Why deformations are cohomological

M. Anel*

Abstract

Deformation of mathematical structures are often classified by some cohomology groups, the purpose of this talk is to understand better why. The general picture is the following:

1. mathematical objects lives in categories,
2. the deformations of an object form a category,
3. first order deformation is linearisation of general deformation problem,
4. the category of first order deformation of an object has a linear structure,
5. "linear categories" are nothing but complexes.

Contents

*CIRGET, UQÀM.

1 Examples of cohomologies involved in deformation problems

Here follows a list of cohomologies related to deformation problems.

1. Geometric examples:
   (a) deformation of a point in a variety: tangent cohomology (tangent complex);
   (b) deformation of a manifold structure: cohomology of the tangent bundle;
   (c) deformation of a vector bundle: cohomology of the bundle of endomorphisms;
   (d) deformation of principal bundle: cohomology of the adjoint bundle;
   (e) deformation of local system: cohomology of π_1 in $Mat_n(\mathbb{K})$;
   (f) ...
2. Algebraic examples:
   (a) deformation of a module structure over a given ring: Ext groups;
   (b) deformation of Lie algebra structure: Lie cohomology;
   (c) deformation of associative algebra structure: Hochschild cohomology;
   (d) deformation of group representation: group cohomology;
   (e) ...

2 Deformation problems

2.1 Examples

Lie algebra structure Deformation of algebraic structures are easier to formalize as deformation of geometric structures so we'll start with that.

Let V be a finite dimensional vector space over $\mathbb{K}$ and e_i a basis of V[1]. A *Lie algebra structure* $\mathfrak{g}$ on V is completely determined by the Lie brackets of the e_i

$$[e_i, e_j] = \sum_j g_{ij}^k e_k,$$

where the g_{ij}^k are number in the base field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ called the structure constants and statisfy a quadratic equation coming from Jacobi identity:

$$[e_i, [e_j, e_k]] = [[e_i, e_j], e_k] + [e_j, [e_i, e_k]] \quad \iff \quad \forall m,\ \sum_\ell g_{jk}^\ell g_{i\ell}^m = \sum_\ell g_{ij}^\ell g_{\ell k}^m + g_{ik}^\ell g_{j\ell}^m. \tag{1}$$

and antisymmetry: $g_{ij}^k = -g_{ji}^k$.

A deformation of the Lie structure $\mathfrak{g} = \{g_{ij}^k\}$ with parameters $(x_1, \dots, x_n) \in \mathbb{R}^n$ is a family of functions $G_{ij}^k(x_1, \dots, x_n)$ such that

1. for all $(x_1, \dots, x_n)$, the $G_{ij}^k(x_1, \dots, x_n)$ satisfy the equation (1) and antisymmetry, and
2. $G_{ij}^k(0, \dots, 0) = g_{ij}^k$.

Let A the ring of functions on parameters $(x_1, \dots, x_n)$, the evaluation at 0 gives a ring morphism $A \to \mathbb{K}$ that can be used to transform any A-module M into a $\mathbb{K}$-module $M \otimes_A \mathbb{K}$. With this in mind, the previous conditions can be restated as: a deformation of the Lie algebra $\mathfrak{g}$ with parameters $(x_1, \dots, x_n) \in \mathbb{R}^n$ is

[1] The same definition works more generally if $\mathbb{K}$ is a ring and V a free $\mathbb{K}$-module, this will be used implicitely.

1. a Lie algebra structure $\mathfrak{G}$ on the free A-module $A \otimes_{\mathbb{K}} V$
2. and an isomorphism $\mathfrak{G} \otimes_A \mathbb{K} \simeq \mathfrak{g}$.

By abuse it is often said that the parameters are *in the ring* A and the parameter space $(B, b) = (\mathbb{R}^n, 0)$ is often replaced by the data $A \to \mathbb{K}$ where A is the rings of functions on B and $A \to \mathbb{K}$ is the ring map given by evaluation at the point $b \in B$.

Vector bundle structure Let X be a manifold and V a locally trivial vector bundle on X of rank r. A deformation of V with parameter (B, b) is the data of

- a vector bundle W of rank r on $B \times X$
- and an isomorphim between V and the restriction of W to $\{b\} \times X$.

By hypothesis, there exists an open covering U_i of X such that the restrictions V_i of V on U_i are trivial bundles ; if, on each U_i, we chose r linearly independant sections $(e_i)_\alpha$ of V_i, we deduce on each $U_{ij} = U_i \cap U_j$ a function $\phi_i^j : U_{ij} \longrightarrow Gl_n(\mathbb{K})$ that gives the coordinates of $(e_j)_\alpha$ in the basis of $(e_i)_\alpha$. These functions satisfies the so-called *cocycle equation* on triple intersections: for all i, j, k, $\phi_i^k = \phi_i^j \phi_k^j$. In analogy with the previous example, let's call the ϕ_i^j the *structure cocycle* of V[2].

This structure cocycle is all that is needed to reconstruct V from the covering U_i up to (non unique) isomorphism. In fact, to any covering U_i and any functions $\phi_i^j : U_{ij} \longrightarrow Gl_n$ satisfying $\phi_i^k = \phi_i^j \phi_k^j$ on triple intersection we can associate a manifold V construct as the quotient of the manifold $\coprod_i U_i \times \mathbb{R}^r$ by the equivalence relation

$$\begin{aligned} \coprod_{i,j} U_{ij} \times \mathbb{R}^r &\longrightarrow \left(\coprod_i U_i \times \mathbb{R}^r \right)^2 \\ (x_{ij}, u) &\longmapsto ((x_{ij}, u), (x_{ij}, \phi_i^j u)). \end{aligned}$$

This manifold has an obvious projection to X that makes it into a vector bundle.

If W is a deformation of V with parameter space B, it may not be possible to trivialise W along the covering $B \times U_i$ of $B \times X$, but it will be if the U_i and B are contractibles. With this extra hypothesis we can understand a deformation of V as a deformation of the ϕ_i^j:

1. functions $\Phi_i^j : (B \times U_i) \cap (B \times U_j) \simeq B \times U_{ij} \to Gl_n(\mathbb{K})$
2. such that $\Phi_i^k = \Phi_i^j \Phi_j^k$ on $B \times U_{ijk}$.

Another way to say this is to use the fact that if A is the ring of functions on B, a function Φ_i^j : $B \times U_{ij} \to Gl_n$ is the same thing as a function $\Phi_i^j : U_{ij} \to Gl_n(A)$ where $Gl_n(A)$ is the space of matrices with coefficients in A. There is the problem of the topology on $Gl_n(A)$ but this can be solved easily when A is of finite dimension (which will be the case with first order deformations, see §2.3).

This approach points out that things can also be formulated using the ring of functions on B, this fact is fundamental to make sense of first order deformations.

[2]The structure cocycle depends on the choice of a local trivialisation of the bundle V as previously the structure constants where depending on the choice of a basis of the vector space V. This fact is the source of an equivalence relation on cocycles essentially given by trivialisation or base change. Equivalence classes for this relation are called *cohomology classes*.

2.2 Axiomatisation

Let (B, b) be a pointed space thought as a space of parameters (typically (B, b) is $\mathbb{R}^n$ pointed at 0) and let X be some structure (e.g. complex manifold structure, Lie algebra structure...). A *deformation of X with parameters in* (B, b) is

1. for every $p \in B$, a structure $X(p)$ of the same type that depends (continuously, differentiably, holomorphically...) on the parameter p (such a data is often called a *family of structures parametrized by B*)
2. and an isomorphism (or an equality, or an equivalence, or whatever is natural to the problem) between X and $X(b)$.

This definition is vague but we are going to precise it in algebraic and geometric contexts.

The term *algebraic structure* is to be understood as a structure having an underlying vector space or abelian group or, more generally, an underlying module over some ring $\mathbb{K}$; the structure will be said to be *over* $\mathbb{K}$. The term *geometric structure* is to be understood as a structure having an underlying topological space (manifold, algebraic variety...). In this sense a Lie group is a geometric structure and not an algebraic structure.[3]

Algebraic deformation problems are the simplest to deal with because structure constants are easy to explicit and often live the base ring $\mathbb{K}$. A family of structure parametrized by B is then a structure with its structure constants in $\mathcal{O}(B)$, the rings of functions on B. It can be axiomatized in the following way: a *deformation of an algebraic structure* $\mathfrak{g}$ *over* $\mathbb{K}$ *with parameters in* (B, b) is

1. an algebraic structure $\mathfrak{G}$ of the same kind over $A = \mathcal{O}(B)$ (for any $p \in B$ the associated evaluation map $A \longrightarrow \mathbb{K}$ can be used to define a specialisation $\mathfrak{G} \otimes_A \mathbb{K}$ of $\mathfrak{G}$ which is the value of the deformation at the parameter p)
2. and an isomorphism between $\mathfrak{g}$ and $\mathfrak{G} \otimes_A \mathbb{K}$ where the augmentation is the one corresponding to $b \in B$ and X.

Deformations of geometric objects will use the idea that a family of geometric objects over a base B is a fibration over B, they can be axiomatized in the following way: a *deformation of a geometric objet V with parameters in* (B, b) is

1. a geometric object W of the same type with a projection over B (the fiber $W(p)$ of this projection over $p \in B$ is the value of the deformation at the parameter p)
2. and an isomorphism between the fiber $W(b)$ and X.

To be dealt with, geometric deformations often need hypothesis of local triviality of the structure.

Algebraic examples In this situation, we will replace the parameter space (B, b) by an augmented commutative[4] algebra $A \to \mathbb{K}$. Often A is $\mathbb{K}[t]$, with augmentation given by evaluation at 0, the deformation is then said to have parameter t, structure constant are polynomial in t.

1. A deformation of an associative $\mathbb{K}$-algebra $\mathfrak{a}$ is the data of an associative A-algebra $\mathfrak{A}$ and an isomorphism $\mathfrak{a} \simeq \mathfrak{A} \otimes_A \mathbb{K}$.
2. Let G be a group and $\rho : G \to Aut(V)$ a representation of G where V is a $\mathbb{K}$-vector space, a deformation of ρ is a representation $R : G \to Aut(W)$ where W is a A-module and an isomorphism $g : W \otimes_A \mathbb{K} \simeq V$ that induces an equality $g^{-1}\rho g = R \otimes_A \mathbb{K}$ where $R \otimes_A \mathbb{K} : G \to Aut(W) \longrightarrow Aut(W \otimes_A \mathbb{K})$.

[3] We forget about the case where the underlying module of an algebraic structure has a topology.

[4] It could be taken non-commutative, but this would be an enlargement of the setting presented here.

The underlying A-module of the A-algebra can be taken of a specific kind: projective, flat, free, coherent... this depends on the type of deformation one is interested in.

Geometric examples Let's fix a pointed space of parameters (B, b).

1. Let X be manifold a deformation of the manifold structure of X is a manifold $\mathcal{X}$ over B and an isomorphism between X and the fiber of $\mathcal{X}$ at b. This idea is implicitely used in Milnor's fiber theory...
2. Let X be a space (manifold, variety...) and x be a point of x, a deformation of x in X is a map $B \to X$ (continuous, differentiable, holomorphic...) such that b is send to x. In particular a one dimensional deformation of x is a path in X going through x and a first order one dimensional deformation will be a tangent vector at x. Every deformation problem can be thought in this way by taking X to be the moduli space of the structure to be deform.

2.3 Infinitesimal deformations

Now it is time to introduce the secret of the differential calculus: the point equipped with the ring $\mathbb{K}[\epsilon] = \mathbb{K}[x]/x^2$ of functions on two infinitesimally closed points.

It is a modern approach to geometry to work with spaces equipped with sheaves of rings, the appendix A.2 gives more details why it is a good idea, for now it is sufficient to say that this context authorize to define the new "space" $U = (pt, \mathbb{K}[\epsilon])$ where pt is the one point space and the ring $\mathbb{K}[\epsilon]$ is viewed as a sheaf of ring over it.

The usual point P is the locally ringed space $(pt, \mathbb{K})$, U has a canonical base point $x : P \to U$ (essentially given by the morphism $\mathbb{K}[\epsilon] \to \mathbb{K}$ that sends ϵ to 0). The space U is to be thought as two points infinitesimally closed, and a morphism from U to a manifold X is nothing than a tangent vector of X (the ringed spaces formalism make this very precise) and the base point U gives the base point of the vector in X (cf. §A.3).

We can now define a *first order (one dimensional) deformation* as a deformation with parameter space (U, x).

In particular for deformation of algebraic structures over $\mathbb{K}$, a first order deformation is an algebraic structure over $\mathbb{K}[\epsilon]$ so it is a perturbation of structure constants by terms of square 0. The situation is analog for geometric structures.

Lie algebras example A first deformation of a Lie $\mathbb{K}$-algebra structure g_{ij}^k is a Lie $\mathbb{K}[\epsilon]$-algebra whose structure constants are deformation of g_{ij}^k by some h_{ij}^k. The structure equations are

$$\begin{array}{rcl} \sum_\ell (g_{jk}^\ell + h_{jk}^\ell \epsilon)(g_{i\ell}^m + h_{i\ell}^m \epsilon)) & = & \sum_\ell (g_{ij}^\ell + h_{ij}^\ell \epsilon)(g_{\ell k}^m + h_{\ell k}^m \epsilon) + (g_{ik}^\ell + h_{ik}^\ell \epsilon))(g_{j\ell}^m + h_{j\ell}^m \epsilon)) \\ & \Longleftrightarrow & \\ \sum_\ell h_{jk}^\ell g_{i\ell}^m + g_{jk}^\ell h_{i\ell}^m & = & \sum_\ell h_{ij}^\ell g_{\ell k}^m + g_{ij}^\ell h_{\ell k}^m + h_{ik}^\ell g_{j\ell}^m + g_{ik}^\ell h_{j\ell}^m \end{array}$$

and antisymmetry $h_{ij}^k = -h_{ji}^k$.

The coefficients h_{ij}^k are the coordinates of a antisymmetric map $h : V \otimes V \to V$ which will deform the bracket and it is more convenient to rewrite the equation under the form

$$[x, h(y, z)] + h(x, [y, z]) = [h(x, y), z] + h([x, y], z) + h(y, [x, z]) + [y, h(x, z)]$$

This is the definition of a Lie 2-cocycle in the Lie cohomology of $\mathfrak{g}$ with coefficients in $\mathfrak{g}$.

Vector bundles example Elements of $Gl_n(\mathbb{K}[\epsilon])$ can be written as $a + b\epsilon$ where $a \in Gl_n(\mathbb{K})$ and $b \in M_n(\mathbb{K})$. In particular, as a space $Gl_n(\mathbb{K}[\epsilon])$ is $Gl_n(\mathbb{K}) \times M_n(\mathbb{K})$ and the evaluation at 0 is the projection to the first factor.

A first order deformation of a structure cocycle ϕ_i^j is a cocycle is then a cocycle $\Phi_i^j = \phi_i^j + \psi_i^j\epsilon \in Gl_n(\mathbb{K}[\epsilon])$. The cocycle equation $\Phi_i^k = \Phi_i^j \Phi_j^k$ gives the equation

$$\phi_i^k + \psi_i^k\epsilon = (\phi_i^j + \psi_i^j\epsilon)(\phi_j^k + \psi_j^k\epsilon) \quad \Longleftrightarrow \quad \psi_i^k = \psi_i^j\phi_j^k + \phi_i^j\psi_j^k.$$

In particular deformation of the trivial bundle are given by ψ_i^j satisfying

$$\psi_i^k = \psi_i^j + \psi_j^k.$$

This is the definition of a 1-cocycle with coefficients in $M_n(\mathbb{K})$.

Remark Vector bundles on X are classified by some pointed space called the non-abelian cohomology of X with values in Gl_n noted $H^1(X, Gl_n)$ (it is pointed at the trivial bundle). First order one parameter deformations of the trivial bundle should define tangent vector to this space at the marked point. The computation gives that this tangent space is $H^1(X, gl_n)$ where gl_n is the Lie algebra of Gl_n. This is general, for any Lie group G, G-bundles are classified by some non-abelian $H^1(X, G)$ and first order deformation of the trivial G-bundle are classified by $H^1(X, \mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra of G. In other words, the tangent to non-abelian cohomology is some abelian cohomology.

2.4 Why 1- and 2- cocycles ?

The degree of a cocycle involved in the deformation of some structure has to do with the degree of the category it is living in. Bundles live in a 1-category and so are their deformations: the 1-cocycle is a deformation and a 0-cocycle is an infinitesimal automorphism.

Lie algebras (also associative algebras) can be seen as living in a 2-category rather than a 1-category, this is why their deformations can be classified by 2-cocycles (see §3.2).

3 Deformations and categories

There is a subtle point that wasn't emphasized while dealing with Lie algebra before, the expression *a Lie algebra structure on a vector space of given dimension* can be understood in two ways:

1. as a structure on one given vector space, or
2. as any Lie algebra whose underlying vector space has the wanted dimension.

In the first case, a Lie algebra structure can be described, once chosen a basis, by structure constants and in the second case, a Lie algebra structure is more an isomorphism class of Lie algebra ; the difference is that the former form a set and the latter a groupoid[5].

This difference is the same as the difference between a cocycle and a cohomology class: a Lie algebra structure in the first sense will give a Lie algebra structure in the second sense and this correspondance is surjective.

In fact, most mathematical objects are living in categories rather than sets (whenever there exists a notion of morphisms between them), this simple fact will have important consequences at the level

[5] As there is no canonical isomorphism between isomorphic Lie algebra, the reasonable thing to do is not to make any choice and keep the information about all isomorphisms.

of deformations: Let X be some geometric object (a variety, a bundle on a given space...) and (B, b) a parameter space, the set of all possible deformations of X with parameters B is in fact a category. [6]

This fact is the source of the non-existence of moduli spaces for most mathematical structure: a space only has an underlying set of points, but moduli problems of objects living in categories ask for a notion of space that has an underlying *category* of points. Several generalisation of topological spaces have been invented to deal with this issue: topos, stacks, orbifolds, Morita equivalence class of groupoids...

But we are not interested here in global moduli problems, only with first order deformations, which can be understood geometrically as tangent vectors on the moduli space: the object to be deformed defines a point on the moduli space and a first order one parameter deformation is a tangent vector at this point.

Following our previous remark, first order deformation will form a category and as it is a tangent space, this category should have a linear structure. We investigate this now.

3.1 What is a linear category ?

Let (X, x) be a pointed manifold, the tangent space to X at x is the set of pointed morphisms $\mathrm{Hom}_\bullet(U, X)$. Let's analyze why it has a vector space structure and why it is natural.

All the structure is coming from U. The ringed space $U \vee U = U \coprod_{pt} U$ is $(pt, \mathbb{K}[\epsilon] \times_{\mathbb{K}} \mathbb{K}[\epsilon])$. It is a simple calculation that $\mathbb{K}[\epsilon] \times_{\mathbb{K}} \mathbb{K}[\epsilon] = \mathbb{K}[x, y]/(x, y)^2$

There is a map $+ : U \to U \vee U$ given by

$$\begin{array}{rcl} \mathbb{K}[x,y]/(x,y)^2 & \longrightarrow & \mathbb{K}[\epsilon] \\ x & \longmapsto & \epsilon \\ y & \longmapsto & \epsilon. \end{array}$$

This map satisfies a symmetry condition: if : $\sigma : U \vee U \longrightarrow U \vee U$ is the exchange of the two U, $\sigma \circ + = +$; and a coassociativity condition: the square

$$\begin{array}{ccc} U & \xrightarrow{+} & U \vee U \\ {\scriptstyle +}\downarrow & & \downarrow{\scriptstyle U \vee +} \\ U \vee U & \xrightarrow{+ \vee U} & U \vee U \vee U \end{array}$$

is commutative.

There is a map $- : U \to U$ given by

$$\begin{array}{rcl} \mathbb{K}[\epsilon] & \longrightarrow & \mathbb{K}[\epsilon] \\ \epsilon & \longmapsto & -\epsilon \end{array}$$

such that the square

$$\begin{array}{ccccc} U & \xrightarrow{+} & U \vee U & \xrightarrow{U \vee -} & U \vee U \\ \downarrow & & & & \downarrow \\ pt & \longrightarrow & & \longrightarrow & U \end{array}$$

is commutative.

[6] It is common to identify two deformations if they are isomorphic, this fact uses implicitely the category structure of all deformations.

Those maps and relations produce an abelian cogroup[7] structure on the pointed space U. The natural abelian group structure on tangent spaces is then deduced formally from the fact that $\mathrm{Hom}_\bullet(-,X)$ is contravariant and $\mathrm{Hom}_\bullet(U\vee U,X)=\mathrm{Hom}_\bullet(U,X)\times\mathrm{Hom}_\bullet(U,X)$.

A $\mathbb{K}$-vector space is an abelian group V with a morphism $\mathbb{K}\longrightarrow \mathrm{End}(V)$. Such a structure is deduced from the fact that $End_\bullet(U)=\mathbb{K}$, indeed an endomorphism of $\mathbb{K}$-algebras of $\mathbb{K}[\epsilon]$ is determined by the image of ϵ in $\mathbb{K}\epsilon$.[8]

From this point it is easy to generalize. If X is not a space any more but a stack, the functor $\mathrm{Hom}_\bullet(-,X)$ takes it values in pointed groupoids and still send $\vee$ to cartesian product of groupoids. Hence the groupoid $\mathrm{Hom}_\bullet(U,X)$ will inherit a vector space structure.

What is a groupoid with a vector space structure ? It is a groupoid $E_1\rightrightarrows E_0$ where E_i are vector spaces and target, source, identity and composition maps are all linear maps.

Now this is a remarkable fact that this data is equivalent to that of a complex of vector spaces of length 2. Indeed for such a groupoid let's consider the source map $s:E_1\to E_0$ and its kernel $\ker(s)$, the image of the identity map $E_0\to E_1$ is a supplement for $\ker(s)$ and $E_1\simeq E_0\oplus\ker(s)$ with this decomposition the source map corresponds the projection on the first factor, the identity to the inclusion of the first factor, the target map is determined by its restriction to $\ker(s)$ via the formula

$$\begin{array}{rcl} t: E_0\oplus\ker(s) & \longrightarrow & E_0 \\ (x,u) & \mapsto & x+t(u). \end{array}$$

and the composition is the map[9]

$$\begin{array}{rcl} E_1\times_{E_0}E_1\simeq E_0\oplus\ker(s)\oplus\ker(s) & \longrightarrow & E_0\oplus\ker(s) \\ (x,u,v) & \mapsto & (x,u+v). \end{array}$$

So everything is determined by the map $t:\ker(s)\longrightarrow E_0$, which defines a complex of length 2.

Here is how a complex $F\xrightarrow{t}E$ (with E in degre 0) can be thought as a groupoid:

1. objects are elements of E;
2. if $x,y\in E$ an arrow from x to y is an element $u\in F$ such that $x+t(u)=y$, its inverse is $-u$;
3. endomorphisms of $x\in E$ are also in bijection with elements of $\ker(t)=H^{-1}(F\to E)$;
4. isomorphisms classes of objects are in bijection with elements of $E/\mathrm{im}(t)=H^0(F\to E)$.

If groupoids were replaced by 2-groupoids or even higher groupoids (simplicial sets), the result would stay the same but the length of complexes would grow: linear 2-groupoids would be equivalent to complexes with three objects (encoding objects, 1-morphisms and 2-morphisms of the 2-groupoid) and linear n-groupoids would be equivalent to complexes of length $n+1$.

[7] Cogroups are the dual notion to group when one reverse the arrows in a category and replace cartesian product for union, they are bizarre objects. Somehow the whole infinitesimal calculus is about cogroups and cotorsors over them... But there is no non-trivial cogroups in the category of sets and the only examples I know are "space" dual to algebras of the kind $\mathbb{K}[\epsilon]$. Maybe it would have been better to say that the algebra $\mathbb{K}[\epsilon]$ is an abelian group object in the category of $\mathbb{K}$-algebra, but I prefer working with the geometric intution.

[8] A very nice way to present this is to uses Lawvere theories: the category generated by the $U\vee\cdots\vee U$ is the dual of the category of $\mathbb{K}$-vector spaces, any contravariant functor sending U to some set V and sending $\vee$ to cartesian products is a vector space structure on V.

[9] More precisely, it can be shown that all groupoid structures on the pair $(\ker(s),E_0)$ are isomorphic this one. Moreover, groupoid structures and category structures on the pair $(\ker(s),E_0)$ coincide.

discrete objects	linear objects
set	vector space T_0
groupoid/category	complex $T_1 \to T_0$
higher groupoid/category (simplicial set)	complex $\dots T_2 \to T_1 \to T_0$
cosimplicial set	complex $T_0 \to T_{-1} \to \dots$

3.2 Examples of deformation cohomologies

We illustrate the previous interpretation of complexes as groupoid. This will give a geometric meaning only the low degree cochains of the complexes computing deformation cohomologies, the geometric interpretation of higher degree cochains is more involved (see §4.1).

Ext cohomology This is the only example we will detail, the computations for other examples are alike.

Let A be a $\mathbb{K}$-algebra and M a A-module, there is a resolution of $\mathrm{Hom}_A(M,M)$ by

$$\mathrm{Hom}(M,M) \longrightarrow \mathrm{Hom}(A\otimes M,M) \longrightarrow \mathrm{Hom}(A\otimes A\otimes M,M) \longrightarrow \dots$$

where the differential is given by

$$\begin{array}{rcl} df(a\otimes x) & = & af(x) - f(ax) \\ dg(a\otimes b\otimes x) & = & ag(b,x) - g(ab,x) + g(a,bx) \\ dh(a\otimes b\otimes c\otimes x) & = & ah(b,c,x) - h(ab,c,x) + h(a,bc,x) - h(a,b,cx) \\ & \dots & \end{array}$$

The n-homology group of this complex is noted $\mathrm{Ext}^n(M,M)$.

The 1-cocycle of this complex are in bijection with first order deformation of the A-module structure M whose underlying $\mathbb{K}[\epsilon]$-module is flat. A flat module over $\mathbb{K}[\epsilon]$ is free and always isomorphic to $M\oplus M\epsilon$ where M is a $\mathbb{K}$-module. A $A[\epsilon]$-module structure on $M\oplus M\epsilon$ is map

$$(A\oplus A\epsilon)\otimes(M\oplus M\epsilon) \longrightarrow M\oplus M\epsilon$$

satisfying the associativity condition. As the previous map is $\mathbb{K}[\epsilon]$-linear, it is entirely determined by two maps

$$\begin{cases} \mu : A\otimes M \longrightarrow M \\ \lambda\epsilon : A\otimes M \longrightarrow M\epsilon \end{cases}$$

satisfying

$$(\mu+\lambda\epsilon)(ab,x) = (\mu+\lambda\epsilon)(a,(\mu+\lambda\epsilon)(b,x))$$

A deformation of a A-module structure on M is thus a map $\lambda : A\otimes M \longrightarrow M$ satisfying

$$\lambda(ab,x) = a\lambda(b,x) + \lambda(a,bx)$$

which is the previous 1-cocycle equation.

An isomorphism between two deformations λ_1 and λ_2 is given by a map

$$f = id_M \oplus \alpha\epsilon : M \oplus M\epsilon \longrightarrow M \oplus M\epsilon$$

satisfying

$$\begin{aligned} f(ax) = af(x) \quad &\Longleftrightarrow \quad (id_M \oplus \alpha\epsilon)(ax + \lambda_1\epsilon(a,x)) = a(id_M \oplus \alpha\epsilon)(x) + \lambda_2\epsilon(a, (id_M \oplus \alpha\epsilon)(x)) \\ &\Longleftrightarrow \quad \alpha(ax) + \lambda_1(a,x) = a\alpha(x) + \lambda_2(a,x) \end{aligned}$$

In other terms α is a 0-cocchain making λ_1 and λ_2 cohomologous. Isomorphism classes of deformation are thus in bijection with $\mathrm{Ext}^1(M,M)$ and elements of $\mathrm{Ext}^0(M,M)$ are in bijection with automorphisms of the trivial deformation ($\lambda = 0$).

Hochschild cohomology In the case the ring A before is isomorphic to $B \otimes B^\circ$ where B is a ring, A-modules are B-bimodules and there exists another resolution that compute the homology of B as a B-bimodule: the Hochschild complex

$$\mathrm{Hom}(k,B) \longrightarrow \mathrm{Hom}(B,B) \longrightarrow \mathrm{Hom}(B\otimes B, B) \longrightarrow \mathrm{Hom}(B \otimes B \otimes B, B) \longrightarrow \dots$$

where the differential is given by

$$\begin{aligned} dx(y) &= xy - yx \\ df(x \otimes y) &= xf(y) - f(xy) + f(x)y \\ dg(x \otimes y \otimes z) &= xg(y,z) - g(xy,z) + g(x,yz) - g(x,y)z \\ &\dots \end{aligned}$$

With a reasonning analogous to the one before, 2-cocycles can be shown to be in bijection with first order deformation of the associative structure of B whose underlying $\mathbb{K}[\epsilon]$-module is free ; and two deformations are isomorphic iff they are cohomologous. Also, 1-cocycles are in bijection with automorphisms of the trivial deformation, which are the same thing as derivations of the algebra B,

The new feature here is the interpretation of equivalence class of 1-cocycles and 0-cocycles: 1-cohomology classes are classes of derivations of B up to internal derivations, and 0-cocycles are elements of the center of B.

That can be understood by viewing the deformation problem of associative algebras not in a 1-category but in a 2-category. Indeed, to any associative algebra one can associate a $\mathbb{K}$-linear category $\underline{B}$ with one object and B as its endomorphisms algebra.[10]

It is possible to show that first order deformations of $\underline{B}$ are the same thing as first order deformations of B are both classified by Hochschild 2-cocycles. Moreover, equivalences of categories between two deformations of $\underline{B}$ are the same thing as 1-cochains making the two 2-cocycles cohomologous. In the new context, it is also possible to interpret 0-cochains as 2-morphisms between two isomorphisms of deformations, so Hochschild 1-cohomology classes are in bijection with isomorphisms classes of equivalences of deformations and 0-cocycle are in bijection with 2-endomorphisms of the identity of the trivial deformation.

The previous considerations are summarized by saying that the groupoid of first order deformations of B corresponds to the complex

$$\mathrm{Hom}(B,B) \longrightarrow HZ^2(B,B)$$

[10] This 2-category can be described more prosaically by defining 2-morphisms in the category of algebra the following way: if $f, g : A \to B$ are two algebras morphisms, a 2-morphism from f to g is an element $b \in B$ such that $bf(a) = g(a)b$ for any $a \in A$.

and the 2-groupoid of first order deformations of $\underline{B}$ corresponds to the complex

$$B \longrightarrow \mathrm{Hom}(B,B) \longrightarrow HZ^2(B,B)$$

where the last terms are the module of Hochschild 2-cocycles of B.

Lie cohomology Let $\mathfrak{g}$ be a Lie $\mathbb{K}$-algebra, $\mathfrak{g}$ is a $\mathfrak{g}$-module for the adjoint action, there is a resolution of $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g},\mathfrak{g})$ by

$$\mathfrak{g} \longrightarrow \mathrm{Hom}(\mathfrak{g},\mathfrak{g}) \longrightarrow \mathrm{Hom}(\wedge^2\mathfrak{g},\mathfrak{g}) \longrightarrow \mathrm{Hom}(\wedge^3\mathfrak{g},\mathfrak{g}) \longrightarrow \dots$$

where the differential is given by

$$\begin{aligned} dx(y) &= [x,y] \\ df(x\otimes y) &= [x,f(y)] - f([x,y]) + [f(x),y] \\ dg(x\otimes y\otimes z) &= [x,f(y,z)] + [y,f(z,x)] + [z,f(x,y)] - f([x,y],z) - f([y,z],x) - f([z,x],y) \\ &\dots \end{aligned}$$

These formulas are related to the coboundary of the Hochschild complex of $U\mathfrak{g}$ the envelopping algebra of $\mathfrak{g}$ Precisely, having in mind that $[-,-]$ is the commutator in $U\mathfrak{g}$, the above formulas are the antisymetrisation of the Hochschild coboundray (taking care of the fact that elements of the complex are already antisymmetric). For example, the formula $[x,f(y)] - f([x,y]) + [f(x),y]$ is written

$$xf(y) - f(y)x - f(xy) + f(yx) + f(x)y - yf(x)$$

in $U\mathfrak{g}$ and is the antisymmetrisation in x and y of the formula $xf(y) - f(xy) + f(x)y$.

As already seen, 2-cocycles of this complex are in bijection with first order deformation of the algebra structure of $\mathfrak{g}$ whose underlying $\mathbb{K}[\epsilon]$-module is free. As before, it can be shown that two deformation are isomorphic iff they are cohomologous and that 1-cocycles are in bijection with automorphisms of the trivial deformation. So the groupoid of first order deformations of $\mathfrak{g}$ is

$$\mathrm{Hom}(\mathfrak{g},\mathfrak{g}) \longrightarrow Z\mathrm{Hom}(\wedge^2\mathfrak{g},\mathfrak{g})$$

where the last term is the module of 2-cocycles.

As for associative algebras, it is possible to include 0-cochains in a deformation complex by enhancing the category of Lie algebras in a 2-category where the 2-morphisms between two morphisms of Lie algebras $f,g:\mathfrak{g}\to\mathfrak{h}$ are elements $b\in\mathfrak{h}$ such that $g=[b,f]$. The complex of length 3

$$\mathfrak{g} \longrightarrow \mathrm{Hom}(\mathfrak{g},\mathfrak{g}) \longrightarrow Z\mathrm{Hom}(\wedge^2\mathfrak{g},\mathfrak{g})$$

is the first order deformation complex of $\mathfrak{g}$ in the 2-category of Lie algebras. It corresponds to a 2-groupoid.

Group cohomology Let G be a group, V a finite dimensional $\mathbb{K}$-vector space and $\rho: G\to \mathrm{End}(V)$ a representation of G in V.

The cohomology of G with coefficients in $\mathrm{End}(V)$ can be computed as the cohomology of the complex

$$\mathrm{End}(V) \longrightarrow \mathrm{Hom}(G,\mathrm{End}(V)) \longrightarrow \mathrm{Hom}(G\times G,\mathrm{End}(V)) \longrightarrow \mathrm{Hom}(G\times G\times G,\mathrm{End}(V))$$

where the differential is given by

$$\begin{array}{rcl} df(g) & = & \rho(g)f - f\rho(g) \\ d\lambda(g,h) & = & \rho(g)\lambda(h) - \lambda(gh) + \lambda(g)\rho(h) \\ d\mu(g,h,k) & = & \rho(g)\mu(h,k) - \mu(gh,k) + \mu(g,hk) + \mu(g,h)\rho(k) \\ & \dots & \end{array}$$

Let's fix a basis in V and identify $\mathrm{End}(V)$ with $M_n(\mathbb{K}) = M_n$. A first order deformation of ρ is a representation of

$$\begin{array}{rcl} R : G & \longrightarrow & M_n(\mathbb{K}[\epsilon]) \simeq M_n \oplus M_n\epsilon \\ g & \longmapsto & \rho(g) + \lambda(g)\epsilon \end{array}$$

R must satisfy the equations $R(gh) = R(g)R(h)$ for all $g, h \in G$, this implies the following equation for λ

$$\lambda(gh) = \rho(g)\lambda(h) + \lambda(g)\rho(h)$$

which is exactly the condition of being a 1-cocycle of G in M_n.

As always, one can show that two deformations are isomorphic iff their cocycles are cohomologous and that the module of 0-cocycles is in bijection with automorphism of the trivial deformation. So the groupoid of first order deformations of ρ is

$$\mathrm{End}(V) \longrightarrow Z\mathrm{Hom}(G, \mathrm{End}(V))$$

where the last term is the module of 1-cocycles.

Čech cohomology Let $U = \{U_i, i \in I\}$ be a given open covering of a space X, and let E be a vector bundle on X, trivialized by U, the adjoint bundle $\mathrm{End}(E)$ is also trivialized by U. For any ordered subset $(i_1, \dots, i_n)$ of I, let $U_{(i_1,\dots,i_n)}$ be the intersection $U_{i_1} \cap \dots \cap U_{i_n}$, the Čech cohomology of (X, U) with coefficients in $\mathrm{End}(E)$ is given by the complex

$$C^0 \longrightarrow C^1 \longrightarrow C^2 \longrightarrow \dots$$

where C^n is the module of sections of $\mathrm{End}(E)$ over $\coprod_{(i_1,\dots,i_n)} U_{(i_1,\dots,i_n)}$ where the disjoint union is taken over all ordered subsets of I of cardinal n.

Let $\{f_{(i_1,\dots,i_n)}\}$ be n-cochain, its coboundary is defined by the formula

$$\begin{array}{rcl} df_{(i_1,\dots,i_{n+1})} : U_{(i_1,\dots,i_{n+1})} & \longrightarrow & \mathrm{End}(E) \\ x & \longmapsto & \sum_k (-1)^k f_{(i_1,\dots,\check{i_k}\dots,i_{n+1})}(x). \end{array}$$

As seen in §2.3, 1-cocycles of this complex are in bijection with first order deformations of E trivialized on U. As before one can check that two deformations are isomorphic iff the corresponding cocycles are cohomologous, in particular the 0-cocycles are in bijection with isomorphisms of the trivial deformation. So the groupoid of first order deformations of E is

$$C^0 \longrightarrow Z^1$$

where the last term is the module of 1-cocycles.

4 Some good questions

...that would deserved longer answers.

4.1 What about higher cohomology groups ?

The interpretation sketched here gives a geometric meaning only to low dimensional cochains and it is natural to ask for a continuation of this interpretation for higher cochains. The answer is not simple and has two steps.

The first step we saw here : the fact that mathematical objects are living in categories forces us to deal with spaces with an underlying category of points. Those spaces are called *stacks* and their definition is involved, but they have the property that their tangent spaces are complexes in non-positive cohomological degree. So this setting gets us half of complexes as tangent spaces.[11] The other half, complexes in non-negative cohomological degree, appear in the so-called *derived geometry*, which definition is as much involved as that of stacks.

At the end of the story, the full deformation cohomology complexes appear as tangent spaces to some *derived stacks*.

type of space	structure of the tangent
manifold	vector space T_0
stack	complex $T_1 \to T_0$
higher stacks	complex $\dots T_2 \to T_1 \to T_0$
derived manifold	complex $T_0 \to T_{-1} \to \dots$
derived stacks	complex $\dots \to T_1 \to T_0 \to T_{-1} \to \dots$

4.2 What about obstruction theory ?

Higher cohomology groups are used to get obstruction classes to glueing of local deformations or their extensions to second order. This structure is often encoded by a bracket operation making two 1-cocycles in a 2-cocycle.

We will not detail the structure of this bracket but mention only that it is a feature (and was actually a motivation) of derived geometry that for any morphisms $x : X \to Y$ and $u : X \to X'$ where X' is is an infinitesimal thickening of X, there exists a natural obstruction in the tangent complex of Y at x which vanishes if x can be extended to a morphism from X'.

4.3 Why is there a dg-Lie structure on deformation complexes ?

The bracket mentionned before is part of a dg-Lie algebra structure on the tangent complexes of any derived stack. Let's explain heuristically why. For a pointed derived stack (X, x) there is an operation that construct a derived stack $\Omega(X, x)$[12] whose points are automorphisms of the point x in X. As expected, $\Omega(X, x)$ has a group structure and its tangent complex $\omega(X, x)$ at the identity has the structure of a Lie algebra.

[11] All, the tangent complexes of the examples have to be reindexed to be in non-positive degree.

[12] $\Omega(X, x)$ is called the loop stack in analogy with the loop space in algebraic topology.

It turns out that $\omega(X,x)$ is the shift of the tangent of X at x[13] and the dg-Lie algebra structure on the latter is nothing than the one of $\omega(X,x)$ transported by this identification.[14]

4.4 Why are deformation cohomologies given by derived endomorphisms ?

This may be the most important question because it gives the way to compute the deformation complexes.

In a number of cases, the complex that computes deformation of an object is the complex of endomorphisms of that object in some derived category:

1. the Lie cohomology complex of $\mathfrak{g}$ computes $R\mathrm{End}_{\mathfrak{g}}(\mathfrak{g})$ the endomorphisms of $\mathfrak{g}$ in the derived category of $\mathfrak{g}$-modules,
2. the Hochschild cohomology complex computes $R\mathrm{End}_{A^\circ\otimes A}(A)$ the endomorphisms of A in the derived category of A-bimoudles,
3. $Ext^*_A(M,M)$ is the homology of $R\mathrm{End}_A(M)$ the endomorphisms of M in the derived category of A-modules,
4. $H^*(G,V)$ is the homology of $R\mathrm{End}_G(V)$ in the derived category of G-modules,
5. $H^*(X,\mathrm{End}(E))$ is the homology of $R\mathrm{End}_X(E)$ the endomorphisms of E in the derived category of sheaves over X, etc.

This fact is another consequence of the fact that the tangent complex of a stack X at some point x is the shift of the tangent complex to the loop stack $\Omega(X,x)$. $\Omega(X,x)$ is the stack classifying automorphisms of x in X, so its tangent at the identity is made out of *infinitesimal automorphisms*. Now, in case the automorphisms of x where naturally the invertible elements in an algebra of endomorphisms, infinitesimal automorphisms are nothing than endomorphims (as the Lie algebra of Gl_n is M_n).

In case the automorphisms of x where not naturally embedded in an algebra, infinitesimal automorphisms are just the Lie algebra (whatever it means in this context) of the group $\Omega(X,x)$. This fact also explain (and make precise) the remark made above that the "tangent" to the non-abelian $H^1(X,G)$ is $H^1(X,\mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra of a Lie group G. [15]

A Some ideas behind deformations

A.1 Moduli problems

As algebraic structure have a base ring (the ring over which the underlying module of the structure is a module) geometric structure have often a base space (this is clear for a bundle). The fundamental thing about these base objects is the possibility to change them:

1. if $\mathbb{K}\to\mathbb{K}'$ is a morphism of commutative algebras, a $\mathbb{K}$-algebra A can be made functorially into a $\mathbb{K}'$-algebra $A\otimes_{\mathbb{K}}\mathbb{K}'$,
2. if $X'\to X$ is a differentiable map of manifolds, a bundle $E\to X$ can be made functorially into a bundle $E\times_X X'\to X'$ over X'.

[13] Looping or delooping in complexes is just a shift of indices.

[14] This explain also why the dg-Lie algebra structure needs a shift of degree to be define nicely.

[15] To be precise, one should add that for Lie cohomology and Hochschild cohomology one needs to loop twice: Hochschild cohomology of A is naturally the Lie algebra of $\Omega_{id}\Omega(LinCat,A)$ where $LinCat$ is the moduli 2-stack of linear categories, this is why tangent vectors to $LinCat$ are calssified by 2-cocycles and not 1-cocycles as it would be with only one looping.

When X is seen a base space, it is convenient for the intuition to refer to a vector bundle over X as a family of vector spaces parametrized by X. The algebraic picture can be made into the geometric one via spectra of commutive rings: a $\mathbb{K}$-algebra is the same thing as a sheaf of rings over spec($\mathbb{K}$), *i.e.* a family of algebras parametrized by points of spec(A).

The existence of these families of structures parametrized by some base space is the source of moduli space for the structure involved. Indeed, the intuition of the existence of a moduli space, *i.e.* of a topology on the set of all possible variation of the structure (or only isomorphisms classes) rises from the existence of continous families of structure: we would very much like to think a bundle over X, *i.e.* a family of vector space over X, as a continous map from X to the *space* of vector spaces.

So we don't know what is the topology of the moduli space but we know what all continous maps from ordinary spaces to it should be. This data is sufficient to construct the topology on the moduli space. [16]

Here is a general idea: in mathematics, any object that has a definition exists, but possibly outside the class of objects one is dealing with. The best known illustration is roots of $x^2 + 1$, another is the double point U of above.

Yet another is that any moduli problem has a solution as some functor[17], this is the so-called functor of points approach to moduli problems. But not all those functors are spaces, and the problem of the existence of the object is transformed (in fact formalized) into the problem of finding a space that represents the moduli functor. If such a space does not exist and we are sad about it, we are left with two options: either find a best spatial approximation (the so-called coarse moduli space) or to enlarge the category of spaces and hope to find a representative there (orbifold, geometric stacks...). I won't say more this is a subject for a whole new talk.

Anyway, the study of infinitesimal deformations is only the study of tangent spaces to these moduli spaces and does not need all that technology.

A.2 Why ringed spaces are a good idea

Topology is about open subsets and continuous functions ; differential, complex or algebraic *geometries* are about specific continuous functions (C^∞, holomorphic, polynomial). The tool used to enhance topology into geometry is the *sheaf of functions* to work with, the so-called *structure sheaf*. An space with a sheaf of rings is called a ringed space. The sheaf of rings is called a sheaf of local rings if its stalks are local rings, a space with a sheaf of local rings is called a locally ringed space. Those are the objects of modern geometry.[18]

Every manifold defines canonically a locally ringed space, but there is more and this is the point. The simplest examples are ringed points: a sheaf of local rings on the point is just one local ring, and the following are of major importance in geometry.

1. $(pt, \mathbb{K})$: the point equiped with the base field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.
2. $(pt, \mathbb{K}[\epsilon])$ where $\mathbb{K}[\epsilon] = \mathbb{K}[x]/(x^2)$: this is the model for two infinitesimally closed points[19]. A

[16] Althought not in the obvious way : the moduli space is not in general the set of equivalences classes endowed with the final topology.

[17] Precisely a presheaf over the category of all parameter spaces.

[18] The condition of locality might read bizarre at first, it can be justified by the facts that sheaves of functions on manifolds do have local rings stalks and that morphisms between manifolds induce local ringed morphisms between stalks, but it can also be justified by C^∞-rings spectral theory which reconstruct a manifold from its C^∞-ring of functions (as a spectrum) this theory uses in a fundamental way the notion of local rings: the spectrum of a ring is the moduli space of all possible localizations of the ring, the sheaf of rings is then the universal family of localizations (no good reference exists).

[19] To get convinced why, show that the rings of (polynomial, C^∞, holomorphic...) functions on two points on the line of abscisses 0 and a is always $\mathbb{K}[x]/x(x-a) \simeq \mathbb{K} \oplus \mathbb{K}$ and then move a to 0.

morphism $(pt, \mathbb{K}[\epsilon]) \longrightarrow (X, \mathcal{O}_X)$ of locally ringed spaces is the same thing as a point of X and a tangent vector at this point (cf. §A.3).

3. $(pt, \mathbb{K}[\eta])$ where $\mathbb{K}[\eta] = \mathbb{K}[x]/(x^3)$: this is the model for three points infinitesimally closed aligned in a single direction. A morphism $(pt, \mathbb{K}[\eta]) \longrightarrow (X, \mathcal{O}_X)$ of locally ringed spaces is the same thing as 2-jet of arc in X. It is clear how one could continue in this direction to encodes jets of any order and any dimension.

4. Germs of functions at some point x of some space are thought as functions on the intersection of the filter all neighourhoods x, but in a separated space, this intersection is only the point. In ringed spaces, this intersection is not isomorphic to the point but is a point equipped with a local ring. Again, this creates an actual object to deal with the intuition.

All the previous examples were constructed as some limit of a situation in ordinary spaces. This is why the category of locally ringed spaces is nice and useful: it is a completion of usual spaces.[20] The completion of spaces by locally ringed spaces is essentially a completion of spaces for *intersection singularities*, *i.e.* multiple points.[21] In particular in such a completion, any fiber of any differentiable map between (C^∞, holomorphic...) manifolds exists.

Remark on generalisations of spaces One can sketch an analogy with number theory here. This analogy is to be taken informally, I do not claim there is a identical structure between the two settings, but I claim the motivations for these generalisations are somehow identical.

- It is known that rational number were invented to close natural numbers under basic operations; in geometry, manifolds have been invented because not everything was an affine space.
- Real numbers are limits of rational numbers, so are locally ringed spaces for manifold, their category is complete in some sense.
- Complex number were invented as formal solutions to algebraic equations; stack and derived manifolds have been invented to be formal solutions to moduli problems.
- When compared to $\mathbb{Q}$, the whole of complex numbers is wild and the field of algebraic numbers (solution to algebraic equations in $\mathbb{Q}$) is a smaller nicer one to work with ; when compare to manifolds, the whole category of derived stacks is wild too, the subcategory of Deligne-Mumford stacks is nicer. As algebraic numbers are some specific completion so are Deligne-Mumford stacks (completion by etale quotients).
- One could extend the previous analogy between periods numbers (values of integration of algebraic differential forms on algebraic cycles) and Artin stacks (completion by submersive quotients).

A.3 $\mathbb{K}[\epsilon]$ and tangent vectors

If A is a commutative ring and M a A-module, a *derivation of A in M* is a map $d : A \to M$ such that $d(ab) = d(a)b + ad(b)$.

For example, let X be a vector field on a manifold M and let $O(M)$ be the ring of functions on M, the derivation along X define a derivation of $O(M)$ in $O(M)$ (viewed as a module over itself):

$$\begin{aligned} X : O(M) &\longrightarrow O(M) \\ f &\longmapsto df(X) : x \mapsto df_x(X(x)) \end{aligned}$$

[20]This is the motto of any enlargement of topological spaces: the new spaces (infinitesimally closed points, singular manifolds, orbifolds, stacks, topos...) are always limit situations of the classical ones.

[21]A better completion for intersection singularities exists, this is the purpose of derived geometry. Also a completion for *quotient singularities* exists, this is the geometry of orbifolds and stacks. It is only in this double context that the whole complexes of deformation cohomologies appear as tangent spaces.

it is a derivation as $d(fg) = (df)g + fdg$.

It turns out there is a isomorphism between the module of derivations of $O(M)$ in $O(M)$ and the module of vector fields on M. It is sufficient to prove it locally, *i.e.* when $M \simeq \mathbb{R}^n$: let U_i be the canonical basis for vector fields on $\mathbb{R}^n$, the derivation associated to U_i is $\frac{\partial}{\partial x^i}$ and the derivation associated to any vector field $X = \sum_i \xi^i U_i$ is $d_X = \sum_i \xi^i \frac{\partial}{\partial x^i}$; reciprocally, a derivation d of $O(\mathbb{R}^n)$ in $O(\mathbb{R}^n)$ can be evaluated on the coordinate functions x^i to give n functions $\xi^i = dx^i(X)$, and the vector field corresponding to d is $\sum_i \xi^i U_i$.

Let x be a point of M and $ev_x : O(M) \longrightarrow \mathbb{K}$ the evaluation of functions at x, this morphism induces a $O(M)$-module structure on $\mathbb{K}$. Let V be a tangent vector at x, in local coordinates, V can be written $\sum_i v^i U_i$ (where $v^i \in \mathbb{K}$). V defines a derivation d_V of $O(M)$ in $\mathbb{K}$ by the formula

$$d_V(f) = df_x(V).$$

This correspondance induces a $\mathbb{K}$-linear isomorphism between the vector space of derivations of $O(M)$ in $\mathbb{K}$ (view as an $O(M)$-module via the map ev_x) and the tangent vector space at x.

Finally, let's see that a derivation into $\mathbb{K}$ is the same thing as a ring morphism $O(M) \longrightarrow \mathbb{K}[\epsilon]$. Let ϕ be a ring morphism $O(M) \longrightarrow \mathbb{K}[\epsilon]$, using $\mathbb{K}[\epsilon] \simeq \mathbb{K} \oplus \mathbb{K}\epsilon$, $\phi = \alpha \oplus \beta\epsilon$ for two maps $\alpha : O(M) \longrightarrow \mathbb{K}$ and $\beta : O(M) \longrightarrow \mathbb{K}$. ϕ being a ring morphism is equivalent to the following equations on α and β:

$$\begin{aligned} \alpha(ab) &= \alpha(a)\alpha(b) \\ \beta(ab) &= \beta(a)\alpha(b) + \alpha(a)\beta(b) \end{aligned}$$

that is α is a ring morphism and β is a derivation of $O(M)$ in $\mathbb{K}$ for the module structure on $\mathbb{K}$ given by α.