Enriching algebras over coalgebras and operads over cooperads

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Foreword

This is a work in progress with A. Joyal.

We are trying to understand Koszul duality from a conceptual point of view.

We still don't understand Koszul duality, but we discovered some category theory underlying the bar and cobar constructions.

Main theorem

Let $(\mathbf{V}, \otimes, \mathbf{1}, [-,-])$ be a symmetric monoidal closed locally presentable category and let P be a cocommutative Hopf colored operad in $\mathbf{V}$.

Theorem (A-J)

1. *The category* P-**Coalg** *is symmetric monoidal closed.*
2. *The category* P-**Alg** *is enriched, tensored, cotensored and symmetric monoidal over* P-**Coalg**.

Corollary

Let $P = As$ *the associative operad.*

1. *The category* **Coalg** *of coassociative coalgebras is symmetric monoidal closed.*
2. *The category* **Alg** *of associative algebras is enriched, tensored, cotensored and symmetric monoidal over* **Coalg**.

Main theorem

Corollary

Let $P = \mathbf{K}$ a category (in **Set***).*

1. *The category of functors $[\mathbf{K}^{op}, \mathbf{V}]$ is symmetric monoidal closed.*
2. *The category of functors $[\mathbf{K}, \mathbf{V}]$ is enriched, tensored, cotensored and symmetric monoidal over $[\mathbf{K}^{op}, \mathbf{V}]$.*

Corollary

Let $P = OP$ be the operad of K-colored operads.

1. *The category $\mathbf{coOp}(K)$ of K-colored cooperads is symmetric monoidal closed.*
2. *The category $\mathbf{Op}(K)$ of K-colored operads is enriched, tensored, cotensored and symmetric monoidal over $\mathbf{coOp}(K)$.*

Part I - Hopf operads

Colored operad

Let K be a set (could be a category). We put $S(K)$ for the free symmetric monoidal category on K.
Let $(\mathbf{V}, \otimes)$ be a symmetric monoidal category.

A *K-colored operad* P in $\mathbf{V}$ is the data of a functor

$$P : S(K)^{op} \times K \to \mathbf{V}$$

which is a monoid for the substitution monoidal structure

$$P \circ P \to P \qquad \text{and} \qquad I \to P.$$

Colored operad

Concretly, this amounts to the data of

- objects

$$P_k^{\overline{k}} = P_k^{k_1,\dots,k_n} \in \mathbf{V}$$

  (where the k, k_i are in K)

- actions of symmetric groups related to repetition of elements in $\overline{k}$
- and maps

$$P_k^{k_1,\dots,k_n} \otimes P_{k_1}^{\overline{\ell_1}} \otimes \cdots \otimes P_{k_n}^{\overline{\ell_n}} \longrightarrow P_k^{\overline{\ell_1} \oplus \cdots \oplus \overline{\ell_n}}$$

$$\mathbf{1} \to P_k^k$$

satisfying associativity and unitality conditions.

Colored operad - examples

- If $P[n]$ is a unisorted operad (Associative, Commutative, Poisson, Lie, L_∞, A_∞...)
  we put $K = \{*\}$ and

$$P_*^{\overbrace{*,\dots,*}^{n \text{ times}}} := P[n]$$

- If B is an associative algebra, we put $K = \{*\}$,

$$P_*^* := B$$

  and all Ps are other 0.
- If $\mathbf{K}$ is a category, we put $K = ob(\mathbf{K})$,

$$P_{k'}^k := \mathbf{K}(k, k')$$

  and all other Ps are 0.

Colored P-algebra

For a covariant functor $A : K \to \mathbf{V}$ we shall denote the value at $k \in K$ by A_k.

For $\overline{k} = (k_1, \dots, k_n)$ we put $A_{\overline{k}} = A_{k_1} \otimes \cdots \otimes A_{k_n}$.

Let P a K-colored operad.
A P-algebra is a functor $A : K \to \mathbf{V}$ together with maps

$$P_k^{\overline{k}} \otimes A_{\overline{k}} \to A_k$$

satisfying associativity and unitality conditions.

Colored P-algebras - examples

- If P is a unisorted operad, an algebra A is a unisorted P-algebra
$$P[n] \otimes A^{\otimes n} \longrightarrow A$$
- If $P = B$ is an associative algebra, an algebra A is a left module
$$B \otimes A \longrightarrow A$$
- If $P = \mathbf{K}$ is a category, an algebra A is a covariant functor $\mathbf{K} \to \mathbf{V}$.

Colored P-coalgebra

For a contravariant functor $C : K^{op} \to \mathbf{V}$ we shall denote the value at $k \in K$ by C^k

For $\overline{k} = (k_1, \dots, k_n)$ we put $C^{\overline{k}} = C^{k_1} \otimes \cdots \otimes C^{k_n}$.

Let P a K-colored operad.
A P-coalgebra is a functor $C : K^{op} \to \mathbf{V}$ together with maps

$$P_k^{\overline{k}} \otimes C^k \to C^{\overline{k}}$$

satisfying coassociativity and counitality conditions.

Colored P-algebras - examples

- If P is a unisorted operad, a coalgebra C is a unisorted P-coalgebra
$$P[n] \otimes C \longrightarrow C^{\otimes n}$$
- If $P = B$ is an associative algebra, a coalgebra C is a right module.
$$B \otimes C \longrightarrow C$$
- If $P = \mathbf{K}$ is a category, a coalgebra C is a contravariant functor $\mathbf{K}^{op} \to \mathbf{V}$.

Hadamard product

If P and Q are two K-colored operad their Hadamard product of $P \otimes Q$ is defined by

$$(P \otimes Q)_k^{\overline{k}} := P_k^{\overline{k}} \otimes Q_k^{\overline{k}}$$

This is again an operad:

$$\left(P_k^{\overline{k}} \otimes Q_k^{\overline{k}}\right) \otimes \left(P_{k_1}^{\overline{\ell_1}} \otimes Q_{k_1}^{\overline{\ell_1}}\right) \otimes \cdots \otimes \left(P_{k_n}^{\overline{\ell_1}} \otimes Q_{k_n}^{\overline{\ell_n}}\right)$$

$$= \left(P_k^{\overline{k}} \otimes P_{k_1}^{\overline{\ell_1}} \otimes \cdots \otimes P_{k_n}^{\overline{\ell_n}}\right) \otimes \left(Q_k^{\overline{k}} \otimes Q_{k_1}^{\overline{\ell_1}} \otimes \cdots \otimes Q_{k_n}^{\overline{\ell_n}}\right)$$

$$\longrightarrow P_k^{\overline{\ell_1} \oplus \cdots \oplus \overline{\ell_n}} \otimes Q_k^{\overline{\ell_1} \oplus \cdots \oplus \overline{\ell_n}}$$

Hopf operad

The category **Op**(K) of K-colored operad is symmetric monoidal for the Hadamard product.

A (cocommutative) Hopf operad is an operad which is a cocommutative comonoid for the Hadamard product.

Equivalently, this says that all $P_k^{\overline{k}}$ are cocommutative comonoids and that the compositions an unit maps are coalgebra maps.

Examples:

- all operads in **Set** (Associative, Commutative, any category, the operad of K-colored operads, ...)
- all operads in **Top** (E_n, John's *Phyl*...)
- the Poisson operad
- any cocommutative bialgebra

(co)algebras over Hopf operad

Let P be a Hopf operad.

If A and B are P-algebras, their Hadamard product $A \otimes B$ is defined by

$$(A \otimes B)_k := A_k \otimes B_k$$

it is again a P-algebra.

$$P_k^{\bar{k}} \otimes A_k \otimes B_k \longrightarrow P_k^{\bar{k}} \otimes P_k^{\bar{k}} \otimes A_k \otimes B_k =$$

$$P_k^{\bar{k}} \otimes A_k \otimes P_k^{\bar{k}} \otimes B_k \longrightarrow A^{\bar{k}} \otimes B^{\bar{k}} = (A \otimes B)^{\bar{k}}$$

Similarly, if C and D are P-coalgebras, their Hadamard product $C \otimes D$ defined by

$$(C \otimes D)^k := C^k \otimes D^k$$

is again a P-coalgebra.

Part II - SWEEDLER THEORY

Sweedler theory

Let P be a colored operad in a symmetric monoidal closed locally presentable category $\mathbf{V}$.

Let P-**Alg** and P-**Coalg** be the categories of P-algebras and of P-coalgebras.

Theorem (folklore)

1. *P-**Alg** and P-**Coalg** are locally presentable.*
2. *There exists a monadic adjunction*

$$U : P\text{-}\mathbf{Alg} \rightleftarrows \mathbf{V}^K : P.$$

3. *There exists a comonadic adjunction*

$$P^\vee : \mathbf{V}^K \rightleftarrows P\text{-}\mathbf{Coalg} : U.$$

$P^\vee$ is not an analytic comonad (cooperad), hence difficult to describe explicitly.

Sweedler theory of a Hopf operad

Let P be a colored Hopf operad, there exists six functors

tensor product	$\otimes$	:	P-**Coalg** $\times$ P-**Coalg** $\to$ P-**Coalg**
internal hom	$\textsc{Hom}$	:	P-**Coalg**^{op} $\times$ P-**Coalg** $\to$ P-**Coalg**
Sweedler hom	$\{-,-\}$	:	P-**Alg**^{op} $\times$ P-**Alg** $\to$ P-**Coalg**
Sweedler product	$\rhd$	:	P-**Coalg** $\times$ P-**Alg** $\to$ P-**Alg**
convolution	$[-,-]$	:	P-**Coalg**^{op} $\times$ P-**Alg** $\to$ P-**Alg**
tensor product	$\otimes$	:	P-**Alg** $\times$ P-**Alg** $\to$ P-**Alg**

such that

Theorem (A-J)

1. (P-**Coalg**, $\otimes$, $\textsc{Hom}$) *is symmetric monoidal closed.*
2. (P-**Alg**, $\{-,-\}$, $\rhd$, $[-,-]$, $\otimes$) *is enriched, tensored, cotensored and symmetric monoidal over* **Coalg**.

Sweedler theory of the associative operad

For $P = As$ the associative operad, there exists six functors

tensor product	$\otimes$	:	$\mathbf{Coalg} \times \mathbf{Coalg} \to \mathbf{Coalg}$
internal hom	HOM	:	$\mathbf{Coalg}^{op} \times \mathbf{Coalg} \to \mathbf{Coalg}$
Sweedler hom	$\{-,-\}$	:	$\mathbf{Alg}^{op} \times \mathbf{Alg} \to \mathbf{Coalg}$
Sweedler product	$\triangleright$	:	$\mathbf{Coalg} \times \mathbf{Alg} \to \mathbf{Alg}$
convolution	$[-,-]$	:	$\mathbf{Coalg}^{op} \times \mathbf{Alg} \to \mathbf{Alg}$
tensor product	$\otimes$	:	$\mathbf{Alg} \times \mathbf{Alg} \to \mathbf{Alg}$

such that

Theorem

(Porst) $(\mathbf{Coalg}, \otimes, \mathrm{HOM})$ *is symmetric monoidal closed.*

(A-J) $(\mathbf{Alg}, \{-,-\}, \triangleright, [-,-], \otimes)$ *is enriched, tensored, cotensored and symmetric monoidal over* **Coalg**.

Sweedler theory of the associative operad

If we choose $(\mathbf{V}, \otimes) = (\mathbf{Set}, \times)$, then P-$\mathbf{Alg} = Mon$ and P-$\mathbf{Coalg} = \mathbf{Set}$. and the enrichment is trivial.

If we choose $(\mathbf{V}, \otimes) = (\mathbf{Vect}, \otimes)$, then the enrichment is not trivial.

$P^\vee = T^\vee$ is the cofree coalgebra functor (much bigger than the tensor coalgebra).

$\textsc{Hom}$ and $\{-,-\}$ do not have a simple presentation but

$$\textsc{Hom}(C, T^\vee(X)) = T^\vee([C, X])$$

$$\{T(X), A\} = T^\vee([X, A]).$$

Sweedler theory of the associative operad

An atom of a coalgebra C is an element e such that $\Delta(e) = e \otimes e$ and $\epsilon(e = 1)$
A primitive element u of C with respect to some atom e is an element e such that $\Delta(u) = u \otimes e + e \otimes u$

Proposition

- $atom(\mathrm{Hom}(C, D)) = hom(C, D)$
- $prim_f(\mathrm{Hom}(C, D)) = Coder_f(C, D)$
- $atom(\{A, B\}) = hom(A, B)$
- $prim_f(\{A, B\}) = Der_f(A, B)$

Sweedler theory of the associative operad

The operation $[-,-]$ is the convolution algebra.

If C is a coalgebra and A an algebra, $[C,A]$ is an algebra for the product

$$[C,A]\otimes[C,A]\xrightarrow{can}[C\otimes C,A\otimes A]\xrightarrow{[\Delta,m]}[C,A].$$

A map $C\otimes A\to B$ in $\mathbf{V}$ is called a measuring if the corresponding map $A\to[C,B]$ is an algebra map.

Sweedler theory of the associative operad

$\mu : C \otimes A \to B$ is a measuring iff the following diagram commutes

$$\begin{array}{ccccc}
C \otimes A \otimes A & \xrightarrow{\Delta_C \otimes A^2} & C \otimes C \otimes A \otimes A & \xrightarrow{\simeq} & C \otimes A \otimes C \otimes A \\
\Big\downarrow{\scriptstyle C \otimes m_A} & & & & \Big\downarrow{\scriptstyle \mu \otimes \mu} \\
 & & & & B \otimes B \\
 & & & & \Big\downarrow{\scriptstyle m_B} \\
C \otimes A & & \xrightarrow{\qquad\qquad \mu \qquad\qquad} & & B
\end{array}$$

In terms of elements, this gives the formula in B

$$\mu(c, aa') = \sum \mu(c^{(1)}, a)\mu(c^{(2)}, a')$$

(where $\Delta(c) = \sum c^{(1)} \otimes c^{(2)}$)

Sweedler theory of the associative operad

The algebra $C \triangleright A$ can be defined as the quotient of $T(C \otimes A)$ given by coequalizing the two sides of

$$
\begin{array}{ccccc}
C \otimes A \otimes A & \xrightarrow{\Delta_C \otimes A^2} & C \otimes C \otimes A \otimes A & \xrightarrow{\simeq} & C \otimes A \otimes C \otimes A \\
\Big\downarrow {\scriptstyle C \otimes m_A} & & & & \Big\downarrow {\scriptstyle \iota \otimes \iota} \\
 & & & & T(C \otimes A) \otimes T(C \otimes A) \\
 & & & & \Big\downarrow {\scriptstyle m} \\
C \otimes A & & \xrightarrow{\qquad \iota \qquad} & & T(C \otimes A) \\
 & & & & \vdots \downarrow \\
 & & & & C \triangleright A
\end{array}
$$

In particular we have

$$C \triangleright T(X) = T(C \otimes X).$$

Sweedler theory of the associative operad

Let C be a coalgebra and A, B be two algebras, we have bijection between the following sets

measurings	$C \otimes A \to B$
algebra maps	$A \to [C, B]$
algebra maps	$C \triangleright A \to B$
coalgebra maps	$C \to \{A, B\}$.

Sweedler theory of the associative operad

Let C be a coalgebra and A an algebra,

we deduce three kinds of adjunctions

$$\text{type I} \qquad C \triangleright - : \mathbf{Alg} \rightleftarrows \mathbf{Alg} : [C, -]$$

$$\text{type II} \qquad [-, A] : \mathbf{Coalg} \rightleftarrows \mathbf{Alg}^{op} : \{-, A\}$$

$$\text{type III} \qquad - \triangleright A : \mathbf{Coalg} \rightleftarrows \mathbf{Alg} : \{A, -\}$$

Sweedler theory of the associative operad

Type I adjunctions are quite frequent: if $\mathbf{V} = \mathbf{Vect}$

- E finite algebra, $E^\star \rhd -$ is left adjoint to $E \otimes -$,
- $C = k \oplus k\delta$ with $\Delta(\delta) = \delta \otimes 1 + 1 \otimes \delta$
  $[C, A] = A[\epsilon]$ and $C \rhd A = T_A(\Omega_A)$,
- $C = T^c(x)$ (tensor coalgebra)
  $[C, A] = A[t]$ and $C \rhd A = J(A)$ (jet ring of A).

Type II encompasses Sweedler duality: if $\mathbf{V} = \mathbf{Vect}$ and $A = k$, we have bijection between

algebra maps $\quad B \to C^\star = [C, k]$

and coalgebra maps $\quad C \to B^\circ = \{B, k\}$.

Type III encompasses the bar-cobar constructions (if $\mathbf{V} = \mathbf{dgVect}$).

Back to the general theory

The six Sweedler operations of a Hopf operad P:

$$
\begin{array}{rcl}
\otimes & : & P\text{-}\mathbf{Coalg} \times P\text{-}\mathbf{Coalg} \to P\text{-}\mathbf{Coalg} \\
\textsc{Hom} & : & P\text{-}\mathbf{Coalg}^{op} \times P\text{-}\mathbf{Coalg} \to P\text{-}\mathbf{Coalg} \\
\{-,-\} & : & P\text{-}\mathbf{Alg}^{op} \times P\text{-}\mathbf{Alg} \to P\text{-}\mathbf{Coalg} \\
\triangleright & : & P\text{-}\mathbf{Coalg} \times P\text{-}\mathbf{Alg} \to P\text{-}\mathbf{Alg} \\
[-,-] & : & P\text{-}\mathbf{Coalg}^{op} \times P\text{-}\mathbf{Alg} \to P\text{-}\mathbf{Alg} \\
\otimes & : & P\text{-}\mathbf{Alg} \times P\text{-}\mathbf{Alg} \to P\text{-}\mathbf{Alg}
\end{array}
$$

Back to the general theory

The tensor products are computed termwise (Hadamard).

So is the convolution algebra: for C a P-coalgebra and A a P-algebra, we have

$$[C, A]_k = [C^k, A_k].$$

This is a P-algebra for the product

$$P_k^{\overline{k}} \otimes [C, A]_{\overline{k}} \longrightarrow P_k^{\overline{k}} \otimes P_k^{\overline{k}} \otimes [C^{\overline{k}}, A_{\overline{k}}] \longrightarrow$$

$$[C^k, C^{\overline{k}}] \otimes [C^{\overline{k}}, P_k^{\overline{k}} \otimes A_{\overline{k}}] \longrightarrow [C^k, A_k]$$

A map $C \otimes A \to B$ in $\mathbf{V}^K$ is called a measuring if the corresponding map $A \to [C, B]$ is a P-algebra map.

Back to the general theory

For associative algebras $\mu : C \otimes A \to B$ is a measuring iff the following diagram commutes

$$\begin{array}{ccccc}
C \otimes A \otimes A & \xrightarrow{\Delta_C \otimes A^2} & C \otimes C \otimes A \otimes A & \xrightarrow{\simeq} & C \otimes A \otimes C \otimes A \\
\downarrow {\scriptstyle C \otimes m_A} & & & & \downarrow {\scriptstyle \mu \otimes \mu} \\
 & & & & B \otimes B \\
 & & & & \downarrow {\scriptstyle m_B} \\
C \otimes A & & \xrightarrow{\qquad \mu \qquad} & & B
\end{array}$$

Back to the general theory

$\mu : C \otimes A \to B$ is a measuring iff the following diagram commutes

$$
\begin{array}{ccccc}
P_k^{\overline{k}} \otimes C^k \otimes A_{\overline{k}} & \xrightarrow{\Delta_P} & P_k^{\overline{k}} \otimes P_k^{\overline{k}} \otimes C^k \otimes A_{\overline{k}} & \xrightarrow{\simeq} & P_k^{\overline{k}} \otimes C^k \otimes P_k^{\overline{k}} \otimes A_{\overline{k}} \\
 & & & & \downarrow{\scriptstyle \Delta_C} \\
 & & & & C^{\overline{k}} \otimes P_k^{\overline{k}} \otimes A_{\overline{k}} \\
 & & & & \downarrow{\scriptstyle \simeq} \\
\downarrow{\scriptstyle m_A} & & & & P_k^{\overline{k}} \otimes C^{\overline{k}} \otimes A_{\overline{k}} \\
 & & & & \downarrow{\scriptstyle \mu^{\otimes n}} \\
 & & & & P_k^{\overline{k}} \otimes B^{\overline{k}} \\
 & & & & \downarrow{\scriptstyle m_B} \\
C^k \otimes A_k & & \xrightarrow{\qquad\qquad\mu\qquad\qquad} & & B_k
\end{array}
$$

Back to the general theory

The P-algebra $C \triangleright A$ can be defined as the quotient of $P(C \otimes A)$ given by coequalizing the two sides of

$$
\begin{array}{ccccc}
P_k^{\overline{k}} \otimes C^k \otimes A_{\overline{k}} & \longrightarrow & P_k^{\overline{k}} \otimes P_k^{\overline{k}} \otimes C^k \otimes A_{\overline{k}} & \xrightarrow{\simeq} & P_k^{\overline{k}} \otimes C^k \otimes P_k^{\overline{k}} \otimes A_{\overline{k}} \\
\Big\downarrow & & & & \Big\downarrow \\
 & & & & C^{\overline{k}} \otimes P_k^{\overline{k}} \otimes A_{\overline{k}} \\
 & & & & \Big\downarrow \simeq \\
 & & & & P_k^{\overline{k}} \otimes C^{\overline{k}} \otimes A_{\overline{k}} \\
 & & & & \Big\downarrow \\
 & & & & P_k^{\overline{k}} \otimes P(C \otimes A)_{\overline{k}} \\
 & & & & \Big\downarrow \\
C^k \otimes A_k & & \longrightarrow & & P(C \otimes A)_k .
\end{array}
$$

Sweedler theory of a category $\mathbf{K}$

For $P = \mathbf{K}$ a category with set of objects K, we have

$$P\text{-}\mathbf{Alg} = [\mathbf{K}, \mathbf{V}] \qquad \text{and} \qquad P\text{-}\mathbf{Coalg} = [\mathbf{K}^{op}, \mathbf{V}].$$

There exists six functors

$$\begin{array}{ccl} \otimes & : & [\mathbf{K}^{op}, \mathbf{V}] \times [\mathbf{K}^{op}, \mathbf{V}] \to [\mathbf{K}^{op}, \mathbf{V}] \\ \textsc{Hom} & : & [\mathbf{K}^{op}, \mathbf{V}]^{op} \times [\mathbf{K}^{op}, \mathbf{V}] \to [\mathbf{K}^{op}, \mathbf{V}] \\ \{-,-\} & : & [\mathbf{K}, \mathbf{V}]^{op} \times [\mathbf{K}, \mathbf{V}] \to [\mathbf{K}^{op}, \mathbf{V}] \\ \rhd & : & [\mathbf{K}^{op}, \mathbf{V}] \times [\mathbf{K}, \mathbf{V}] \to [\mathbf{K}, \mathbf{V}] \\ [-,-] & : & [\mathbf{K}^{op}, \mathbf{V}]^{op} \times [\mathbf{K}, \mathbf{V}] \to [\mathbf{K}, \mathbf{V}] \\ \otimes & : & [\mathbf{K}, \mathbf{V}] \times [\mathbf{K}, \mathbf{V}] \to [\mathbf{K}, \mathbf{V}] \end{array}$$

By symmetry between $\mathbf{K}$ and $\mathbf{K}^{op}$ we have

Theorem (?)

1. $[\mathbf{K}, \mathbf{V}]$ *and* $[\mathbf{K}^{op}, \mathbf{V}]$ *are symmetric monoidal closed*
2. *and are enriched, tensored and cotensored over each other.*

Sweedler theory of a category **K**

For $A, B : \mathbf{K} \longrightarrow \mathbf{V}$ and $C, D : \mathbf{K}^{op} \longrightarrow \mathbf{V}$ we have:

$$
\begin{aligned}
(C \otimes D)^k &= C^k \otimes D^k \\
\mathrm{H\small{OM}}(C, D)^k &= \int_{k' \in k/(\mathbf{K}^{op})} [C^{k'}, D^{k'}] \\
\{A, B\}^k &= \int_{k' \in \mathbf{K}/k} [A_{k'}, B_{k'}] \\
(C \triangleright A)_k &= \int^{k' \in \mathbf{K}/k} C^{k'} \otimes A_{k'} \\
[C, A]_k &= [C^k, A_k] \\
(A \otimes B)_k &= A_k \otimes B_k
\end{aligned}
$$

Sweedler theory of left and right modules over B

Let $P = B$ a cocommutative bialgebra, we have

$$P\text{-}\mathbf{Alg} = B\text{-Mod} \qquad \text{and} \qquad P\text{-}\mathbf{Coalg} = \text{Mod-}B.$$

There exists six functors

$$\begin{array}{ccl} \otimes & : & \text{Mod-}B \times \text{Mod-}B \to \text{Mod-}B \\ \textsc{Hom} & : & (\text{Mod-}B)^{op} \times \text{Mod-}B \to \text{Mod-}B \\ \{-,-\} & : & B\text{-Mod}^{op} \times B\text{-Mod} \to \text{Mod-}B \\ \triangleright & : & \text{Mod-}B \times B\text{-Mod} \to B\text{-Mod} \\ [-,-] & : & (\text{Mod-}B)^{op} \times B\text{-Mod} \to B\text{-Mod} \\ \otimes & : & B\text{-Mod} \times B\text{-Mod} \to B\text{-Mod} \end{array}$$

such that

Theorem

1. $(\text{Mod-}B, \otimes, \textsc{Hom})$ *is symmetric monoidal closed.*
2. $(B\text{-Mod}, \{-,-\}, \triangleright, [-,-], \otimes)$ *is enriched, tensored, cotensored and symmetric monoidal over* Mod-B.

Sweedler theory of left and right modules over B

For M, N two left B-modules and Q, R two right B-modules

$$\begin{aligned}
\mathrm{Hom}(Q,R) &= \int_{(B/\star)^{op}} [Q,R] \\
\{M,N\} &= \int_{B/\star} [M,N] \\
(Q \triangleright M) &= \int^{B/\star} Q \otimes M \\
[Q,M] &= [Q,M]
\end{aligned}$$

where $B/\star$ is the division category of the ring B

- objects = elements of B
- arrows $a \to b$ = elements c s.t. $a = bc$

Sweedler theory of operads

For $P = OP(K)$ the operad of K-colored operads, there exists six functors

$$
\begin{array}{ccl}
\otimes & : & \mathbf{coOp}(K) \times \mathbf{coOp}(K) \to \mathbf{coOp}(K) \\
\textsc{Hom} & : & \mathbf{coOp}(K)^{op} \times \mathbf{coOp}(K) \to \mathbf{coOp}(K) \\
\{-,-\} & : & \mathbf{Op}(K)^{op} \times \mathbf{coOp}(K) \to \mathbf{coOp}(K) \\
\triangleright & : & \mathbf{coOp}(K) \times \mathbf{Op}(K) \to \mathbf{Op}(K) \\
[-,-] & : & \mathbf{coOp}(K)^{op} \times \mathbf{Op}(K) \to \mathbf{Op}(K) \\
\otimes & : & \mathbf{Op}(K) \times \mathbf{Op}(K) \to \mathbf{Op}(K)
\end{array}
$$

such that

Theorem (A-J)

1. $(\mathbf{coOp}(K), \otimes, \textsc{Hom})$ *is symmetric monoidal closed.*
2. $(\mathbf{Op}(K), \{-,-\}, \triangleright, [-,-], \otimes)$ *is enriched, tensored, cotensored and symmetric monoidal over* $\mathbf{coOp}(K)$.

Sweedler theory of operads

The monoidal structures are the Hadamard tensor products.

If C is a cooperad and A an operad, $[C, A]$ is the convolution operad of Berger-Moerdijk.

We have formulas

$$\begin{aligned} \mathrm{Hom}(C, OP^{\vee}(X)) &= OP^{\vee}([C, X]) \\ \{OP(X), A\} &= OP^{\vee}([X, A]) \\ C \triangleright OP(X) &= OP(C \otimes X) \end{aligned}$$

Part III - MAURER-CARTAN THEORY

Maurer-Cartan theory of algebras

Let $\mathbf{V} = \mathbf{dgVect}$ (= chain complexes),
then $\mathbf{Alg} = \mathbf{dgAlg}$ and $\mathbf{Coalg} = \mathbf{dgCoalg}$.

For A a dg-algebra, an element $a \in A_{-1}$ is said to be Maurer-Cartan if it satisfies the equation

$$da + a^2 = 0.$$

Let MC be the dg-algebra generated by a universal Maurer-Cartan element:

$$\mathrm{MC} = k[u]$$

with $|u| = -1$ and $du = -u^2$.

Maurer-Cartan elements of A are in bijection with algebra maps $\mathrm{MC} \to A$.

Maurer-Cartan theory of algebras

Let C be a dg-coalgebra and A be a dg-algebra.

A twisting cochain from C to A is defined to be a Maurer-Cartan element of the convolution algebra $[C, A]$

Let $Tw(C, A)$ be the set of twisting cochains from C to A. It is in bijection with the set of algebra maps $\mathrm{MC} \to [C, A]$.

Maurer-Cartan theory of algebras

The bar construction $B : \mathbf{dgAlg} \to \mathbf{dgCoalg}$ and the cobar construction $\Omega : \mathbf{dgCoalg} \to \mathbf{dgAlg}$ are defined to be the functors representing

$$\begin{aligned} \mathbf{dgCoalg}^{op} \times \mathbf{dgAlg} &\longrightarrow \mathbf{Set} \\ (C, A) &\longmapsto Tw(C, A) \end{aligned}$$

In other words B and Ω are such that there exists natural bijections between

twisting cochains	$C \to A$
algebra maps	$\Omega C \to A$
coalgebra maps	$C \to BA.$

Maurer-Cartan theory of algebras

A twisting cochain is an algebra map $\mathrm{MC} \to [C, A]$.

Using Sweedler operations, we have bijection between the following sets

$$\begin{array}{ll} \text{algebra maps} & \mathrm{MC} \to [C, A] \\ \text{algebra maps} & C \triangleright \mathrm{MC} \to A \\ \text{coalgebra maps} & C \to \{\mathrm{MC}, A\}. \end{array}$$

We deduce that the adjunction of type III

$$- \triangleright \mathrm{MC} : \mathbf{dgCoalg} \rightleftarrows \mathbf{dgAlg} : \{\mathrm{MC}, -\}$$

is the bar-cobar adjunction

$$\Omega : \mathbf{dgCoalg} \rightleftarrows \mathbf{dgAlg} : B$$

(up to a subtlety about conilpotent coalgebras).

Maurer-Cartan theory of algebras

Recall that $MC = T(u)$ is free as a graded algebra.
The formulas

$$\begin{aligned} \{T(X), A\} &= T^{\vee}([X, A]) \\ C \triangleright T(X) &= T(C \otimes X) \end{aligned}$$

gives the classical construction of the bar and cobar functors

$$\begin{aligned} BA = \{MC, A\} &= T^{\vee}(u^{\star} \otimes A) \\ \Omega C = C \triangleright MC &= T(C \otimes u) \end{aligned}$$

The internal and external part of the differentials come respectively from the differential of A (or C) and of MC.

Operadic Maurer-Cartan theory

Let P be an operad (with one color), the invariant space is

$$Inv(P) = \prod_n P[n]^{\Sigma_n}$$

is a pre-Lie algebra.
A Maurer-Cartan element of P is a Maurer-Cartan element in this pré-Lie algebra.

It is a family of elements $u_n \in P(n)_{-1}$ such that

$$du_n = \sum u_k \circ_i u_{n-k+1}$$

Operadic Maurer-Cartan theory

Let MC be the graded operad freely generated by u_n in arity n and degree -1 with differential generated by

$$du_n = \sum u_k \circ_i u_{n-k+1}$$

An operad map $MC \to P$ is the same thing as a Maurer-Cartan element of P.

We called MC the Maurer-Cartan operad.

Operadic Maurer-Cartan theory

An operadic twisting cochain $C \to A$ is a Maurer-Cartan element in the convolution operad $[C, A]$.

The operadic bar and cobar constructions are defined to represent the functor

$$\begin{array}{rcl} \mathbf{dgCoop}^{op} \times \mathbf{dgOp} & \longrightarrow & \mathbf{Set} \\ (C, A) & \longmapsto & Tw(C, A) \end{array}$$

The Sweedler theory of operads gives us bijections between

operadic twisting cochains	$C \to A$
operad maps	$\Omega C = C \triangleright MC \to A$
cooperads maps	$C \to BA = \{MC, A\}$.

Operadic Maurer-Cartan theory

Recall that $MC = OP(u)$ is free as a graded operad.
The formulas

$$\begin{aligned} \{OP(X), A\} &= OP^\vee([X, A]) \\ C \triangleright OP(X) &= OP(C \otimes X) \end{aligned}$$

gives the classical construction of the bar and cobar functors

$$\begin{aligned} BA = \{MC, A\} &= OP^\vee(u^\star \otimes A) \\ \Omega C = C \triangleright MC &= OP(C \otimes u) \end{aligned}$$

The internal and external part of the differentials come respectively from the differential of A (or C) and of MC.

Operadic Maurer-Cartan theory

What is MC ?

In the symmetric operadic case, an MC algebra structure on X is the same thing as a curved L_∞-algebra structure on $s^{-1}X$.

(In the non-symmetric operadic case, an MC algebra structure on X is the same thing as a curved A_∞-algebra structure on $s^{-1}X$.)

Hence, the curved L_∞ (or A_∞) operads governs the bar and cobar constructions through the Sweedler operation.
With a slight abuse of notation:

$$BA = \{cL_\infty, A\} \qquad \text{and} \qquad \Omega C = C \rhd cL_\infty .$$

NEXT

Develop the formalism of Maurer-Cartan for general colored operads.

Apply it to recover all known bar-cobar constructions, including the bar-cobar construction for (co)algebras relative to an operadic twisting cochain.

Understand Koszul complexes and Koszul duality.

Thank you.