrightarrow & S \end{array} \qquad \begin{array}{ccc} \mathbb{N} & \longrightarrow & \mathbb{Q}_{\geq 0} \\ \downarrow & \ulcorner & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{R} \end{array}$$

$$\begin{array}{ccc} Set_{fin} & \longrightarrow & Set_{fin}[\{K(G,n)\}] \\ \downarrow & \ulcorner & \downarrow \\ Set_{fin}[S^1] & \longrightarrow & S \end{array} \qquad \begin{array}{ccc} \mathbb{N} & \longrightarrow & \mathbb{N}[\{\frac{1}{p}\}] \\ \downarrow & \ulcorner & \downarrow \\ \mathbb{N}[-1] & \longrightarrow & \mathbb{R} \end{array}$$

The universe of bounded coherent objects in S

A map $X \to Y$ is bounded coherent if its fibers are bounded coherent spaces.

Let $BCoh(X)$ be the subcategory of $S_{/X}$ spanned by BC maps.

If X is $S_{coh}^{<\infty}$, than

$$BCoh(X) = (S_{coh}^{<\infty})_{/X}$$

Theorem

Bounded coherent maps are stable by composition.

Proof.

This is equivalent to the stability by extension. □

The universe of bounded coherent objects in S

Bounded coherent maps are under by base change and define a subfibration of

$$\begin{array}{ccc} BCoh & \hookrightarrow & S^{\to} \\ & \searrow & \downarrow{\scriptstyle cod} \\ & & S \end{array}$$

It is in fact a substack (it has descent).

The universe of bounded coherent objects in S

Theorem (Descent for bounded coherent maps)

For any diagram $X : I \to S$,

$$BCoh(\operatorname{colim} X_i) = \lim_i \; BCoh(X_i)$$

Proof.

Because the coherence condition is on fibers, the descent adjunction of S (which is an equivalence)

$$S_{/\operatorname{colim} X_i} \leftrightarrows \lim_i \; S_{/X_i}$$

restricts to bounded coherent maps. □

Extensive sums

Theorem

Finite sums are extensive in $S^{<\infty}_{coh}$:

$$(S^{<\infty}_{coh})_{/0} = 1$$

$$(S^{<\infty}_{coh})_{/X+Y} = (S^{<\infty}_{coh})_{/X} \times (S^{<\infty}_{coh})_{/Y}$$

Proof.

Can use $(S^{<\infty}_{coh})_{/X} = BCoh(X)$ and descent.
Or a direct computation using the extensivity of sums in S. □

Quotients of Segal groupoids

Recall that a Segal groupoid in S is a simplicial space $X_\bullet : \Delta^{op} \to S$ satisfying

$$X_n = X_1 \times_{X_0} \cdots \times_{X_0} X_1 .$$

Let $|X|$ be the colimit, then $|X|$ is called the quotient and $q : X_0 \to |X|$ the quotient map.

Recall that quotient Segal groupoids are effective in S (part of Giraud axioms for ∞-topoi):
if $|X|$ is the colimit of $X_\bullet$, the following square is cartesian

$$\begin{array}{ccc} X_1 & \longrightarrow & |X| \\ \big\downarrow & \ulcorner & \big\downarrow \Delta_{|X|} \\ X_0 \times X_0 & \xrightarrow{(q,q)} & |X| \times |X| \end{array}$$

Quotients of Segal groupoids

Theorem

$S_{coh}^{<\infty}$ is stable by quotients of Segal groupoids.

Proof.

The quotient map $X_0 \to |X|$ is a cover, hence $\pi_0(X)$ is finite.
The cartesian diagram

$$\begin{array}{ccccc} \Omega_{x,y}|X| & \longrightarrow & X_1 & \longrightarrow & |X| \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \Delta \\ 1 & \xrightarrow{(x,y)} & X_0 \times X_0 & \xrightarrow{(q,q)} & |X| \times |X| \end{array}$$

proves that the spaces $\Omega_{x,y}|X|$ are in $S_{coh}^{<\infty}$.
Hence all $\pi_n(X, x)$ are finite. □

Universal property of $S^{<\infty}_{coh}$

Theorem ($S^{<\infty}_{coh}$ = exact completion of Set_{fin})

$S^{<\infty}_{coh}$ is the smallest category of S containing Set_{fin} and closed under quotient of Segal groupoids.

Proof.

Let C be this smallest class.
We saw that $C \subset S^{<\infty}_{coh}$.
For the reverse, we use an induction.
The subcategory spanned by quotients of Segal groupoids in Set_{fin} is that of 1-truncated bounded coherent spaces.
The subcategory spanned by quotients of Segal groupoids in k-truncated bounded coherent spaces, is that of $(k+1)$-truncated bounded coherent spaces.
Eventually, all $S^{<\infty}_{coh}$ ends up in C. □

Descent for Segal groupoids

Recall

Theorem (Descent for bounded coherent maps)

For any diagram $X : I \to S$,

$$BCoh(\operatorname{colim} X_i) = \lim_i \; BCoh(X_i).$$

Using $BCoh(X) = (S^{<\infty}_{coh})_{/X}$ if X is bounded coherent, we get

Corollary (Descent in $S^{<\infty}_{coh}$)

For any diagram $X : I \to S^{<\infty}_{coh}$, whose colimit (computed in S) is in $S^{<\infty}_{coh}$, we have

$$(S^{<\infty}_{coh})_{/\operatorname{colim} X_i} = \lim_i \; (S^{<\infty}_{coh})_{/X_i}$$

This applies in particular to Segal groupoids.

Truncation modalities

Theorem

$S_{coh}^{<\infty}$ is stable by the n-connected/n-truncated factorization systems.

Proof.

$S_{coh}^{<\infty}$ is stable by truncation. □

Cartesian closed

Theorem

$S_{coh}^{<\infty}$ is cartesian closed.

Proof.

$S_{coh}^{<\infty} \subset S$ is stable by products, sufficient to show stability by exponential.
We use presentation by Kan complexes to get

$$X^Y = \lim_n X^{Y_n}.$$

All X^{Y_n} are in $S_{coh}^{<\infty}$ by stability by products.
Since X is k-truncated (for some k), so are the X^{Y_n}.
The limit is a priori infinite, but it is a limit of k-truncated spaces, so $\Delta_{\leq k+1} \to \Delta$ is coinitial for diagram of k-truncated spaces. □

Locally cartesian closed

Theorem

$S^{<\infty}_{coh}$ is locally cartesian closed.

Proof.

If $X = E$ is a finite set, $(S^{<\infty}_{coh})_{/X} = (S^{<\infty}_{coh})^E$ is CC.
For a general X, we use Kan complexes and descent

$$(S^{<\infty}_{coh})_{/X} \;=\; \lim_n \,(S^{<\infty}_{coh})_{/X_n}$$

the result follows because a limit of CC categories is CC. □

The universe of bounded coherent spaces

For a bounded coherent space X, we know $End(X) = X^X$ is bounded coherent.

Then so is $Aut(X) \subset End(X)$ (since we just discard some connected components).

Then so is $BAut(X)$ as quotient of a Segal groupoid.

Let BC be the set of isomorphism classes of objects in $S_{coh}^{<\infty}$, we saw it is countable.
This proves:

Theorem

The fibration in groupoids $BCoh \to S$ is representable by the space

$$\mathbb{U}_{coh}^{<\infty} := \coprod_{X \in BC} BAut(X).$$

(Moreover, this space can be presented by a countable simplicial set.)

Enough univalent families

If $X \to Y$ is a map in $S^{<\infty}_{coh}$, it is classified (in S) by a unique square

$$\begin{array}{ccccc} X & \longrightarrow & (\mathbb{U}^{<\infty}_{coh})' & & := \coprod_{X \in BC} X/Aut(X) \\ \downarrow & \ulcorner & \downarrow & & \\ Y & \longrightarrow & \mathbb{U}^{<\infty}_{coh} & & := \coprod_{X \in BC} BAut(X) \end{array}$$

Because $\pi_0(Y)$ is finite this maps factors through a square

$$\begin{array}{ccc} X & \longrightarrow & \coprod_{X \in E} X/Aut(X) \\ \downarrow & \ulcorner & \downarrow \\ Y & \longrightarrow & \coprod_{X \in E} BAut(X) \end{array}$$

for a finite subset $E \subset BC$.

Enough univalent families

The object $\mathbb{U}^{<\infty}_{coh}$ is too big to be in $S^{<\infty}_{coh}$, but only because of its π_0.

But we can approximate it by bounded coherent subobjects.

A map $X \to Y$ in $S^{<\infty}_{coh}$ is univalent if the classifying map $Y \to \mathbb{U}^{<\infty}_{coh}$ is a mono.

Theorem

Any map in $S^{<\infty}_{coh}$ is the pullback of a univalent map in $S^{<\infty}_{coh}$.

Proof.

$\coprod_{X \in E} BAut(X) \to \mathbb{U}^{<\infty}_{coh}$ is a mono.
We saw that $BAut(X)$ is in $S^{<\infty}_{coh}$.
So is $X/Aut(X)$ (quotient of the Segal groupoid $Aut(X) \times X \rightrightarrows X$).
Then, so is the map

$$\coprod_{X \in E} X/Aut(X) \longrightarrow \coprod_{X \in E} BAut(X).$$

□

Enough univalent families

Theorem
The object $\mathbb{U}^{<\infty}_{coh}$ *is stable under* Σ *and* Π*.*

Proof.
Σ = stability by extension.
Π = $S^{<\infty}_{coh}$ is locally cartesian closed. □

It is easy to see that no univalent map $X \to Y$ in $S^{<\infty}_{coh}$ can be closed under sums (this would imply a countable $\pi_0(Y)$).

It seems unlikely that any univalent map be closed under Σ and Π.

Subobject classifier

Recall that $2 = \{0, 1\}$ is a subobject classifier in Set_{fin}.

A map $X \to Y$ in S is a monomorphism iff the map $\pi_0(X) \to \pi_0(Y)$ is injective.

Any subobject of a bounded coherent space X is bounded coherent.

Let $Sub(X)$ be the set of subobjects of X, we have bijections

$$Sub(X) = 2^{\pi_0(X)} = 2^X.$$

Theorem

2 is a subobject classifier in $S_{coh}^{<\infty}$.

Summary

The category $S_{coh}^{<\infty}$ of bounded coherent spaces

- has finite limits;
- has finite sums, and they are extensive;
- has quotients of Segal groupoids, and they satisfy descent;
- is idempotent complete;
- has all truncation modalities;
- is locally cartesian closed;
- has enough univalent families;
- and has a subobject classifier.

All this with

- without pushouts!

Moreover, we have seen that the embedding $S_{coh}^{<\infty} \subset S$ preserves all these structures, and that

- $S_{coh}^{<\infty}$ has universe $\mathbb{U}_{coh}^{<\infty}$ in S stable under Σ and Π.

Thanks!