Enveloping ∞-topoi

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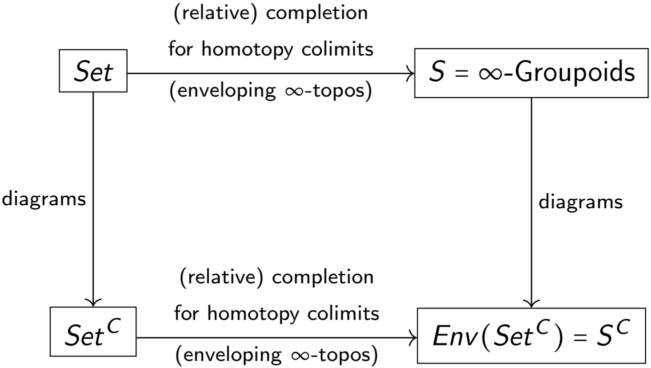
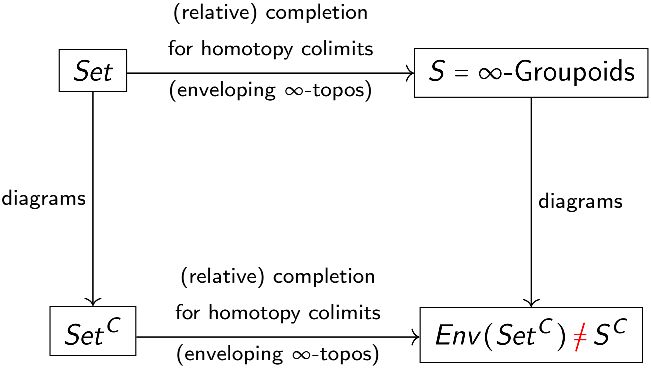
A question



should commute

A question



but *does not* commute !

A question

This raises a distressing question:

What is the enveloping ∞-topos of the 1-topos of simplicial sets?

Is it simplicial spaces?

$$Env\left(Set^{\Delta^{op}}\right) \quad \stackrel{?}{=} \quad S^{\Delta^{op}}$$

If not, then something is very wrong in the practice of higher categories...

Fortunately, the answer is yes, but the proof is not trivial.

Plan

1. Enveloping ∞-topoi
2. The problem
3. The explanation
4. The envelope of simplicial sets

∞-Topoi

One of the big achievement of higher category theory has been the definition of the notion of ∞-topos, which is a higher analog of the classical notion of topos.

1-Category Theory	Sets	Topos
∞-Category Theory	∞-Groupoids	∞-Topos

For an introduction:
http://mathieu.anel.free.fr/mat/doc/Anel-Joyal-Topo-logie.pdf
Chapter on New spaces in Mathematics and Physics (2 vol., CUP 2021)

∞-Topoi

A 1-topos is a presentable 1-category such that

1. coproduct are universal and disjoint

$$E_{/\coprod X_i} = \prod E_{/X_i}$$

2. quotients of equivalence relations are universal and effective

$$E_{/\operatorname{colim}(X_1 \rightrightarrows X_0)} = \lim \left(E_{/X_0} \rightrightarrows E_{/X_1} \right)$$

More concretely, a topos is a cc lex localization

$$Pr_0(C) = [C^{op}, Set] \longrightarrow L^{lex}_{cc}(P_0(C), W) = E$$

∞-Topoi

The definition of an ∞-topos is somehow simpler.

An ∞-topos is a presentable ∞-category E such that

1. all colimits are universal and effective (= so-called descent axiom)

$$E_{/\operatorname{colim} X_i} = \lim E_{/X_i}$$

More concretely, an ∞-topos is a cc lex localization

$$Pr(C) = [C^{op}, S] \longrightarrow L^{lex}_{cc}(P(C), W) = E$$

∞-Topoi

Sets and ∞-groupoids are very close formally.

The language of Martin-Löf type theory, which was invented for sets, has been discover to be quite a comprehensive language for ∞-groupoids (HoTT).

Similarly, 1-topoi and ∞-topoi are very close.

Essentially one thing is new with ∞-topoi:

the existence of ∞-connected maps.

∞-Topoi

Any map $f : X \to Y$ in an ∞-topos has a (fiberwise) Postnikov tower

$$X \to \cdots \to P_n(f) \to \cdots \to P_0(f) \to P_{-1}(f) \to Y$$

A map f is called n-connected if $P_n(f) \simeq Y$.

A map f is called ∞-connected if $P_n(f) \simeq Y$, for all n.

There exists ∞-topoi with ∞-connected maps which are not equivalences $X \xrightarrow{\not\simeq} Y$ (e.g. parametrized spectra).

∞-Topoi

An ∞-topos is hypercomplete if all ∞-connected are equivalences.

Any presheaf ∞-topoi $Pr(C)$ is hypercomplete.

The hypercompletion of an ∞-topos E is the cc lex localization inverting all ∞-connected maps.

Enveloping ∞-topos

$$\mathit{Locale} \overset{\text{(−1)-truncated objects}}{\longleftarrow} \mathit{Topos} \overset{\text{0-truncated objects}}{\longleftarrow} \infty\text{-}\mathit{Topos}$$

$$\mathit{Locale} \underset{\text{enveloping topos}}{\hookrightarrow} \mathit{Topos} \underset{\text{enveloping } \infty\text{-topos}}{\hookrightarrow} \infty\text{-}\mathit{Topos}$$

$$E^{\leq -1} \longleftarrow\!\shortmid\; E^{\leq 0} \longleftarrow\!\shortmid\; E$$

$$O(X) \longmapsto Sh_0(X)$$

$$Sh_0(\mathbf{X}) \longmapsto Sh_\infty(\mathbf{X})$$

$$Sh_0(X)^{\leq -1} = O(X) \qquad \text{and} \qquad Sh_\infty(\mathbf{X})^{\leq 0} = Sh_0(\mathbf{X})$$

Enveloping ∞-topos

How to construct the enveloping ∞-topos of a 1-topos?

Quite straightforward.

Recall that

$$Sh_0(X) = [O(X)^{op}, Set]^{\text{sheaf}}$$

where $F : O(X)^{op} \to Set$ is a sheaf iff

$$F(U) = \lim \left(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j) \right)$$

for any covering family $U_i \to U$.

Enveloping ∞-topos

Similarly

$$Sh_\infty(\mathbf{X}) \;=\; [Sh_0(\mathbf{X})^{op}, S]^{\text{sheaf}}$$

where $F : Sh_0(\mathbf{X})^{op} \to S$ is a higher sheaf iff

$$F(U) = \lim \left(\underbrace{\prod F(U_i) \rightleftarrows \prod F(U_{ij}) \;\substack{\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow}\; \prod F(U_{ijk}) \;\substack{\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow}\; \cdots}_{\text{full simplicial digram}} \right)$$

for any covering family $U_i \to U$.

The problem

Things would be pretty smooth if it wasn't for the fact that

the enveloping ∞-topos of $[C^{op}, Set]$ *need not be* $[C^{op}, S]$.

Which is quite disturbing...

Even more, if we know that, for $-1 \leq n < \infty$

the enveloping n-topos of $[C^{op}, Set]$ is $[C^{op}, S^{\leq n}]$.

But so is life at ∞, plenty of surprises.

The problem

A counter-example can be found in

- Dugger, Hollander, Isaksen, *Hypercovers and simplicial presheaves* (2004)
- Rezk, *Toposes and homotopy toposes* (2005)

but it is not stated explicitly as such.

They were just trying to construct examples of ∞-connected maps.

I owe to Jonas Frey the remark that the construction is done in the enveloping ∞-topos of a presheaf 1-topos.

The problem

Let J be the poset (J is for Jonas)

$$J = \begin{array}{ccccccc} x_0 & \longleftarrow & x_1 & \longleftarrow & x_2 & \longleftarrow & \cdots \\ & \swarrow\!\!\nwarrow & & \swarrow\!\!\nwarrow & & \swarrow\!\!\nwarrow & \\ y_0 & \longleftarrow & y_1 & \longleftarrow & y_2 & \longleftarrow & \cdots \end{array}$$

DHI & R prove that the envelope of $[J^{op}, Set]$ has a non-trivial ∞-connected object (i.e. is not hypercomplete).

Therefore, it cannot be a presheaf category (which are always hypercomplete).

(Somehow, this has to do with *stable homotopy theory*, see appendix)

The problem

The envelope of $[C^{op}, Set]$ need not be the ∞-topos $[C^{op}, S]$.

In consequence,

the envelope of $Sh_0(C, \tau)$ need not be the ∞-topos $Sh_\infty(C, \tau)$.

This is a bit of a problem.

How to compute the envelope of a 1-topos E if one cannot use a presentation by a site?

The problem

Fortunately, we have the following result.

Proposition (Lurie HTT)

The envelope of $[C^{op}, Set]$ *is* $[C^{op}, S]$ *if* C *has finite limits.*

Proof.

Let E be an ∞-topos and $E^{\leq 0} \subset E$ the subcategory of discrete objects.

$[C^{op}, Set] \to E^{\leq 0}$	cc lex functors
$C \to E^{\leq 0}$	lex functors
$C \to E$	lex functors
$[C^{op}, S] \to E$	cc lex functors.

□

The problem

This result is fortunate because

any 1-topos can be presented by a site with finite limits.

When C is a lex category, the envelope of $Sh_0(C,\tau)$ is $Sh_\infty(C,\tau)$.

All seems good

but not quite yet.

The problem

Many 1-topoi of interest are not naturally presented by means of a lex category:

1. Set^{G} G-sets
2. $Set^{\Delta^{op}}$ simplicial sets
3. $Set^{\square^{op}}$ cubical sets
4. $Set^{\mathbb{T}^{op}}$ classifier of flat algebras of an algebraic theory

It can be quite difficult to produce a lex site presenting these examples.

So what are their envelope?

So what's going on?

The main questions are

1. why is the envelope of $[C^{op}, Set]$ not always $[C^{op}, S]$?
2. when is the envelope of $[C^{op}, Set]$ actually $[C^{op}, S]$?

So what's going on?

Going back to the proof for lex C, we get for an arbitrary C

$[C^{op}, Set] \to E^{\leq 0}$	cc lex functors
$C \to E^{\leq 0}$	~~lex~~ flat functors
$C \to E$	~~lex~~ flat ∞-functors
$[C^{op}, S] \to E$	cc lex functors.

So what's going on?

The answer to the question of why is essentially the following.

Let C be a 1-category and E an ∞-topos.

A flat 1-functor

$$C \longrightarrow E^{\leq 0}$$

need not induce a flat ∞-functor

$$C \longrightarrow E^{\leq 0} \hookrightarrow E$$

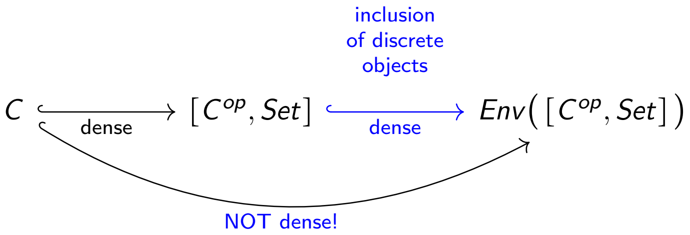
if E is not hypercomplete.

(see Anel, *Flat ∞-functors*, work in progress)

We're gonna take another path today.

So what's going on?

Another way to understand the problem is the following



Not all objects of the envelope are colimits of representables.

Why?

Because the inclusion of discrete objects does not preserves colimits.

So what's going on?

Not all objects of the envelope are colimits of representables.

In fact, the culprits are discrete presheaves!

$$C \xhookrightarrow[\text{~~dense~~}]{} [C^{op}, Set] \xhookrightarrow[\text{dense}]{} Env([C^{op}, Set])$$

Not all objects of $[C^{op}, Set]$ are colimits of representables

in $Env([C^{op}, Set])$.

(Ain't it outrageous...)

So what's going on?

For any object in $[C^{op}, Set]$, there is a canonical map

$$\eta_F : \operatorname*{colim}_{C_{/F}} c \to F$$

where the colimit is computed in $Env([C^{op}, Set])$.

I call good a discrete object F such that η_F is an isomorphism.

Any representable functor is good.

So what's going on?

Theorem (A. *Enveloping ∞-topoi*, work in progress)

1. *The maps* $\eta_F : \operatorname{colim}_{C_{/F}} c \to F$ *are* ∞*-connected.*
2. *The hypercompletion of* $Env\big([C^{op}, Set]\big)$ *is generated by the map* η_F *and is the topos* $[C^{op}, S]$.
3. *The hypercompletion of* $Env\big([C^{op}, Set]\big)$ *is* $[C^{op}, S]$.
4. *The envelope of* $[C^{op}, Set]$ *is* $[C^{op}, S]$ *iff all discrete presheaves are good.*

Simplicial sets

What is the envelope of simplicial sets?

Is it the ∞-topos of simplicial spaces?

Yes!

(phew...)

Simplicial sets

Theorem (A.)

The envelope of $[\Delta^{op}, Set]$ *is* $[\Delta^{op}, S]$.

Proof.

All simplicial sets are good. □

Simplicial sets

Proposition

Good objects are stable by

1. *Giraud colimits:*
   1.1 *coproduct and*
   1.2 *quotients by equivalence relations;*
2. *and pushout along monomorphisms.*

Proof.

Discrete sums, quotients by equivalence relations and pushout along a mono are preserved by the inclusion

$$[C^{op}, Set] \hookrightarrow Env\big([C^{op}, Set]\big).$$

□

Proof that all simplicial sets are good

All representable $\Delta[n]$ are good.

All $\partial\Delta[n]$ are good.
By induction, using pushouts along monos:

- OK for $\partial\Delta[1] = \Delta[0] \coprod \Delta[0]$
- $\partial\Delta[n]$ is a pasting of $\Delta[n-1]$ along $\Delta[n-2]$ (all good), all but the last face, pasted along $\partial\Delta[n-1]$, which is good by induction hypothesis.

Then, all simplicial sets are iterated pushouts along monos

$$\begin{array}{ccc} \partial\Delta[n] & \longrightarrow & X \\ \downarrow & & \downarrow \\ \Delta[n] & \dashrightarrow & X' \end{array}$$

Other examples

How about G-sets? (G a discrete group)

All G-sets are good:

The generator is G acting on itself.

Any coproduct of G is good.

Any orbit G/H is a quotient of an equivalence relation in good objects

$$\coprod_H G \rightrightarrows G \to G/H$$

Any G-set is a coproduct of orbits.

Hence

$$Env\left(Set^G\right) = S^G.$$

Other examples

How about cubical sets?

Dedekind cube $= \{0 < 1\}^n$ full subcat of *Poset*.

All Dedekind cubical sets are good and

$$Env\left(Set^{\square^{op}}\right) = S^{\square^{op}}.$$

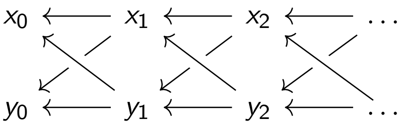
(see Anel, *Enveloping ∞-topoi*, work in progress)

Thanks!

Bonus

DHIR counter-example

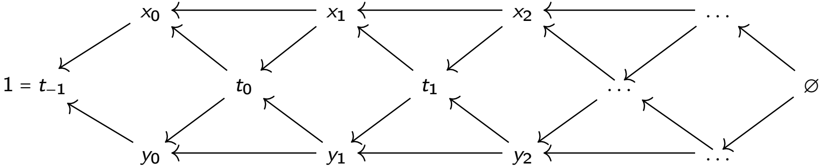
Recall the poset J



and the frame $F = [J^{op}, \underline{2}]$.

DHIR counter-example

$F = [J^{op}, \underline{2}]$ has an explicit description:

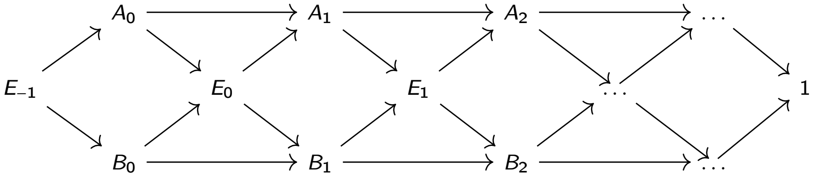


where all squares are bicartesian:

$$x_{n+1} \vee y_{n+1} = t_n = x_n \wedge y_n$$

DHIR counter-example

A sheaf on F is a diagram



where all squares are cartesian

$$E_{n-1} = A_n \times_{E_n} B_n.$$

DHIR counter-example

In *Set* we have the following formula

$$\begin{aligned} E_{n-1} &= A_n \times_{E_n} B_n \\ &= A_n \times_{A_{n+1} \times_{E_{n+1}} B_{n+1}} B_n \\ &= A_n \times_{A_{n+1} \times B_{n+1}} B_n \end{aligned}$$

Because

$$A_{n+1} \times_{E_{n+1}} B_{n+1} \rightarrowtail A_{n+1} \times B_{n+1}$$

is always a mono.

Hence, the sets E_ns are completely determined by the A_ns and B_ns.

DHIR counter-example

Another way to understand the formula

is to look at it as a double path space.

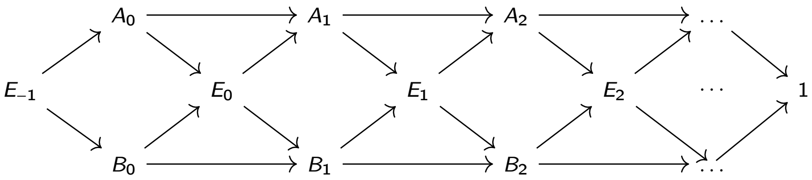
$$\begin{aligned} E_{n-1} &= \Omega_{A_n,B_n} E_n \\ &= \Omega_{A_n,B_n} \left(\Omega_{A_{n+1},B_{n+1}} E_{n+1} \right) \end{aligned}$$

But in a 1-category double path spaces are trivial

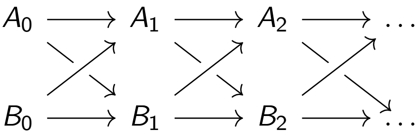
$$\Omega_{A_n,B_n} \Omega_{A_{n+1},B_{n+1}} E_{n+1} \;=\; \Omega_{A_n,B_n} \Omega_{A_{n+1},B_{n+1}} 1$$

DHIR counter-example

In other words, a sheaf of sets on F



is the same thing as a presheaf on J



DHIR counter-example

No mystery there, we just computed that

the enveloping 1-topos of the presheaf frame

$$[J^{op}, \underline{2}]$$

is the presheaf 1-topos

$$[J^{op}, Set]$$

DHIR counter-example

The reasoning is the same for sheaves with values in k-groupoids

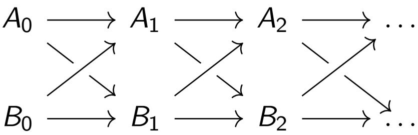
$$
\begin{aligned}
E_{n-1} &= A_n \underset{E_n}{\times} B_n \\
&= A_n \underset{A_{n+1} \underset{E_{n+1}}{\times} B_{n+1}}{\times} B_n \\
&= A_n \underset{A_{n+1} \underset{A_{n+2} \underset{E_{n+2}}{\times} B_{n+2}}{\times} B_{n+1}}{\times} B_n \\
&\vdots \\
&= A_n \underset{\substack{\vdots \\ A_{n+k} \times B_{n+k}}}{\times} B_n
\end{aligned}
$$

(using that $(k+2)$-iterated path spaces are trivial)

The E_ns are still determined by the A_ns and B_ns.

DHIR counter-example

And sheaves of k-groupoids on F are the same as presheaves of k-groupoids on J



The enveloping k-topos of the presheaf frame

$$[J^{op}, \underline{2}]$$

is the presheaf k-topos

$$\left[J^{op}, S^{\leq k}\right]$$

DHIR counter-example

But the reasoning fails for $k = \infty$.

The E_ns can no longer be written in terms of the A_ns and B_ns.

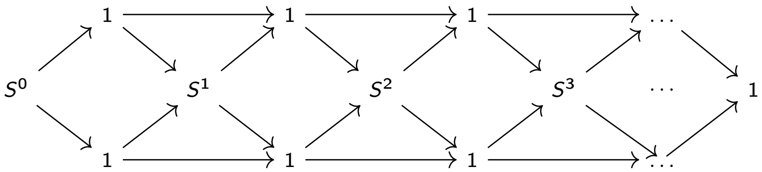
Here is the proof.

DHIR counter-example

There is a sheafification

$$[F, S] \longrightarrow Sh_\infty(F)\,.$$

The sheafification of



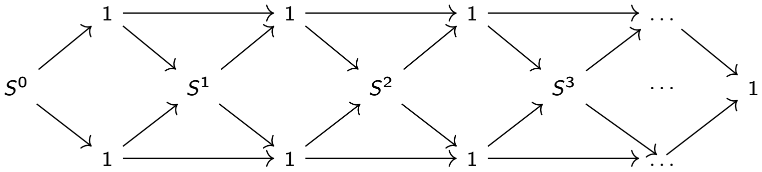
is not terminal, even though its n-truncations are all terminal (see Rezk).

This is an example of an ∞-connected object.

Notice that all A_ns and B_ns are 1.

DHIR counter-example

Sheafification = shift left and loop, repeat.



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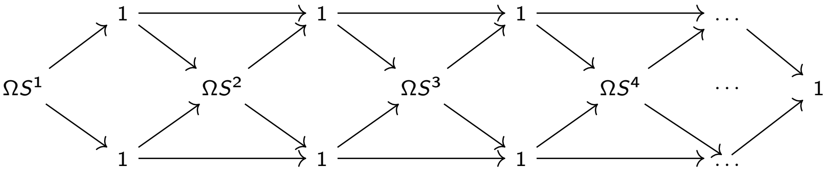
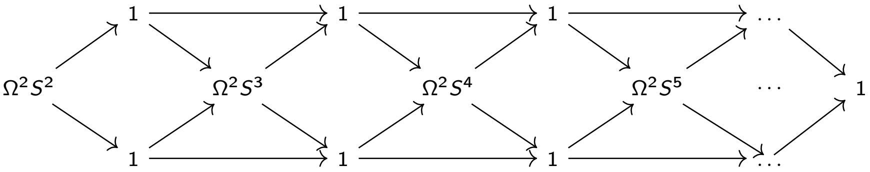


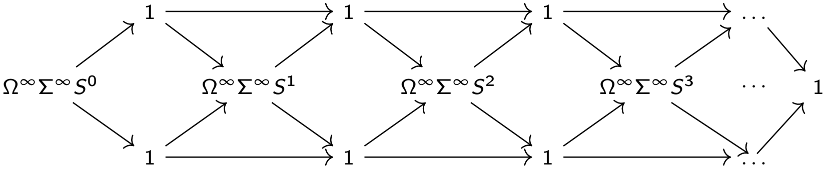
DHIR counter-example

Sheafification = shift left and loop, repeat.



DHIR counter-example

At the limit:



The homotopy of the space $QS^0 = Q^\infty\Sigma^\infty S^0$ is the stable homotopy of spheres.

It is not contractible.

Hence the associated sheaf is not terminal.

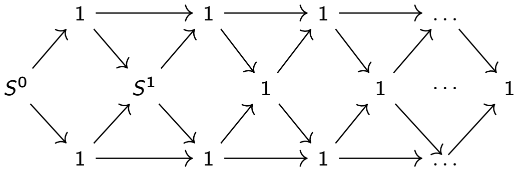
This sheaf is in fact $\eta_1 = \mathrm{colim}_{C/1}\, c$ in $Env([J^{op}, Set])$.

DHIR counter-example

Let us see that it is ∞-connected.

The truncation of a sheaf is the sheafification of the truncation.

For example, the P_1-truncation is



whose sheafification is 1.

This is similar with other P_n because $P_n S^N = 1$, for $N \gg n$.

This proves that the envelope of $[J^{op}, Set]$ is not hypercomplete

and cannot be a presheaf topos.