What is a space?

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6th Workshop on Formal Topology
Birmingham, April 8, 2019

These slides cover the first part of my talk at the workshop.

In particular, the definition of an ∞-topos, which was my second part, is not given here (for this, I refer to other notes that I have written, available on my website).

However, this slides present

- the context in which the notions of (1-)topos and ∞-topos make sense,
- some reasons for why they are needed,
- and some explanation as to why they are defined the way they are.

A simple question

The theme of this workshop is the question

What is a space?

This is one of my favorite questions.

What is a space?

My favorite answer:

A space is a collection of *different* things.

Spatiality is about being different!

What is a space?

If there is only one thing, there is no space.

With only one thing, nothing can move, nothing can change, nothing is relative.

What is a space?

Only when I have two things, a space appears.

The common language reflects this by talking about the space between the two dots.

Mathematicians prefer to talk about the space of the two dots.

I think the first view is deeper. I think the fact that the two dots are separated is the fundamental idea.

What is a space?

Intuitively, this separation means that a "wall", or a "cut", or something, can be built between the two points.

What is a space?

In logic difference is opposed to equality.

But in "topo-logic", where difference = separation, it is opposed to juxtaposition.

This is the basis for two characteristic operations of topology: cutting and pasting.

Logic	difference	equality
Topo-logic	separation (cutting)	juxtaposition (pasting)

Separation

How to cut a space?

No choice:

by means of another space.

We are going to see how this idea leads naturally to the encoding of space by means of algebras of functions.

Separation – Hausdorff spaces

The most common notion of separation is given by Hausdorff spaces.

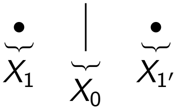
A space X is Hausdorff if, for any two points x, x', there exists a partition

$$X = X_1 + X_0 + X_{1'}$$

such that

1. X_1 and X_1 are open
2. x is in X_1
3. x' is in $X_{1'}$

X_0 is the "wall" between X_1 and $X_{1'}$



Separation – Hausdorff spaces

Let $\$$ be the Sierpiǹski space

$$\$ = \{0, 1\}$$

$$O(\$) = \Big\{\varnothing, \{1\}, \{0,1\}\Big\}.$$

0 is a closed point and 1 an open point.

A continuous map $X \to \$$ is an open-closed partition.

$$\begin{array}{ccc} X_0 + X_1 & \longrightarrow & \{0\} + \{1\} \\ \big\downarrow & \lrcorner & \big\downarrow \\ X & \longrightarrow & \$ \end{array}$$

Separation – Hausdorff spaces

Let $\mathbb{S} \vee_0 \mathbb{S}$ be the glueing of two Sierpiǹski spaces along the closed point 0.

$$\mathbb{S} \vee_0 \mathbb{S} = \{1, 0, 1'\}$$

0 is a closed point and 1 and $1'$ are open points.

A continuous map $X \to \mathbb{S} \vee_0 \mathbb{S}$ is a partition

$$X = X_1 + X_0 + X_{1'}$$

as before:

$$\begin{array}{ccc} X_0 + X_1 + X_{1'} & \longrightarrow & \{1\} + \{0\} + \{1'\} \\ \big\downarrow & \ulcorner & \big\downarrow \\ X & \longrightarrow & \mathbb{S} \vee_0 \mathbb{S} \end{array}$$

Separation – Hausdorff spaces

A space X is Hausdorff iff, for any two points x, x', there exists a commutative diagram

$$\begin{array}{ccc} \{x\} + \{x'\} & \longrightarrow & \{1\} + \{1'\} \\ \downarrow & & \downarrow \\ X & \xrightarrow[\text{cut}]{} & \$ \vee_0 \$ \end{array}$$

Separation – T_0-Spaces

Another classical notion of separation is the condition T_0.

A space X is T_0 if, for any two points x, x', there exists a partition

$$X = X_1 + X_0$$

such that

1. X_1 is open and X_1 is closed
2. x is in X_1 and x' is in X_0

   OR
3. x' is in X_1 and x is in X_0

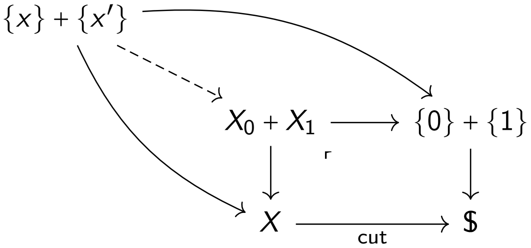
Separation – T_0-Spaces

A space X is T_0 iff, for any two points x, x', there exists a commutative diagram

$$\begin{array}{ccc} \{x\} + \{x'\} & \xrightarrow{\simeq} & \{0\} + \{1\} \\ \downarrow & & \downarrow \\ X & \xrightarrow[\text{cut}]{} & \$ \end{array}$$

Separation – T_0-Spaces

The partition $X = X_0 + X_1$ is obtained by pullback

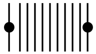


Separation – Completely Hausdorff spaces

A space X is completely Hausdorff iff, for any two points x, x', there exists a commutative diagram

$$\begin{array}{ccc} \{x\} + \{x'\} & \longrightarrow & \{0\} + \{1\} \\ \downarrow & & \downarrow \\ X & \xrightarrow[\text{cut}]{} & [0,1] \subset \mathbb{R} \end{array}$$

Geometrically, this looks like this



Separation – Disconnected space

A space X is disconnected iff, for any two points x, x', there exists a commutative diagram

$$\begin{array}{ccc} \{x\} + \{x'\} & \longrightarrow & \{0\} + \{1\} \\ \big\downarrow & & \big\| \\ X & \underset{\text{cut}}{\longrightarrow} & \mathbb{B} = \{0, 1\} \end{array}$$

Intuitively, x and x' are in different connected components.

Separation – Gauge space

In all cases, we have a space $\mathbf{A}$ with two points $\mathbf{1} \xrightarrow{0} \mathbf{A} \xleftarrow{1} \mathbf{1}$.

$$
\begin{array}{ccc}
\{1\} + \{1'\} & \longrightarrow & \mathbb{S} \vee_0 \mathbb{S} \\
\\
\{0\} + \{1\} & \longrightarrow & \mathbb{S} \\
\\
\{0\} + \{1\} & \longrightarrow & [0, 1] \\
\\
\{0\} + \{1\} & \longrightarrow & \mathbb{R} \\
\\
\{0\} + \{1\} & \longrightarrow & \mathbb{B}
\end{array}
$$

Separation – Gauge space

I'm going to call $\mathbf{A}$ a gauge space.

It is used to separate, cut, or slice, other spaces X into level sets.

$$\begin{array}{ccc} X_a & \longrightarrow & X \\ \big\downarrow & \ulcorner & \big\downarrow \\ \mathbf{1} & \xrightarrow[\{a\}]{} & \mathbf{A} \end{array}$$

Separation – Gauge space

Different choices for **A** give different notions of separation.

Gauge space **A**	*Separation*
Sierpiǹski space $\mathbb{S} = \{0 < 1\}$	T_0-spaces
Boole space $\mathbb{B} = \{0, 1\}$	disconnected spaces
$\mathbb{S} \vee_0 \mathbb{S} = \{1' > 0 < 1\}$	Hausdorff spaces (T_2)
$\mathbb{S} \vee_0 \mathbb{S} \vee_1 \mathbb{S} \vee_0 \mathbb{S}$ $= \{1' > 0' < 1 > 0'' < 1''\}$	Urysohn spaces ($T_{2\frac{1}{2}}$)
Interval $[0, 1]$/Real numbers $\mathbb{R}$	completely Hausdorff spaces

Coordinates

A system of $\mathbf{A}$-coordinates is set N of slicings such that

$$X \longrightarrow \mathbf{A}^N$$

is an embedding.

Intuitively, a coordinate system splits the space X in transversal slices (cf. latitude and longitude) whose intersection is at most a point.

Examples

- any completely Hausdorff space can be embedded into some $[0,1]^N$
- any T_0 space can be embedded into some $\$^N$

Coordinates

I denote $[X, \mathbf{A}]$ the set of maps $X \to \mathbf{A}$.

Existence of coordinates can always be tested with $N = [X, \mathbf{A}]$ and the canonical map

$$X \longrightarrow \mathbf{A}^{[X,\mathbf{A}]}$$

A space X be said to be **A**-separated if this map is an embedding.

This is a better notion of separation than the one defined before using pairs of points.

It says that there exists enough slicing not only to separate the points but to reconstruct the topology.

Coordinates

$[X, \mathbf{A}]$ is the set of coordinates (or cuts, or slicings).

Can a space be reconstructed from its coordinates?

The functor

$$\begin{aligned} [-, \mathbf{A}] : Top^{op} &\longrightarrow Set \\ X &\longrightarrow [X, \mathbf{A}] \end{aligned}$$

is never fully faithful.

Coordinates

A map $f : X \to Y$ in *Top* induces a map (pullback of functions)

$$f^* : [Y, \mathbf{A}] \to [X, \mathbf{A}] .$$

The functor $[-, \mathbf{A}]$ would be fully faithful if there was only the maps f^* between the $[X, \mathbf{A}]$.

But there are "dummy maps" between the $[X, \mathbf{A}]$.

Idea: use the natural algebraic structure of $\mathbf{A}$ to reduce the number of morphisms between the $[X, \mathbf{A}]$.

Algebra of coordinates

For each $\mathbf{A} \to \mathbf{A}$, we have $[X, \mathbf{A}] \to [X, \mathbf{A}]$ (reindexing of coordinates).

For each $\mathbf{A}^2 \to \mathbf{A}$, we have $[X, \mathbf{A}]^2 \to [X, \mathbf{A}]$ (composition of coordinates).

More generally, any map $\mathbf{A}^n \to \mathbf{A}$ induces a natural map

$$[X, \mathbf{A}]^n \to [X, \mathbf{A}] \, .$$

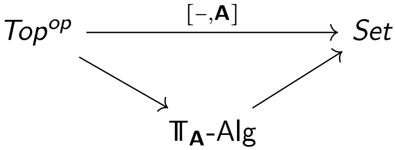
Algebra of coordinates

The full subcategory generated by the $\mathbf{A}^n$ is a Lawvere theory

$$\mathbb{T}_{\mathbf{A}} = \{\ \mathbf{A}^n \mid n \in \mathbb{N}\ \} \subset \ \mathit{Top}.$$

For any X, $[X, \mathbf{A}]$ is an $\mathbb{T}_{\mathbf{A}}$-algebra.

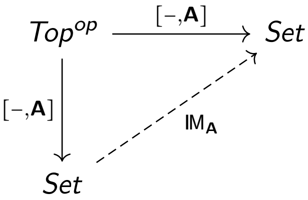
We get a factorization



Algebra of coordinates

More generally, the functor $[-,\mathbf{A}] : Top^{op} \to Set$ has a monad of endomorphisms $\mathbb{M}_{\mathbf{A}}$ (enhancing $\mathbb{T}_{\mathbf{A}}$ by the operations of infinite arity).

$\mathbb{M}_{\mathbf{A}}$ is the right Kan extension of $[-,\mathbf{A}]$ along itself



We have

$$\mathbb{M}_{\mathbf{A}}(E) = \lim_{E \to [X,\mathbf{A}]} [X,\mathbf{A}] = \lim_{(X \to \mathbf{A}^E)^{op}} [X,\mathbf{A}] = \left[\mathbf{A}^E, \mathbf{A}\right]$$

Algebra of coordinates

In practice (*), we use also notions of algebras that are between $\mathbb{T}_{\mathbf{A}}$ and $\mathbb{M}_{\mathbf{A}}$. That it is that have all operations of finite arities, but only some of the operations of infinite arity.

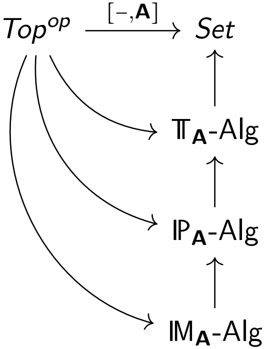
I denote by $\mathbb{P}_{\mathbf{A}}$ such a notion of algebra.

We'll see examples shortly.

(*) I have not find a better justification than a pragmatic one for these intermediate notions. That's why I used the letter $\mathbb{P}$, as in "practice".

Algebra of coordinates

Finally, we get factorizations of the coordinate functors through several levels of algebraic structures

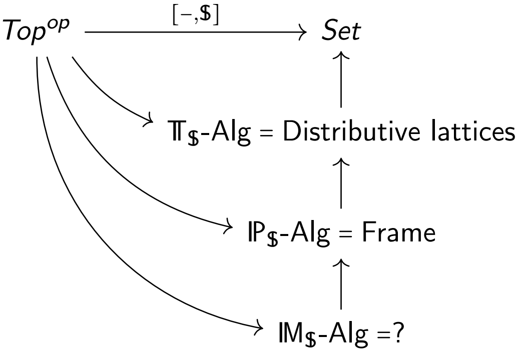


Each of the vertical functor is faithful. They remove more and more maps between the $[X, \mathbf{A}]$, making the functors from Top^{op} more and more fully faithful. This does not always product a fully faithful functor in the end, but this is to be taken as a feature of this process and not a defect.

The $-algebra of coordinates

Let us turn to examples.

What does this gives when **A** = $?



The $\$$-algebra of coordinates

If $\mathbf{A} = \$$ the Sierpiǹski space, we get the theory of frames.

Among the the continuous maps on $\$$

$$\$^E \longrightarrow \$$$

are

1. arbitrary suprema $\bigvee : \$^E \to \$$
2. finite infima $\wedge : \$^E \to \$$

These maps generate the theory $\mathbb{P}_\$$ of Frames.

The $\mathbb{S}$-algebra of coordinates

A frame is

- a poset F
- with arbitrary suprema $\bigvee$
- finite infima $\wedge$
- satisfying a distributivity law

$$a \wedge \bigvee_i b_i = \bigvee_i a \wedge b_i$$

A morphism of frames is a map $F \to F'$ preserving order, suprema and infima.

The frames form a category Frame. It is naturally enriched over posets.

A frame look like a commutative ring. We shall come back to this comparison.

The $\mathbb{S}$-algebra of coordinates

The free frame $\mathbb{P}_{\mathbb{S}} : Set \to Set$ is

$$E \longmapsto \left[\mathbb{S}^{(E)}, \mathbb{S}\right]$$

where $\mathbb{S}^{(E)}$ is the space of maps $E \to \mathbb{S}$ with a finite number of values 1 (=poset of finite subsets of E).

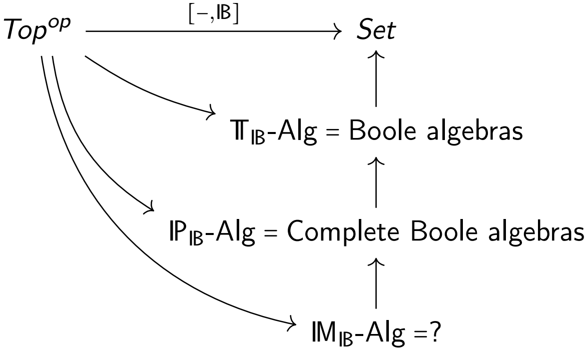
The monad $\mathbb{M}_{\mathbb{S}} : Set \to Set$ is

$$E \longmapsto \left[\mathbb{S}^{E}, \mathbb{S}\right]$$

They coincide only when E is finite.

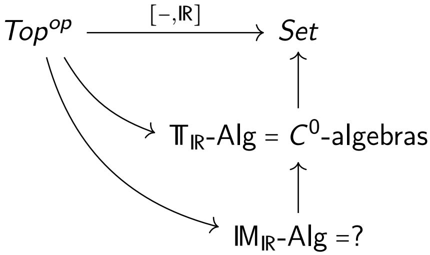
The $\mathbb{B}$-algebra of coordinates

What does this gives when $\mathbf{A} = \mathbb{B}$?



The $\mathbb{R}$-algebra of coordinates

What does this gives when $\mathbf{A} = \mathbb{R}$?



The $\mathbb{R}$-algebra of coordinates

If $\mathbf{A} = \mathbb{R}$, we get the theory of C^0-algebras

The continuous operations on $\mathbb{R}$ of finite arities

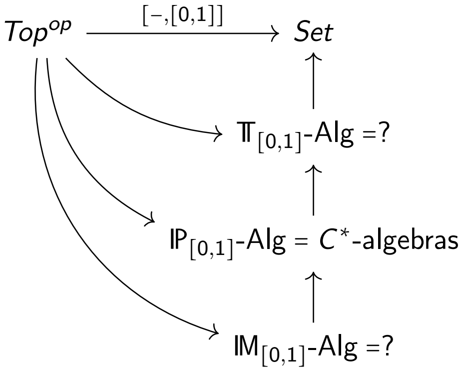
$$\mathbb{T}_{\mathbb{R}}(n) = [\mathbb{R}^n, \mathbb{R}]$$

do not have a nice set of generators.

$[\mathbb{R}^n, \mathbb{R}]$ is the free C^0-algebra on n generators.

The [0, 1]-algebra of coordinates

What does this gives when $\mathbf{A} = [0,1]$?

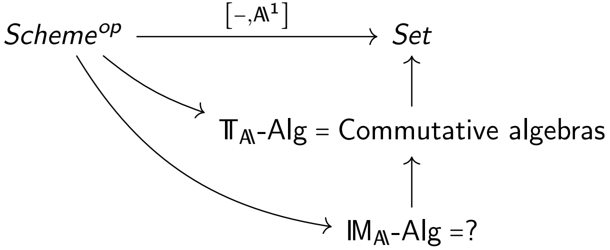


Classically, C^*-algebras are presented by bounded functions with values in $\mathbb{C}$. But functions with values in $[0,1] \subset \mathbb{R} \subset \mathbb{C}$ characterize also these algebras.

Coordinates in algebraic geometry

The same setting applies to other categories than topological spaces.

For example, with the category of schemes and $\mathbf{A} = \mathbb{A}^1$ the affine line, we would get



Algebras of coordinates

Gauge space **A**	*Algebraic structure*	*separation degree*
Boole space $\mathbb{B}$	Boolean algebra	disconnected
Sierpiǹski space $\mathbb{S}$	Frames	T_0
$\mathbb{R}$	C^0-rings	regular Hausdorff ($> T_2$)
$[0,1]$	C*-algebras ?	regular Hausdorff ($> T_2$)

Algebras of coordinates

We saw that coordinates, or slicings, $X \to \mathbf{A}$ have natural algebraic structures.

This is because any object $\mathbf{A}$ has natural algebraic structures (the choice of which depends on the context).

Geometrically, this says that cuts or slicings can be composed.

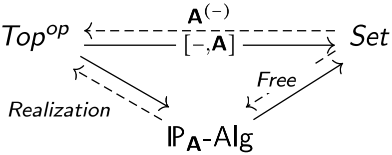
This explains why algebraic devices are always there when manipulating topological objects.

But there is still the question of the faithfulness of the algebraic description.

Now we can reformulate it.

The space/algebra adjunction

There exists left adjoint functors



The functor *Realization* is constructed as follows

- for a free algebra $Free(E)$, we put

$$Realization(Free(E)) = \mathbf{A}^E$$

  These are the affine spaces.

- then we extend by commutation to colimits.

$$Realization(\operatorname{colim} FE_i) = \lim_i \mathbf{A}^{E_i}$$

The space/algebra adjunction

The functor *Realization* is sometimes called the *Spectrum* of the algebra.

But this is a misleading name and a misunderstood notion.

So I will not use it.

The space/algebra adjunction

For $\mathbf{A} = \$$, we get the usual adjunction

$$Top^{op} \underset{\mathbf{O}(-)=[-,\$]}{\overset{Realization}{\leftrightarrows}} \text{Frame}$$

For a space X, $\mathbf{O}(X)$ is the frame of open subset of X.

For a frame F, $Realization(F)$ is the space of its points.

None of these two functors are fully faithful.

But the comparison is still interesting.

The space/algebra adjunction

For $\mathbf{A} = \mathbb{B}$, we get the adjunction

$$\mathit{Top}^{op} \underset{\mathbf{O}(-)=[-,\mathbb{B}]}{\overset{\mathit{Realization}}{\rightleftarrows}} \text{Boole algebras}$$

For a space X, $\mathbf{O}(X)$ is the Boole algebra of clopen of X.

For a frame F, $\mathit{Realization}(F)$ is the corresponding Stone space.

This last functor is fully faithful.

The space/algebra adjunction

In algebraic geometry, we get the usual adjunction

$$Scheme^{op} \underset{\mathbf{O}(-)}{\overset{Realization}{\rightleftarrows}} CommRing$$

Where the functor *Realization* is the Zariski spectrum, which is fully faithful.

Its image is spanned by affine schemes.

The endofunctor $Scheme \to Scheme$ is the "affine reflection".

The space/algebra adjunction

$$Top^{op} \underset{\mathbf{O}(-)}{\overset{Realization}{\leftrightarrows}} \mathbb{P}_{\mathbf{A}}\text{-Alg}$$

An object X of *Top* is called

- **A**-separated if $X \to Realization\ \mathbf{O}(X)$ is an embedding.
- **A**-proper if $X \to Realization\ \mathbf{O}(X)$ is an isomorphism.

It happens sometimes that *Realization* $\mathbf{O}$ is a closure operator (=idempotent).

In examples, $X \to Realization\ \mathbf{O}(X)$ corresponds to some sort of compactification.

The space/algebra adjunction

Gauge space **A**	*Algebraic structure*	*Separated spaces*	*Proper spaces*	*Completion*
Boole space $\mathbb{B}$	Boolean algebras	Discon-nected spaces	Stone spaces	"Stone-ification?"
Sierpiǹski space $\mathbb{S}$	Frames	T_0-spaces	Sober spaces	Soberi-fication
$[0,1]$	C*-algebras	Regular Hausdorff	Compact Hausdorff	Stone-Čech compacti-fication?
$\mathbb{R}$	C^0-rings	?	?	?

The space/algebra duality

Topology	*Algebra*	*Gauge*
Stone spaces	Boole algebras	$\mathbb{B}$
Compact Hausdorff spaces	C^*-algebras	$[0,1]$
Locales	Frames	$\mathbb{S}$
Affine scheme	Comm. rings	$\mathbb{A}^1$

Geometry v. algebra

I have explained why the study of spaces leads naturally to some algebraic objects.

But the correspondance is not perfect.

The category Top^{op} is not a category of algebras of some kind.

So we are facing a choice:

- keep topological spaces
- or prefer the algebraic side.

The second choice is better.

It frees the notion of space from the narrow view that are topological spaces.

Geometry v. algebra

Why prefer an algebraic point of view on space ?

- perfect duality between algebraic and geometric objects
- nicer category (monadic, accessible)
- distinguished class of objects (affine spaces/free algebras)
- algebraic toolbox
  - presentations (generators and relations)
  - subspaces = quotients = ideals (differential calculus)
  - modules...
- and most of all nothing is lost:

the whole of the category *Top* can be reconstructed by algebraic means, though not in a direct way.

Locales v. Topological spaces

A locale is an object in the category Frame^{op}.

A morphism of locales is a morphism in Frame^{op}.

The category of locales is defined as

$$\mathrm{Locale} = \mathrm{Frame}^{op}.$$

Locales v. Topological spaces

I'm not going to develop the theory of locales.

For an explanation on why locales are geometric objects, see my paper Topo-logie (2019, Anel-Joyal)

I just want to explain how the (1-)category of all topological spaces can be reconstructed from the (2-)category Locale.

Locales v. Topological spaces

The adjunction

$$\mathit{Top}^{op} \underset{O(-)}{\overset{\mathit{Realization}}{\rightleftarrows}} \text{Frame}$$

provide an adjunction, and two idempotent endofunctor

$$\mathit{soberification} \circlearrowright \mathit{Top} \underset{\rho}{\overset{\iota}{\rightleftarrows}} \text{Locale} \circlearrowleft \mathit{spatialization}$$

which induces an equivalence between the subcategories of

$$\text{Sober spaces} \simeq \text{Locales with enough points}$$

From this point of view, the theory of topological spaces seems to have some residue (non-sober spaces) invisible from locale theory.

This is because it is the wrong comparison between the two notions.

Locales v. Topological spaces

A locale is discrete if the dual frame is of the type $P(E)$ for some set E.

A topological space X is

- a set E
- a subframe $O(X) \subset P(E)$

The inclusion $O(X) \subset P(E)$ is a morphism of frame.

Dually, it corresponds to a surjection of locales

$$E \twoheadrightarrow X$$

where E is a discrete locale.

Locales v. Topological spaces

Proposition

The 1-category of all *topological spaces is equivalent to the full sub 2-category of Locale*$^{\rightarrow}$ *(morphisms of locales) spanned by surjections with discrete domain.*

In other words, a topological space is a locale with the extra-structure of a set of points.

From this point of view, the functor

$$Top \xrightarrow{\iota} \text{Locale}$$

is simply the functor forgetting the set of points.

Locales v. Topological spaces

Morale 1: nothing is lost by replacing topological spaces with locales.

Morale 2: Locales elucidate the nature of topological spaces.

Morale 3: Locales do provide a setting simpler, more general and more powerful than topological spaces.

More degrees of separation

So far, I should have convinced you that the encoding of space in terms of algebraic structure is both

- meaningful (separation of the space)
- natural (algebraic structure of the gauge space)
- and faithful (encompasses most approaches to topology).

I'm going to defined the notion of topos as dual to the algebraic notion of logos.

But before, I will explain why the problem of separating spaces is the source (*) of the notion of topoi.

(*) I am talking about the conceptual source, not the historical source.

More degrees of separation

T_0-spaces are those that can be studied by means of functions in the Sierpiǹski space $\mathbb{S}$.

Classically, T_0 is the lowest of degrees of separation.

Problem: many interesting spaces are below T_0!

1. "bad" quotients ($\mathbb{R}/\mathbb{Z} \oplus \alpha\mathbb{Z}$ for $\alpha \notin \mathbb{Q}$)
2. foliation spaces ($\mathbb{T}^2/\mathbb{R}_\alpha$)
3. orbifolds ($\mathbb{R}^2/\ \mathbb{Z}/n\mathbb{Z}$)
4. moduli spaces (curves, bundles...)
5. the space of sets (sheaf = continuous family of sets)
6. the space of models of a logical theory
7. the space of ∞-groupoids (stack = cont. fam. of ∞-gpd)
8. ...

Need other gauge spaces relative to which these spaces would be separated. This gauge cannot be a T_0-space. It has to be a one of the sub-T_0-space.

More degrees of separation

Why is the space of sets not separated?

And why is there even a space of sets?

The intuition that there is a topology on the "set of sets" comes from the idea that a sheaf on a space X is a continuous family of sets.

If there was a space of sets $\mathbb{A}$, then a sheaf F on X should be a continuous map $F : X \to \mathbb{A}$.

More degrees of separation

If $X = \$ = \{0 < 1\}$ is the Sierpiǹski space, a sheaf on $\$$ is a map of sets $E_0 \to E_1$.

This is the same thing as a functor

$$Pt(\$) = \{0 \to 1\} \quad \longrightarrow \quad Set.$$

This suggest to see the maps between sets as specialization morphisms.

More degrees of separation

The new idea is to look as $\mathbb{A}$ has having a category of points and not only a set.

$$Pt(\mathbb{A}) \quad = \quad Set$$

It is in this sense that the space $\mathbb{A}$ is not T_0.

I call T_{-n} a space whose points form an n-category.

More degrees of separation

From the perspective of classical spaces (even non-T_0), the separation structure of $\mathbb{A}$ is a nightmare:

- points can be specialized in several ways,
- two points can be specialization of each other,
- and a point can even be its own specialization!

Topos

But, precisely because the separation structure of $\mathbb{A}$ is so bad, many badly separated spaces will be nicely $\mathbb{A}$-separated!

This is the intuition of what a topos is.

A *topos* is the kind of space which can be *separated from the gauge* $\mathbb{A}$.

But what is the corresponding algebraic structure?

Topos

Since the object $\mathbb{A}$ is not yet constructed, it is not clear how to compute its natural algebraic structure.

But I have mentioned what a continuous map $X \to \mathbb{A}$ should be: a sheaf on X.

We can use this to define $\mathbb{A}$ as a functor

$$\begin{array}{rcll} \mathbb{A} : Top^{op} & \longrightarrow & CAT & \\ X & \longmapsto & Sh(X) & \text{(cat. of sheaves on } X\text{)} \\ f : X \to Y & \longmapsto & f^* : Sh(Y) \to Sh(X) & \text{(p.b. of sheaves)} \end{array}$$

Topos

The 2-category $[\,Top^{op}, CAT\,]$ provide a category where the object $\mathbb{A}$ exists.

We can look at the theory generated by $\mathbb{A}$ inside $[\,Top^{op}, CAT\,]$.

A classical theory would look for operations $\mathbb{A}^N \to \mathbb{A}$ for N a set (finite or not).

But we are in a 2-category, cotensored over the category Cat of small categories,

so it is more natural to look at operations $\mathbb{A}^C \to \mathbb{A}$ indexed by a small category C (finite or not), i.e. operations whose arities are small categories.

Topos

For C a category, $\mathbb{A}^C$ is the functor sending a space X to the category $Sh(X)^C$ of C-diagrams of sheaves.

A natural transformation $\mathbb{A}^C \to \mathbb{A}$ is a functor

$$Sh(X)^C \longrightarrow Sh(X)$$

which is natural on X.

The simplest of such operations that can be considered are limits and colimits.

Topos

For a continuous map $f : X \to Y$, it is a fact that the functor $f^* : Sh(Y) \to Sh(X)$ preserves in general all small colimits but only finite limits.

This is a 1-categorical analog of the property that pulling back open subsets preserves all union but only finite intersections.

The algebraic structure of $\mathbb{A}$ is defined by the natural transformations $\mathbb{A}^N \to \mathbb{A}$ that can be obtained by composing small colimits and finite limits.

I call logos this algebraic structure.

Topos

By definition, the structure of logos is generated by the operations of colimits and finite limits in categories of sheaves, but it is not obvious what are the relations between these operations.

It should be given by a distributivity relation.

Ideally, one would like a distributivity relation between colimits and finite limits

$$\lim_{i \in I} \operatorname{colim}_{j \in J} X_{i,j} \quad = \quad \operatorname{colim}_{k : I \to J} \lim_{i} X_{i,k(i)}$$

But I do not know of such formula.

Another way is to consider not (co)limits of diagrams, but the monad completing a category for (co)limits.

Topos

Let $C \to C^{lex}$ the free completion for finite limits, a category is lex (has finite limits) iff the functor $C \to C^{lex}$ has a right adjoint.

Let $C \to P(C)$ the free completion for small colimits, a category is cocomplete iff the functor $C \to P(C)$ has a left adjoint. If C is small $P(C) = [C^{op}, Set]$.

These two monads on CAT have a Beck distributivity relation (see Garner-Lack "lex colimits"): if D is a lex category, then $P(D)$ is also a lex category, i.e. there exists a natural lex functor

$$P(D)^{lex} \to P(D).$$

Topos

The free logos $S[C]$ on a category C is defined to be

$$S[C] := P\left(C^{lex}\right)$$

Because of Beck distributivity, the functor

$$\begin{array}{ccc} CAT & \longrightarrow & CAT \\ C & \longmapsto & S[C] \end{array}$$

is a monad, and its algebras are the logoi.

Topos

Any object of $S[C] = P\left(C^{lex}\right)$ defines a natural transformation $\mathbb{A}^C \to \mathbb{A}$:

$$\alpha : S[C] \times Sh(X)^C \quad \longrightarrow \quad Sh(X)$$

If $F \in C^{lex}$, there exists a finite diagram $x : I \to C$ such that $F = \lim x(i)$. Then, for any $E : C \to Sh(X)$

$$\alpha(F, E) \quad := \quad \lim_i E(x(i))$$

If $F \in P(()C^{lex})$, there exists a small diagram $y : J \to C^{lex}$ such that $F = \operatorname{colim} y(j)$. Then, for any $E : C \to Sh(X)$

$$\alpha(F, E) \quad := \quad \operatorname*{colim}_i \; \alpha(y(j), E \circ y)$$

Topos

Finally, the theory of logoi is defined by

$$\mathbb{P}_{Logos}(C) \;:=\; S[C] \quad \subset \quad Nat(\mathbb{A}^C, \mathbb{A}) \;=:\; \mathbb{M}_{Logos}(C)$$

I will not detail here the theory of logos and topos, nor their higher analogs.

I refer to my paper Topo-logie (2019, Anel-Joyal).

The following table summarizes most of what I have said so far.

Summary

Gauge space **A**	*Functions to* **A**	*Algebraic structure*	*Space*	*Degree of separation*
Boole space $\mathbb{B}$	Clopens	Boolean algebra	Stone space	$T_{>2}$
Interval $[0,1]$	Bounded real functions	$C^\star$-algebras	Compact Hausdorff	T_2
Sierpiǹski space $\mathbb{S}$	Open subsets	Frames	Locales	T_0
Space of sets $\mathbb{A}$	Étale maps (sheaves)	Logos	Topos	T_{-1}
Space of ∞-gpd $\mathbb{A}_\infty$	∞-Étale maps (∞-sheaves)	∞-Logos	∞-Topos	$T_{-\infty}$

The geography of spaces

One of the points I want to make in this talk, is that the notion of space is diverse.

Any choice of a gauge space provide a spatial notion.

Classically, all the notion of spaces are reduced to the central notion of topological space.

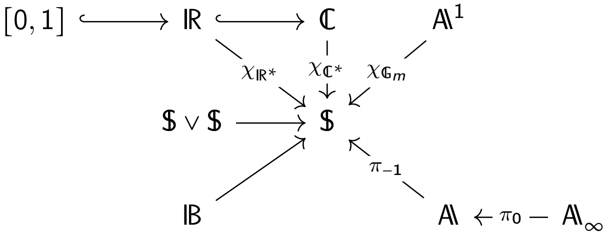
I think this is a mistake:

1. some features about classical spaces are better understood when several gauge space and the corresponding categories of spaces are introduced,
2. such a view articulates topological spaces better to algebraic geometry (via commutative rings) and differential geomtryc (via C^{∞}-rings),
3. and some spaces (like topoi) are outside the reach of locales.

The geography of spaces

But there is a very good reason for the central situation of topological space (or locales):

in practice, gauge spaces have always a canonical morphism to the Sierpiǹski space.



This is why every notion of space has an underlying locale.

The geography of spaces

The "same" object can be equipped with several spatial structures:

1. $\mathbb{R}_{C^0} = \mathbb{R}$ viewed with its continuous structure.
2. $\mathbb{R}_{C^\infty} = \mathbb{R}$ viewed with its differentiable structure.
3. $\mathbb{R}_{\mathrm{alg}} = \mathbb{R}$ viewed with its algebraic structure (only polynomial and rational functions).
4. $\mathbb{R}_{\mathrm{meas}} = \mathbb{R}$ viewed with its (Borel) measurable structure.
5. $\mathbb{R}_{dis} = \mathbb{R}$ viewed with its discrete structure.

In fact they should not be looked as the same object at all!

They rather are subobjects of one another.

The geography of spaces

These structures compare the following way

$$\mathbb{R}_{\mathrm{alg}} \quad \subset \quad \mathbb{R}_{C^\infty} \quad \subset \quad \mathbb{R}_{C^0} \quad \subset \quad \mathbb{R}_{\mathrm{meas}} \quad \subset \quad \mathbb{R}_{dis}.$$

Any rational function is differentiable. Any differentiable function is continuous. Any continuous function is measurable. Any measurable function is a function.

This recovers the different kinds of geometries: algebraic geometry, differential geometry, topology, measure theory...

From this point of view, the classical difference (since Riemann) between topology and geometry, or between the different kinds of geometries is very thin. All can be presented in a common framework.

Thanks!