Computing enveloping ∞-topoi

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A question

On going work inspired by conversations with Steve Awodey, Jonas Frey and Andrew Swan.

What is the enveloping ∞-topos of the 1-topos of simplicial sets?

$$Env\left(Set^{\Delta^{op}}\right) \quad \stackrel{?}{=} \quad \mathcal{S}^{\Delta^{op}}$$

Yes, of course!

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Plan

1. Enveloping ∞-topoi
2. The problem
3. The explanation
4. The envelope of simplicial sets

Enveloping ∞-topos

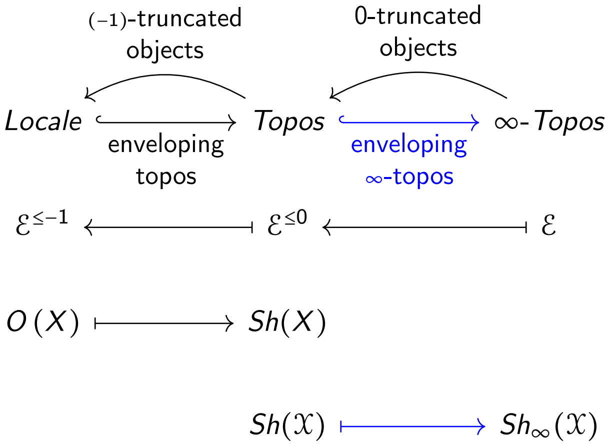
One of the big achievement of higher category theory has been the definition of the notion of ∞-topos, which is a higher analog of the classical notion of topos.

1-Category Theory	Sets	Topos
∞-Category Theory	∞-Groupoids	∞-Topos

For an introduction:

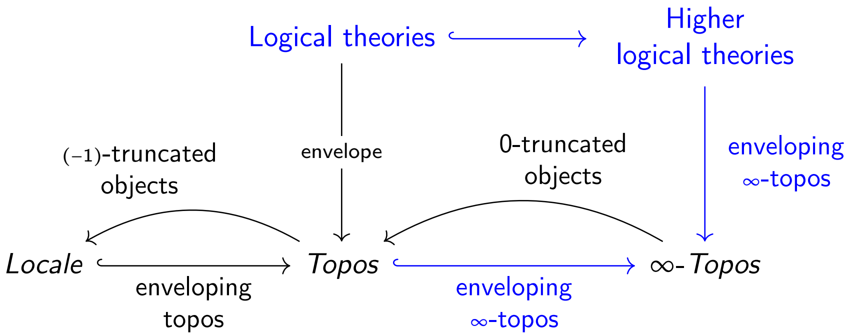
http://mathieu.anel.free.fr/mat/doc/Anel-Joyal-Topo-logie.pdf

Enveloping ∞-topos



$Sh(X)^{\leq -1} = O(X)$ and $Sh_\infty(\mathcal{X})^{\leq 0} = Sh(\mathcal{X})$

Enveloping ∞-topos



Enveloping ∞-topos

How to construct the enveloping ∞-topos of a 1-topos?

Quite straightforward.

Recall that

$$Sh(X) \subset [O(X)^{op}, Set]^{\text{sheaf}}$$

where $F : O(X)^{op} \to Set$ is a sheaf iff

$$F(U) = \lim \left(\prod_i F(U_i) \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}} \prod_{i,j} F(U_i \times_U U_j) \right)$$

for any covering family $U_i \to U$.

Enveloping ∞-topos

Similarly

$$Sh_\infty(\mathcal{X}) \subset [Sh(\mathcal{X})^{op}, \mathcal{S}]^{\text{sheaf}}$$

where $F : Sh(\mathcal{X})^{op} \to \mathcal{S}$ is a higher sheaf iff

$$F(U) = \lim \left(\underbrace{\prod F(U_i) \rightleftarrows \prod F(U_{ij}) \ \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \ \prod F(U_{ijk}) \ \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \ \dots}_{\text{full simplicial digram}} \right)$$

for any covering family $U_i \to U$.

The problem

Things would be pretty smooth if it wasn't for the following remark:

the enveloping ∞-topos of $[C^{op}, Set]$ need not be $[C^{op}, \mathcal{S}]$.

This is quite bizarre because, for $-1 \leq n < \infty$

the enveloping n-topos of $[C^{op}, Set]$ is $[C^{op}, \mathcal{S}^{\leq n}]$.

(But so is life at ∞...)

The problem

A counter-example is given in

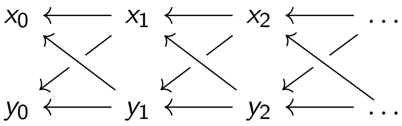
Dugger, Hollander, Isaksen, *Hypercovers and simplicial presheaves* (2004)

and

Rezk, *Toposes and homotopy toposes* (2005)

The problem

Let J be the poset

$J =$ 

D-H-I & R prove that the envelope of $[J^{op}, Set]$ has a non-trivial ∞-connected objet (i.e. is not hypercomplete).

Therefore, it cannot be a presheaf category (which are always hypercomplete).

The problem

The envelope of $[C^{op}, Set]$ need not be the ∞-topos $[C^{op}, \mathcal{S}]$.

The envelope of $Sh(C, \tau)$ need not be the ∞-topos $Sh_\infty(C, \tau)$.

This is a bit of a problem.

How to compute the envelope of a 1-topos $\mathcal{E}$ if one cannot use a presentation by a site?

The problem

Fortunately, we have the following result.

Proposition (Lurie HTT)

The envelope of $[C^{op}, Set]$ *is* $[C^{op}, \mathcal{S}]$ *if* C *has finite limits.*

Proof.

Let $\mathcal{E}$ be an ∞-topos and $\mathcal{E}^{\leq 0} \subset \mathcal{E}$ the subcategory of discrete objects.

$$
\begin{aligned}
[C^{op}, Set] &\to \mathcal{E}^{\leq 0} && \text{cc lex functors} \\
C &\to \mathcal{E}^{\leq 0} && \text{lex functors} \\
C &\to \mathcal{E} && \text{lex functors} \\
[C^{op}, \mathcal{S}] &\to \mathcal{E} && \text{cc lex functors.}
\end{aligned}
$$

□

The problem

This is fortunate because any 1-topos can be presented by a site with finite limits.

But not so much.

The problem

Many 1-topoi of interest are not naturally presented by means of a lex category:

1. Set^G — G-sets
2. $Set^{\Delta^{op}}$ — simplicial sets
3. $Set^{\square^{op}}$ — cubical sets
4. $Set^{\mathbb{T}^{op}}$ — classifier of flat algebras of a theory

It can be quite difficult to produce a lex site presenting these examples.

So what are their envelope?

So what's going on?

The main questions are

1. why is the envelope of $[C^{op}, Set]$ not always $[C^{op}, \mathcal{S}]$?
2. when is the envelope of $[C^{op}, Set]$ actually $[C^{op}, \mathcal{S}]$?

So what's going on?

- we go back to the proof for lex C, we get for a general C

$[C^{op}, Set] \to \mathcal{E}^{\leq 0}$	cc lex functors
$C \to \mathcal{E}^{\leq 0}$	~~lex~~ flat functors
$C \to \mathcal{E}$	~~lex~~ flat ∞-functors
$[C^{op}, \mathcal{S}] \to \mathcal{E}$	cc lex functors.

So what's going on?

The answer to the first question is essentially the following.

Let C be a 1-category and $\mathcal{E}$ an ∞-topos.

A flat 1-functor

$$C \longrightarrow \mathcal{E}^{\leq 0}$$

need not induce a flat ∞-functor

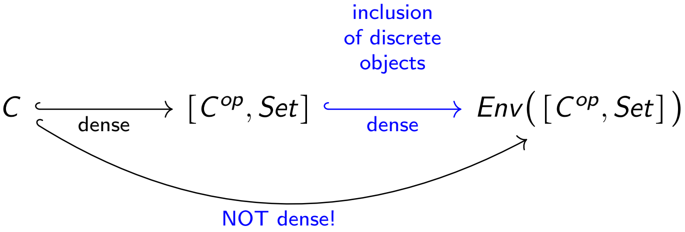
$$C \longrightarrow \mathcal{E}^{\leq 0} \hookrightarrow \mathcal{E}$$

if $\mathcal{E}$ is not hypercomplete.

(see Anel, *Flat ∞-functors*, work in progress)

So what's going on?

Another way to understand the problem is the following



Not all objects of the envelope are colimits of representables.

Why?

Because the inclusion of discrete objects does not preserves colimits.

So what's going on?

Not all objects of the envelope are colimits of representables.

In fact, the culprits are discrete presheaves!

$$C \xhookrightarrow[\text{~~dense~~}]{} [\,C^{op}, Set\,] \xhookrightarrow[\text{dense}]{} Env\big(\,[\,C^{op}, Set\,]\,\big)$$

Not all objects of $[\,C^{op}, Set\,]$ are colimits of representables

in $Env\big(\,[\,C^{op}, Set\,]\,\big)$.

(Ain't it outrageous...)

So what's going on?

Let me call good a discrete object F such that

$$\operatorname{colim}_{C_{/F}} c = F$$

in $Env\big([C^{op}, Set]\big)$.

Theorem (A.)

The envelope of $[C^{op}, Set]$ is $[C^{op}, \mathcal{S}]$ iff all discrete presheaves are good.

(see Anel, *Enveloping ∞-topoi*, work in progress)

Simplicial sets

What is the envelope of simplicial sets?

Is it the ∞-topos of simplicial spaces?

Yes!

(phew...)

Simplicial sets

Theorem (A.)

The envelope of $[\Delta^{op}, Set]$ *is* $[\Delta^{op}, \mathcal{S}]$.

Proof.

All simplicial sets are good. □

Lemma

Good objects are stable by Giraud colimits:

1. *discrete sums and*
2. *quotients by equivalence relations.*

Proof.

Discrete sums and quotients by equivalence relations are preserved by the inclusion

$$[C^{op}, Set] \hookrightarrow Env\big([C^{op}, Set]\big).$$

□

Proof that all simplicial sets are good

A simplicial set X can be covered by simplices. This provides an equivalence relation:

$$\coprod_{ij} \Delta[n_i] \times_X \Delta[n_j] \rightrightarrows \coprod_i \Delta[n_i] \twoheadrightarrow X$$

All simplices $\Delta[n_i]$ are good, then so is $\coprod_i \Delta[n_i]$.

If $\Delta[n_i] \times_X \Delta[n_j]$ is good then so is $\coprod_{ij} \Delta[n_i] \times_X \Delta[n_j]$, and so is X.

Proof that all simplicial sets are good

$\Delta[n_i] \times_X \Delta[n_j]$ is a subobject of the prism $\Delta[n_i] \times \Delta[n_j]$.

All subobjects of $\Delta[n_i] \times \Delta[n_j]$ are good

because they are all pasting of simplices along faces maps,

that is, they are quotients of equivalence relations in simplices.

End of the proof.

Other examples

How about G-sets?

All G-sets are good and

$$Env\left(Set^G\right) = \mathcal{S}^G.$$

How about cubical sets?

All Dedekind cubical sets are good and

$$Env\left(Set^{\square^{op}}\right) = \mathcal{S}^{\square^{op}}.$$

(see Anel, *Enveloping ∞-topoi*, work in progress)

Thanks!