The Geometry of Ambiguity

An introduction to the ideas of Derived Geometry

M. Anel

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1 Introduction

Derived Geometry is a theory of geometry designed in order to have a better behavior of singular points than in Algebraic, Complex or Differentiable Geometries. It is named after derived categories, derived functors, derived tensor products, and pretty much anything "derived", because it proposes a setting where the natural constructions of all the notions give directly the derived versions. Within Derived Geometry, nothing has to be derived anymore.

As often in history of Mathematics, most of the computing methods to deal with singular points were invented before the proper formalization of a theory organizing and justifying them (Koszul resolutions, Chevalley complexes, equivariant methods...). Perhaps, what is really new in Derived Geometry is not so much its methods, but the new understanding it proposes. Derived Geometry has successfully interpreted *in terms of geometry* these previously ad hoc constructions. We shall come back to this in our conclusion.

Tangent complexes The easiest way to introduce Derived Geometry is probably the following analogy. Recall that homological algebra can be read as the enhancement of the theory of vector spaces into the theory of chain complexes, then Derived Geometry is to geometry (ordinary topological spaces, manifolds, schemes...) what chain complexes are to vector spaces. This analogy is good enough because a number of features of chain complexes do have analogs in "derived spaces". For example, complexes in positive or negative degrees have corresponding derived spaces called respectively *stacks* and *derived schemes* (or *derived manifolds*, depending on the context). It is also possible to truncate derived spaces and extract analogs of Z_0 and H_0.

In fact, there exists a precise comparison between the two theories: it happens that the tangent spaces to derived spaces are naturally chain complexes and no longer vector spaces. The relationship between the different derived spaces and their tangent is summarized in the following table which is useful to keep in mind.

Type of space	Structure of the tangent		
scheme/manifold	vector space		T_0
stacks	chain complex	$\cdots \to T_1 \to$	T_0
derived scheme/manifold	cochain complex		$T_0 \quad \to T_{-1} \to \ldots$
general derived space	unbounded cx.	$\cdots \to T_1 \to$	$T_0 \quad \to T_{-1} \to \ldots$

It turns out that these tangent complexes are not so difficult to compute in practice. Examples involve Hochschild complexes, tangent sheaf cohomology, group cohomology... Actually, they are somehow so easy to compute that people stumbled upon them before realizing what they were. For example, it is a

classical fact of deformation theory that the tangent space at a point of a moduli space[1] is given by a homology group of some complex, but the rest of these complexes was for a long time overlooked. It is only by realizing that they were more regular as a whole (they have a Lie algebra structure and are often perfect complexes) than the sole homology group of interest that people came up with the idea that moduli spaces would be more regular if they could be defined such that the whole complex would be the tangent space. This tangent Lie structure was used in deformation theory [20], emphasized by Drinfeld [12] and theorized by the "derived deformation theory" (DDT) [25, 24]; and the "perfection" of tangent complexes can be understood as the meaning of the "hidden smoothness" of moduli spaces [9]. Altogether, these ideas sprung mostly in the late '80s-'90s.

Singularities   This new geometry conserves the old one, classical manifolds or schemes embed faithfully in derived spaces. The new features concerns in fact only singular points and thus spaces with singularities: smooth manifolds and smooth schemes behaved as they always did in this new geometry.

Singular points are classically defined as the points whose tangent space has a dimension bigger than expected, which is a way to say that not all tangent vectors can be integrated into a curve. They are essentially of two kinds: *quotient and intersections singularities*. For example, quotient singularities appear in a quotient by a group action when a point has a stabilizer under the action, and intersection singularities appear when the intersection is not transverse. We shall see in section 2 how these two kinds of singularities create respectively a positive and a negative part in the tangent complexes. This is actually the first insight about tangent complexes: a point is singular iff its tangent complex has non-zero homology. This extra tangent structure says something about the singularity and in fact, the whole formal neighborhood of a point can be reconstructed from its tangent complex (see 2.2.3).

Stacks   This daring idea of a new kind of space whose tangent spaces would be chain complexes, probably best summarized in the introduction of [9], could only be imagined because of a very mature ground. First of all, homological algebra had spread in Algebraic Geometry where it had rebirthed with *Derived Categories* and *total derived functors*, and since the '60s every geometer was accustomed to the "derived philosophy" which asserts that objects could have more regular properties if they were replaced by some "derived" version, enhanced with some "hidden" structure.

But most significant was the influence of *Stack Theory*, which had risen from an obscure topic for descent and non-abelian cohomology [19] to a proper geometrical theory after the works of Deligne, Mumford and Artin [3, 10] in the late '60s-'70s. Stacks are a notion of highly unseparated space where points can

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[1] A *moduli space* is a space classifying something, it could be the solutions to some equations but also a structure such as the space of curves, the space of vector spaces... Inspired by the example of elliptic curves, *moduli* is the general name for the coordinates on those spaces.

form categories instead of just sets or posets and this was perfectly suited to encode the singular structures of moduli spaces.[2]

Stacks were for some time a kind of necessary evil in Algebraic Geometry, something efficient but whose geometrical nature was unclear. The proper context to understand them is in fact ∞-categories and an analogy with homotopy theory. Implicit in [19], this analogy was fully devised in the '80s by Grothendieck in Pursuing Stacks and successfully formalized in the '90s by Hirschowitz and Simpson [26, 54] by using the ideas and methods of Algebraic Topology (simplicial presheaves and model categories).[3]

The importance of stacks for Derived Geometry was that they were already proposing a notion of space with tangent complexes. Within stack theory, a point can have a symmetry group and the Lie algebra of this group is a part of the tangent structure at the point [38]. In other words, the tangent complex of a stack at some point encodes not only first order deformations of this point but also the symmetries of these deformations (which are in bijection with first order deformations of the identity endomorphism). The tangent complexes of stacks, a notion that would eventually be better formalized by Simpson [54] in the context of higher stacks, are concentrated in (homological) non-negative degrees and have been eventually recognized to explain half of the tangent complexes of the DDT. In fact, stack theory has been recognized to be one half of the pursued derivation of geometry, precisely the part regularizing the properties of quotient singularities [9, 24]. The other half, corresponding to intersection singularities and the negative part of the tangent complex, would take some new ideas.

The good formalism   The first formalisms of Derived Geometry were purely algebraic: since they were made to extract tangent chain complexes, they were using algebraic structures on chain complexes (Lie dg-algebras, commutative dg-algebras and even dg-coalgebras) together with the natural extension of commutative algebra to this setting.[4] This was efficient, but the spatial langage was rather a psychological trick to justify the formal manipulations than yet a proper insight on a new geometry. Also, the formalisms were either for formal neighborhoods only [24, 35] or too specific [9, 32], and were calling for a better theory.

The good idea was eventually found in the working philosophy of algebraic geometers, which, from the formalisation of algebraic groups [11] to the formalisation of moduli problems and algebraic stacks [10, 54], were constructing the objects of Algebraic Geometry in two steps: first, by defining a notion of affine scheme (encoded faithfully by a commutative ring of functions), and then, by completing the category of affine schemes by the quotient stacks of (etale or smooth) groupoids (we shall detail this in section 3.1). The recipe for Derived Geometry was then to do the same starting with some "derived rings" (commu-

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[2]It was folkloric at the time that the symmetries of points in moduli spaces were an obstruction to describe them locally by affine schemes or manifolds.

[3]These methods were actually available to Grothendieck, but he did not like them and refused to used them.

[4]Perhaps the first occurence of this is to be looked in Physics, see 3.5.

tative dg-algebras). Hinich formalized this in the infinitesimal scale [24]. The attempt of Behrend [5], although correctly conceived on a conceptual level, was wronged by the technical limitations due to the use of 2-category theory instead of higher category theory. It was finally Toën and Vezzosi, both trained in homotopical algebra and ∞-categories, who were the first to formalize a proper setting by working out Algebraic Geometry within model categories [61, 62, 63].

Later, Lurie, with a better background in categorical logic, understood how to apply this approach to differentiable and complex geometries [43, 56, 50]. He also improved the presentation of the theory by working out fully the higher categorical background. This eventually led him to re-prove and improve a number of major theorems of Algebraic Geometry in the derived setting and to develop a tremendous amount of higher categorical notions on the way [39, 40, 41].

Nowadays, Derived Geometry is out there and its methods are spreading from Algebraic Geometry to other mathematical fields. The most important exportation of its ideas may yet be Derived Symplectic Geometry, where it provides a definition of symplectic structures for singular spaces [6, 7, 49, 60].

Derived rings The nature of those "derived rings" with which to start Derived Geometry is actually an important degree of freedom of the theory and this was the reason for the very general setting of Toën and Vezzosi in [63]. If the first motivation of Derived Geometry was to improve the tools of Algebraic Geometry to work on singular spaces, Toën and Vezzosi were also motivated by an original application in Algebraic Topology where a "Brave New Algebraic Geometry" was nascent. The application was regarding elliptic cohomology [63, ch. 2.4] and would eventually be worked out fully by Lurie [42].

The possibility of this choice of derived rings is also what authorizes the definition of derived differential geometry and derived complex geometry [43, 50, 56]. We shall say more about this in 3.2.6.

Why derivation ? Why stacks ? Why derived rings ? How can we have a geometric intuition of these objects ? We shall try to answer these questions through the text. Let us only say for now that the deep reasons for the necessity for the derivation of geometry (and all derivations) is not to be looked for within geometry or algebra but within an insufficiency of Set Theory and even of its extension Category Theory. We shall come back to this in our conclusion.

Other texts Previous texts have been written to explain the ideas of Derived Geometry, the main ones would be Toën's surveys [57, 59] and Lurie's introductions to his DAG series and to his book [41]. The present text has been written as a complement to these texts, I have tried to emphasize the conceptual guidelines of the theory rather than the applications (which are detailed in the aforementioned texts) and to give a more global view on the matter of derivation.

Notations Through all the text, the word *space* shall be used in an informal way to refer to the general idea of space, independently of any mathematical formalization (topological space, topos, manifold, scheme, stack...). A basic knowledge of category theory is assumed, in particular regarding limits and colimits.

For the differential geometer We shall limit the presentation of derived *algebraic* geometry, but all considerations can be transposed into differential and complex geometries. For the reader uncomfortable with the notion of scheme, it is enough to know that the notion of a scheme differs from that of a manifold by the fact that it is allowed to have singularities. In the whole text the word "scheme" can be always be understood as "manifold with possible singularities". The expression "affine scheme" means a scheme which can be defined as the zeros of some functions in some affine space $\mathbb{A}^n$, "general" schemes are constructed by glueing affine schemes. Contrary to Differential Geometry where any manifold can be embedded in $\mathbb{R}^n$, not every scheme is affine, i.e. a subspace of some $\mathbb{A}^n$ (the projective spaces, for example). This explain the double vocabulary of "affine/non-affine" scheme in Algebraic Geometry and in this text, it can be ignored by the differential geometer.

2 Tangent complexes

The easiest way to get familiar with derived Algebraic Geometry features is the computation of the so-called *(co)tangent complexes*. The tangent complex is

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[5]In particular the notion of homotopy colimit that I have chosen to put at the heart of this text (see 3.2.1).

an enhancement of the tangent vector space into a chain complex, it is defined at every point of a space and globally as a bundle. The cotangent complex is the corresponding algebraic object, it is to the tangent complex what the module of differential forms is to tangent spaces and, analogously, it is more suitable for computations.

The theory of cotangent complexes for affine schemes is just a more geometric name for André-Quillen cohomology of commutative rings. The motivation was the study and classification of extensions of rings by cohomological methods. The geometric interpretation is the theory of infinitesimal deformations, where, the cotangent complex helps to answer questions such as "can a tangent vector be integrated into a path ?"[6]

Although the motivation was coming from geometry, the geometrical meaning of these complexes (as a whole) was not at all clear at the time (Grothendieck wonders about such an interpretation in the introduction of [21]). Derived Algebraic Geometry has given a clear answer to this question: within Derived Algebraic Geometry, the natural notion of tangent *is* the tangent complex.

As we will show in examples, tangent complexes are easy enough to compute in practice. For smooth points they are quasi-isomorphic to the tangent spaces but they contain a lot more information at singular points. This latter fact is their whole interest. By focusing on the algebra of functions (rings) and not only on topology (algebra of open subsets), Algebraic Geometry had already established, with the notion of scheme, a good notion of space that could support singularities, but Derived Algebraic Geometry goes further and provides a notion of space with an even better handling of singularities. For example, there is no need of transversality lemmas anymore to compute intersections in Derived Algebraic Geometry (see 3.3.2).[7]

The most important (and the most intriguing, but see §2.2.2) property of tangent complexes, which has no counterpart in Algebraic Geometry, is that they have a natural Lie algebra structure. At smooth points, this structure is trivial (i.e. abelian), but otherwise it contains a lot of structure about the singularity. For example, the whole formal neighborhood of the singular point can be reconstructed from this structure (see §2.2.3). This is particularly useful to study moduli spaces. This Lie structure is computable in practice by mean of L_∞-structure[8] but often reduces to a simple Lie dg-algebra structure. The

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[6]The problem is obvious on a smooth manifold, but not if singular points are allowed, as in schemes. A point is singular if its tangent space has a dimension bigger than the dimension of the space. This means precisely that not all tangent vectors are tangent to paths. The structure of the tangent complex, in particular its Lie structure help to describe the subset (a cone) of vectors that can be integrated. This is the so-called cohomological obstruction calculus.

[7]Difficulty in mathematics might be transformed but never really cancelled, the problem of having transversal intersections is replaced by that of finding a nice resolution (for example a Koszul resolution) of the rings at hand. However we have transformed a geometrical problem into an algebraic problem, which is always more suited for computations.

[8] We shall not give the full definition of a L_∞-structure on a chain complex $\mathfrak{g}$ and we refer to [46] for details. It will be sufficient to know that L_∞-structure are essentially a Lie structure but where all equations have been relaxed to hold only up to homotopy. Such a

reason to be of this Lie structure will be explained in 2.2.2.

We shall not give a proper definition of the tangent complex (this would require the introduction of too many technical notions) but a few words will be said in 3.2.5 about the cotangent complex. The reader can see [59] for an introduction and references therein for precise definitions.

2.1 Examples

2.1.1 Subschemes and intersections

In this section we compute the tangent complex of an affine subscheme defined by some equations. If there is more than one equation, this case encompasses the intersection of subschemes.

Let $\mathbb{A}^n$ be the affine plane of dimension n, we consider the affine scheme Z of zeros of a polynomial function $f : \mathbb{A}^n \to \mathbb{A}^m$, Z is defined by the fiber product

$$
\begin{array}{ccc}
Z & \longrightarrow & \mathbb{A}^n \\
\downarrow & \ulcorner & \downarrow f \\
1 & \overset{0}{\longrightarrow} & \mathbb{A}^m.
\end{array}
$$

The function f is given by a ring map $f : \mathbb{C}[y_1, \ldots y_m] \to \mathbb{C}[x_1, \ldots x_n]$. The ring of functions on Z is the ring A, quotient of $\mathbb{C}[x_1, \ldots x_n]$ by the ideal generated by the elements $f(y_j)$.

Let x be a point of Z, it determines a point of $\mathbb{A}^n$, that we still call x, whose image by f is 0 in $\mathbb{A}^m$. The tangent space $T_x Z$ to $x \in Z$ is defined as the kernel of the differential $df_x : T_x \mathbb{A}^n \to T_0 \mathbb{A}^m$. The point x is called a *regular point* of Z if df_x is surjective. In this case, the dimension of $T_x Z$ is $n - m$ and it can be proven that so is the dimension of Z around x. If df_x is not surjective, x is called a *singular point*. If we are thinking Z as the intersection of the $Z_j = \{f_j = 0\}$, a point x is singular precisely when the intersection is not transverse.

The *tangent complex* of Z at x is defined as the chain complex

$$
\mathbb{T}_x Z \quad = \quad T_x \mathbb{A}^n \to T_0 \mathbb{A}^m
$$

where $T_x \mathbb{A}^n$ is in (homological) degree 0 and $T_0 \mathbb{A}^m$ in degree -1. The tangent space $T_x Z$ is the H_0 of this complex and x is regular iff the homology of $\mathbb{T}_x Z$ is concentrated in degree 0. The H_{-1} measures the defect of submersivity of f at x, and the excess of dimension of $T_x Z$, i.e. the singularity of x. Also the Euler characteristic of $\mathbb{T}_x Z$ is always the expected dimension of Z around x. In

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structure is given by higher order brackets $[-, \ldots, -]_n : \mathfrak{g}^{\otimes n} \to \mathfrak{g}[2 - n]$ having homological weight $2 - n$. The bracket of arity 1 is a differential, the bracket of arity 2 is the Lie bracket, the bracket of arity 3 is called the jacobiator and produces a chain homotopy ensuring Jacobi identity, higher brackets are encoding higher coherence conditions. The structure is that of Lie dg-algebra if the brackets in arity > 2 are zero. Altogether the brackets can be written as the homogeneous components of a single map $S(\mathfrak{g}[-1]) \to \mathfrak{g}$, where $S(\mathfrak{g}[-1])$ is the symmetric algebra on $\mathfrak{g}[-1]$.

particular, it is locally constant as a function of x (and not only semi-continuous as is the dimension of $T_x Z$).

As we mentioned, the main feature of $\mathbb{T}_x Z$ is that it is endowed with a L_∞-structure. This structure exists in fact on the shifted complex[9] $\mathbb{T}_x Z[1]$ concentrated in degrees -1 and -2 and is particularly simple to make explicit in this example. It turns out that an L_∞-structure on a complex concentrated in such degrees is given by a single map (satisfying no condition),

$$S(T_x \mathbb{A}^n) \to T_0 \mathbb{A}^m$$

where $S(T_x \mathbb{A}^n)$ is the symmetric algebra on $T_x \mathbb{A}^n$. Then the L_∞-structure is simply given by the Taylor series of f, the brackets $[-, \ldots, -]_n$ of the L_∞-structure are then given by the homogeneous components, i.e. the higher differential $D^n f$ of f viewed as symmetric functions of n variables.[10]

2.1.2 Quotients

We shall now describe the tangent complex of the quotient of a smooth scheme by a group action.

Let G be a group with Lie algebra $\mathfrak{g}$, acting on a smooth scheme X. Let x be a point of X and $\bar{x}$ the corresponding point in the quotient X/G. The infinitesimal action induces a map $\mathfrak{g} \to T_x X$, where $T_x X$ is the tangent space at x. The image of this map is the tangent to the orbit of x and the kernel is the Lie algebra of the stabilizer of x. Let us call x *regular* if $\mathfrak{g} \to T_x X$ is injective and *singular* if not.[11] If x is regular, the action of G is locally free[12] around x and it is possible to find a local transversal section to the orbits. This section can be used to define the local structure around $\bar{x}$ in the orbit space. In particular, it is of dimension $d = \dim X - \dim G$ and we get that $T_{\bar{x}}(X/G) = (T_x X)/\mathfrak{g}$.

The *tangent complex* of X/G at $\bar{x}$ is defined as the chain complex

$$\mathbb{T}_{\bar{x}}(X/G) \quad = \quad \mathfrak{g} \to T_x X$$

where $T_x X$ is in (homological) degree 0 and $\mathfrak{g}$ in degree 1. As before, x is regular iff the homology of $\mathbb{T}_x(X/G)$ is concentrated in degree 0, in which case H_0 is the tangent space in the quotient. The H_1 of this complex is the Lie algebra of the stabilizer of x, it measures the singular nature of $\bar{x}$. The Euler characteristic of

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[9]The reason for this shift will be explained in 2.2.2.

[10]This example shows the versatility of L_∞-structure, they can be quite remote from actual Lie algebra structures. But this ability to interpolate between what could be call "formal neighborhood structures" and Lie algebra structures is somehow the main interest of L_∞-structures.

[11]Because of discrete groups, x can be regular in this sense but still be a singular point in the quotient (an orbifold point). However, this kind of singularity can be unfolded by an etale cover (a local diffeomorphism) and therefore does not count as a singularity from the point of view of differential calculus.

[12]By which we mean that a neighborhood of the identity of G is acting freely around x. In Algebraic Geometry, local henselian rings need to be used.

$\mathbb{T}_{\bar{x}}(X/G)$ is always the expected dimension of X, again it is locally constant as a function of x (and not only semi-continuous as the dimension of $T_{\bar{x}}(X/G)$).

This tangent complex can be proven to be the tangent space of the *quotient stack* $X//G$. In this setting, the points of $X//G$ form a groupoid and not a set. The tangent space at any point is also a groupoid but with an extra linear structure, and such an object is the same thing as a chain complex in homological degrees 0 and 1 [1, 18]. In this tangent groupoid, the H_0 is the part of tangent encoding first order deformation of $\bar{x}$ as an object in $X//G$ and H_1 is the part encoding the symmetries of such deformations (which are equivalent to first order deformations of the identity of $\bar{x}$ as a morphism in $X//G$).

Again, the shifted complex $\mathbb{T}_x(X/G)[1]$, concentrated in degree 0 and -1, is endowed with an L_∞-structure easy to describe explicitly. It turns out that an L_∞-structure on a complex concentrated in degrees 0 and -1 is given by a two maps

$$d' : S(T_x X) \otimes \mathfrak{g} \to T_x X \qquad \text{and} \qquad d'' : S(T_x X) \otimes \Lambda^2 \mathfrak{g} \to \mathfrak{g}.$$

where $S(T_x X)$ is the symmetric algebra on $T_x X$. The brackets $[-, \ldots, -]_n$ of the L_∞-structure are the homogeneous components of these maps, viewed as symmetric functions of n-variables.

Recall that the action of the Lie algebra $\mathfrak{g}$ on X can be encoded by a Lie algebroid [47]. A Lie algebroid structure is characterized by two maps: the anchor and a Lie bracket. The maps d' and d'' are given respectively by the Taylor series of the anchor map and of the Lie bracket at the point x. We claim that the equations of the L_∞-structure give exactly the conditions on the anchor and the bracket of a Lie algebroid.

In particular, if X is a single point $*$, the quotient stack $*//G$ is the stack BG (classifying G-torsors). Its tangent complex at $*$ reduces to $\mathfrak{g}$ in degree 1, and the L_∞-structure is simply given by the Lie algebra structure of $\mathfrak{g}$.[13]

2.1.3 Fiber products and triangles

The previous computations are actually particular case of a general formula for fiber products of derived stacks: a cartesian square of (pointed) derived stacks induces a cartesian square of tangent complexes [63, Lemma 1.4.1.16]

$$
\begin{array}{ccc}
F \longrightarrow X \\
\downarrow \qquad \downarrow f \\
* \longrightarrow Y
\end{array}
\quad \Longrightarrow \quad
\begin{array}{ccc}
\mathbb{T}_x F \longrightarrow \mathbb{T}_x X \\
\downarrow \qquad \downarrow df \\
0 \longrightarrow \mathbb{T}_{f(x)} Y
\end{array}
$$

In other words, $\mathbb{T}_x F \to \mathbb{T}_x X \to \mathbb{T}_{f(x)} Y$ is a distinguished triangle in the category of chain complexes and the left (or right) object can be computed as the

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[13]The proximity of L_∞-structures with Lie structures is clearer in this example than in the previous one, but once again the L_∞-structure encompasses the extra structure of a formal neighborhood.

mapping cone (or the mapping cocone) of the other two.[14] We recover this way the computation of section 2.1.1 as a cone. In the case of a quotient by a group action as in section 2.1.2, we use the fact that the definition of the quotient as a stack $X//G$ provides always a cartesian square

$$
\begin{array}{ccc}
G & \longrightarrow & X \\
\downarrow & {}^{\ulcorner} & \downarrow q \\
* & \xrightarrow{\ \overline{x}\ } & X//G
\end{array}
$$

where the top map is the parametrization of the orbit of x.[15] Then the construction of the tangent complex follows from a mapping cocone construction.

For the corresponding L_∞-structures the reader can look [46].

2.1.4   Tangent complexes in deformations

Perhaps the most common examples of tangent complex are the chain complexes that appear in deformation theory.[16] We shall give some details about the case of principal bundles and only mention a few other.

Let $P \to X$ be a principal G-bundle, on a fixed smooth scheme (or manifold) X. Such a bundle is classified by a 1-cocycle of X with values in G and the first order deformations of P can be shown to be classified by 1-cocycles of X in the adjoint bundle $ad(P) = P \times_G \mathfrak{g}$. This suggests to introduce the cohomology complex $C^*(X, ad(P))$, or, better for our purpose, the chain complex $\mathbb{T}_P$ such that $(\mathbb{T}_P)_i = C^{1-i}(X, ad(P))$.[17] Remark that $\mathbb{T}_P[1]$ has a Lie dg-algebra structure inherited from the Lie structure of $ad(P)$.

The positive part of this bundle is easy to understand: $H_0(\mathbb{T}_P) = H^1(X, ad(P))$ is in bijection with the isomorphism classes of first order deformations of P and $H_1(\mathbb{T}_P) = H^0(X, ad(P))$ classifies the symmetries of such deformations. Let $Bun_G(X)$ be the stack classifying of G-bundles on X, then the truncated complex $(\mathbb{T}_P)_{\geq 0}$ can be proven to be the tangent complex of $Bun_G(X)$ at the point P. This is quite similar to example 2.1.2.

The interpretation of the negative part of $\mathbb{T}_P$ is more subtle. As in example 2.1.1 we can think of it as related to some non-transverse intersection feature.[18]

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[14]The reader not fluent in Homological Algebra can look at [2, 17] for the definitions of these notions.

[15]It is one of the nice features of quotient stacks $X//G$ that the fibers of the quotient map $X \to X//G$ are always isomorphic to G, even if the action is not free. Actually, the map $X \to X//G$ can be proven to be a G-torsor.

[16]Deformation theory deals with the problem of classifying infinitesimal deformations of a given object (a scheme, a bundle, a ring structure, a group representation...). It is always possible to consider such an object as a point in the moduli space for the structure in question, then, the infinitesimal deformations of the object correspond to the study of the infinitesimal neighborhood of the point. In particular, first order deformations correspond to tangent vectors in the moduli space. Moduli spaces are difficult to construct, but first order deformations are relatively easy since they consists in solving some linear equations.

[17]The degrees are changed for convenience, see 2.2.1.

[18]This intersection has two sources, first the cut off given by the cocycle condition and also

It can also be explained in terms of obstruction theory or better in terms of deformations parametrized by dg-algebras but this requires elements of derived geometry. We shall not explain this here (see [59, 65] or the introduction of [41] for details), the only thing we need to know is that from the point P in $Bun_G(X)$, we have again produced a tangent complex with a Lie algebra structure.

We list a few other complexes related to deformation problems.

If X is a scheme (or a manifold), the tangent at X to the space of schemes is the H_0 of the tangent cohomology complex $(\mathbb{T}_X)_i = C^{i-1}(X, TX)$. Then $\mathbb{T}_X[1]$ has a structure of a Lie dg-algebra inherited from the Lie bracket of TX.

If Γ is a discrete group and M a linear representation of Γ, the tangent space to M in the space of representations (or character space) of Γ is the H_0 of the complex $(\mathbb{T}_M)_i = C^{i-1}(\Gamma, End(M))$ computing the cohomology of Γ in the adjoint representation $End(M)$ of M. Remark that the shifted complex $\mathbb{T}_M[1]$ has a structure of a Lie dg-algebra inherited from the Lie structure of $End(M)$. More generally, if X is a topological space (the previous case being $X = B\Gamma$) and M a local system on X, then $End(M)$ is again a local system and the tangent space at M to the space of local systems on X can be enhanced into the complex $(\mathbb{T}_M)_i = C^{i-1}(X, End(M))$ of cohomology of X in M. And again, $\mathbb{T}_M[1]$ is a Lie dg-algebra.

If A is an associative algebra, the tangent space at A to the space of associative algebra is the H_0 of the (shifted) Hochschild complex $(\mathbb{T}_A)_i = C^{i-2}_{Ass}(A, A)$. An L_∞-structure can be proven to exist on $\mathbb{T}_A[1]$ but it is difficult to explicit. Similarly, if $\mathfrak{g}$ is a Lie algebra, the tangent space at $\mathfrak{g}$ to the space of Lie algebra is the H_0 of the (shifted) Chevalley complex $(\mathbb{T}_\mathfrak{g})_i = C^{i-2}_{Lie}(\mathfrak{g}, \mathfrak{g})$. Again, $\mathbb{T}_\mathfrak{g}[1]$ has a Lie algebra structure inherited from that of the coefficient $\mathfrak{g}$.

2.2 Geometry of tangent complexes

2.2.1 The three parts of the tangent complex

We have seen two notions of singularities: intersection and quotient singularities. We have seen in each case that the tangent structure could be encoded in a complex whose homology is concentrated in degree 0 iff the point is regular. The other homology groups are thus a reflection the singular structure of the point. If there exists non-trivial positive homology groups, this means that some *bad quotient* was involved in the singular structure. If there exists non-trivial negative homology groups, this means that some *bad intersection* was involved in the singular structure. Since a general space is constructed by taking both intersections and quotients (typically a symplectic reduction, see 3.5), its singularities will have both positive and negative tangent parts in general.

Altogether, our examples propose the picture of three specific parts of the tangent complex:

the wild limit (indexed by the category of refinement of atlases) that has to be taken when bundles are described in terms of Čech cocycles.

positive part (quotient singularity)		zero part		negative part (intersection singularity)
$\cdots \to T_2 \to T_1$	$\to$	T_0	$\to$	$T_{-1} \to T_{-2} \to \ldots$
internal symmetries		"real" tangent		outer intersection structure
(so-called "stacky structure")				(so-called "derived structure")

2.2.2 Lie structure and loop stacks

We have seen also that tangent complexes $\mathbb{T}$ were always equipped with a kind of Lie structure on the shift $\mathbb{T}[1]$. It turns out that Derived Geometry provide a very nice and quite simple explanation for this fact.

First, recall that in homotopy theory, that for a pointed space $x : * \to X$, the homotopy fiber product $* \times_X *$ is nothing but the loop space $\Omega_x X$ of X at x, which in particular is a group.[19] The interpretation is in fact the same in any ∞-category, so in particular in derived stacks where $\Omega_x X$ is the derived group stack of symmetries of x in X. Now, recall from 2.1.3 that a cartesian square of pointed derived stacks provide a triangle of tangent complexes. Applied to the square

$$
\begin{array}{ccc}
\Omega_x X & \longrightarrow & * \\
\downarrow & & \downarrow x \\
* & \xrightarrow{\ x\ } & X
\end{array}
$$

it gives $\mathbb{T}_{id_x}\Omega_x X = cone(0 \to \mathbb{T}_x X) = \mathbb{T}_x X[1]$, that is, the shifted tangent complex at x is nothing but the tangent complex to the loop stack at the identity of x. Now since the loop stack $\Omega_x X$ is a group, this explains the Lie algebra structure on the tangent.[20]

2.2.3 Lie structure and formal neighborhood

Perhaps the most bizarre consequence of the existence of the Lie structure is the ability to reconstruct the whole formal neighborhood of a derived stack (called a *formal stack* or a *formal moduli problem*) from the tangent complex and its Lie structure. This idea has a long history [12, 20, 24, 25, 51] and was fully formalized within derived geometry in [45].

If a point x in a scheme X is regular, the formal neighborhood of x is the same

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[19]The fiber product $A \times_C B$ of a diagram of sets $f : A \to C \leftarrow B : g$ is the set of pairs (a, b) such that $f(a) = f(b)$ in C. The homotopy fiber product $A \times_C B$ of a diagram of homotopy types $f : A \to C \leftarrow B : g$ is essentially the same thing but where the equality $f(a) = f(b)$ is taken up to homotopy, i.e. replaced by a path in C. More precisely $A \times_C B$ is the homotopy type of the space of triplets (a, b, γ) where γ is a path in C from $f(a)$ to $g(b)$. When $f = g = x : * \to X$, this gives the loop space. See [13] for more details.

[20]Actually, this interpretation is not fully proven yet. Although Lie algebra structure have been proven to exist on tangent complexes [23, 31, 45], the above interpretation rely on a theory of Lie algebras of derived Lie groups that has not been developed yet.

as the formal neighborhood of a point in the tangent space $T_x X$[21] (thus encoded by the algebra of power series $k[\![T_x^* X]\!]$ generated by the cotangent module at x). When x is not smooth, $T_x X$ is too big and the formal neighborhood is a subspace of $T_x X$. The equation of this subspace can be written from the Lie structure of the cotangent complex $\mathbb{T}_x X$ by means of the Maurer-Cartan equation. Example 2.1.1 shows clearly that the L_∞-structure can reconstruct the formal neighborhood, this should also be clear enough in example 2.1.2. We shall not say more about it, the matter is detailed in [59].

3   Derived spaces

In this section, we explain how to built the spaces of Derived Geometry. This will use ideas from Algebraic Geometry crossed with ideas from Higher Category Theory.

3.1   A view on Algebraic Geometry

As we mentioned in the introduction, it is convenient to split Algebraic Geometry in two parts: the theory of *affine schemes* and the theory of *non-affine spaces* (general schemes, algebraic spaces). Affines schemes are those spaces that can be described faithfully by a commutative rings of functions. Non-affine spaces are those spaces without enough functions and must be described by other means (functor of points, atlases).[22]

The theory of affine schemes consists in an almost perfect dictionary between geometric properties (points, open and closed subsets, etale and proper maps, connectedness, separatedness, bundles, dimensions, vector fields...) and features of commutative rings (fields, localisations, quotients and ideals, separable and finite extensions, idempotents, valuations, modules, generating families and presentations, derivations...). The tools involved in this first part are those of commutative algebra: proofs about affine schemes are ultimately proofs about modules over rings. Commutative algebra is greatly computational and improve the tools of the sole topology: the full power of this algebra is notably used in the definition of the infinitesimal calculus (Kähler differentials but also iterated powers of ideals); it is also used in the definition of closed subspaces (defined as quotients of rings), or intersections (defined as tensor products of rings). These ideas lead in particular to a nice formalization of intersection multiplicities, which topology alone is not able to get.[23] All this will generalize well to derived

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[21]This is the algebraic analog of the local homeomorphism between the tangent space and the neighborhood of a point in a manifold.

[22]This general structure has vocation to be used in Differential and Complex Geometries. Rings need simply to be replaced by the appropriate notions: C^∞-rings for differential geometry [48, 56], and a similar notion for complex geometry [44, 50]. See also 3.2.6.

[23]This power of the algebra over geometry is one of the motivations for the extension of differential geometry with C^∞-rings methods [48], where there is an infinitesimal theory handier than in manifolds. In particular, such an extension allows one to apply the ideas and methods of Synthetic Differential Geometry.

rings and derived affine schemes.

The second part of Algebraic Geometry consists in using affines schemes to build more sophisticated spaces. It happen that not every space of interest in Algebraic Geometry is an affine scheme: projective spaces, Hilbert schemes and other moduli spaces do not in general have enough functions into the affine line to be described faithfully by a commutative ring.[24] However, practice proves that these spaces can often be constructed by a pasting of affine schemes, in the same way a manifold is a pasting of charts. This is the theory of *(non-affine) schemes* and of *algebraic spaces* and of general *sheaves of sets*.[25] Contrary to affine schemes which are defined individually by means of a ring, these new objects cannot be defined individually, but only relatively to the previously defined affine objects. The tools here are those of category theory rather than commutative algebra (limits and colimits of diagrams, presheaves, universal properties...). They are the tools to work on a collection of objects rather than on objects individually. These methods and notions will be derived into the theory of stacks.[26]

We shall present now a more conceptual understanding of these two parts. Recall that a commutative ring is always a quotient of a free ring by some system of equations. Geometrically this says that affine schemes are constructed from affine spaces $\mathbb{A}^n$ (the affine schemes corresponding to free rings) as levels sets of functions, i.e. by fiber products, i.e. by categorical limits. Then schemes and algebraic spaces are constructed from affine schemes by pasting, i.e. by categorical colimits. The two steps of the construction of the spaces of Algebraic Geometry can therefore be read as the following procedure: start with affine spaces (free rings), then add some limits, then add some colimits.

Having this in mind, we can say that derived Algebraic Geometry is built with the same procedure but where we are going to change the way to compute level sets and quotients, i.e. limits and colimits. Changing the way to compute limits transforms the theory of affine schemes into that of *derived affine schemes* (technically, it is done by changing the way quotients are computed in rings, see 3.2.3) and changing the way to compute colimits is the replacement of sheaves by *stacks* (see 3.4). Both changes will require the introduction of higher categories.

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[24]Methods of commutative algebra can be extended to projective spaces (through graded rings), but not all schemes are projective so other methods are needed.

[25]Let us recall that sheaves have two different uses in topology: the most common is to use sheaves of abelian groups as coefficients for cohomology, but sheaves of mere sets can be used as generalized spaces. In fact, from this point of view, sheaves (and stacks in a better way) are useful to solve the following conundrum: how to defined a space which is not a manifold, i.e. which does not have an atlas (like orbifold), or, even worst, a space which is not a topological space, i.e. which does not have enough open subspaces (like an unseparated quotient) ? Sheaves and stacks give a setting where to define such spaces, provided we know what a map from a manifold (or an affine scheme, or a topological space, or any other "basic block") to this space is, i.e. provided we can define a functor of points. This is particularly suited to moduli problems.

[26]A good introduction to these ideas is Toën's course [58].

3.2 Homotopy quotients and derived rings

Algebraic Geometry has long enhanced commutative algebra by introducing homological or homotopical methods (chain complexes, dg-algebras, simplicial modules and algebras...). Classical books on the matter justify these enhancements by the powerful computations they allow (essentially, the existence of long exact and spectral sequences) but not so much by principles.[27] The question of these underlying principles was actually a very difficult question for a long time, it has been solved only with the mutation of Homotopical Algebra into Higher Category Theory in the '90s. We shall explain only the part of the story that has to do with the operation of taking quotients.

3.2.1 Quotients of sets

Quotients of sets are classically dealt with via equivalence relations. However, in practice, equivalences relations are often derived from other structures (graphs, group action...) where two elements may have several ways to be identified (in a graph, there might be more than one edge between two vertices x and y; in a group action, there might be more than one element sending an element x to another y if the action is not free). The associated equivalence relation remembers merely the *existence* of an identification between two elements and forget about the potential *ambiguity* (in the sense of non-canonicity) of identifications.

It turns out that forgetting the multiplicity of identifications can generate irregularities. For example, in the case of a group action, it is not true that working equivariantly is the same thing as working over the quotient if the action is not free.

Another notion of quotient has been invented, that takes into account the potentially multiple identifications of elements. However this operation has values not in sets, but in homotopy types.[28] In consequence, it is called the *homotopy quotient*. It has been developed in homotopy theory where it is one of the most basic tool under the name of *homotopy colimit* [8, 13, 16, 27]. Technically, the homotopy colimit is constructed as a simplicial set such that the classical quotient (the set of equivalence classes) can be identified with the set of connected components.[29] The construction procedure is fairly simple, but the statement of its universal property requires some advanced homotopical algebra.[30]

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[27]Who never wonder about the necessity of homological/homotopical apparatus and how to make sense of all these constructions?

[28]Homotopy types are equivalence classes of topological spaces for the (weak) homotopy equivalence relation. They can be viewed as a generalization of the notion of groupoid and are sometimes called ∞-groupoids.

[29]For the reader unfamiliar with simplicial sets, it is sufficient to know that they look like triangulated topological spaces, but defined in a purely combinatorial way. In particular, they have a homotopy theory.

[30]This may explain why this construction is not more widely known. The best formulation of this universal property needs ∞-category theory: within this setting, the notion of homotopy

The principle of the construction is to identify two elements in a set by putting an edge between them instead of equalizing them. Then three elements are identified by putting a triangle, four with a tetrahedron, etc. It should be clear how this produces a simplicial set and that its set of connected components is indeed the classical quotient. The homotopy colimit of a diagram of sets is formally defined as the homotopy type of this simplicial set. We shall see some examples.

Let us consider first, the case of the diagram describing a graph with two edges a and b between two vertices x and y:

$$\{a,b\} \underset{t}{\overset{s}{\rightrightarrows}} \{x,y\}$$

where the two maps are the source and target maps from the set of edges with values in the set of vertices (here s sends a and b to x and t sends them to y). The colimit of this diagram is the quotient of the set $\{x,y\}$ by the relation $x \simeq y$ and the classical quotient is a singleton. But a and b provide two different identifications between x and y and the homotopy quotient is simply the homotopy type of the graph, equivalent to the homotopy type of a circle.
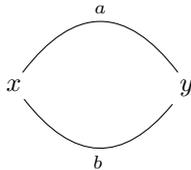
Its set of connected components is, as expected, in bijection with the classical quotient. If the set $\{a,b\}$ is replaced by a set with $n+1$ elements, the homotopy quotient can be computed to have the homotopy type of a wedge of n circles.[31]

Another important example is the homotopy quotient of the trivial action of a group G on a one element set $*$. The recipe of the homotopy quotient gives the so-called *simplicial nerve* of the group, whose geometric realization is a classifying space for G [13, §4]. In consequence, the homotopy quotient is the homotopy type of a classifying space for G. It is connected but has a non-trivial π_1 isomorphic to G.[32]

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colimit is simply the notion of colimit but computed in an ∞-category. We shall talk a bit more about this in 3.2.4.

[31] If both $\{a,b\}$ and $\{x,y\}$ are replaced by singletons, the homotopy colimit is a single vertex with a single edge, i.e. a circle. More generally, leaping from diagrams of sets to diagrams of homotopy types, if the set of edges $\{a,b\}$ is replaced by a homotopy type X of edges, the homotopy colimit is simply the unreduced suspension of X.

[32] The quotient map is a map $* \to BG$. It is funny to remark that with this new notion of quotient, the point has many non-trivial quotients. This example shows that BG is the quotient of a point by the trivial action of the group G, but, in fact, any connected homotopy type can be viewed as a quotient of a single point. This is an important point of view on homotopy types: they are in a sense a structure "below" sets. For example, if G is finite, the cardinality of BG is defined as $1/\#G$ which is indeed below $*$. The reader curious about cardinality of homotopy types can look at [33] and references therein.

Finally, in the general case of a group G acting on a set E, the construction produces the nerve of the associated *action groupoid* $G \times E \rightrightarrows E$ and the homotopy quotient is the homotopy type of the classifying space $E//G$ of this groupoid. If the action is free, this space is homotopy equivalent to the quotient set and the homotopy quotient coincides with the classical one, but if the action is not free, the groupoid remembers the stabilizer of a point as symmetries for this point, thus creating some π_1 in the quotient.[33]

Our examples have only non-trivial homotopy invariants in degree 0 and 1, but homotopy colimits of more complicated diagrams give rise to homotopy types with non-trivial homotopy invariant in any degree. In fact, any homotopy type can be described as the homotopy colimit of a diagram of sets.[34]

The main fact about the homotopy quotient is precisely that its higher homotopy invariants need not be trivial. In this sense, it encodes strictly more structure than the classical quotient. This extra structure is related to some kind of syzygies:[35] the π_1 groups remember the ambiguity to identify elements (redundancy of relations), the π_2 groups remember the ambiguity to identify identifications between elements (redundancy of relations between relations), and so forth. Altogether, the homotopy quotient has the universal property to encode not only equivalence classes of elements, but also the whole ambiguity about the identifications on those elements.[36]

The main reason of considering homotopy quotients is their computational advantages over classical quotients. Let us mention a few. First, it is always true that working equivariantly is equivalent to work over the homotopy quotient (a very important property called *effectivity of groupoid quotients*[37]). Also, a fun-

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[33] The notion of homotopy type can sometimes be replaced by the simpler notion of groupoid (taken up to equivalence of categories) to compute the homotopy quotients. This is the case when the homotopy quotients have trivial π_n for $n \geq 2$, for example, in a group action. Groupoids have a simpler definition than homotopy types, and this explain why part of the literature focus on them instead of full homotopy types. However, groupoids are limited by the fact that homotopy quotients of groupoids may have non-trivial π_2. So it is better, albeit more sophisticated, to work directly with full homotopy types.

[34] Any simplicial set is a diagram of sets and the homotopy colimit of this diagram is the so-called "geometric realization" of the simplicial set. Any homotopy type can be described as a simplicial set (via the nerve of a contractible open covering, see [17]).

[35] This is an analogy with what will be told in 3.2.3.

[36] With this idea in mind, it is possible, and quite fruitful, to read the whole of homotopy theory as an enhancement of the theory of sets incorporating the "ambiguity of identifications". We shall come back to this idea in the conclusion. (This notion of ambiguity is compatible with the way the word is used in Galois theories. I have actually chosen the term *ambiguity* in reference to Galois theory.)

[37] Let R be an equivalence relation on a set E with quotient $Q = E/R$ and quotient map $q : E \to Q$. The map q (or the relation R) is said to be *effective* if R can be reconstructed from q by as $R = E \times_Q E$. Let $G : G_1 \rightrightarrows G_0$ be a groupoid, it can be thought as a generalized equivalence relation on G_0 (where elements have several ways to be equivalent). There exists a notion of quotient for groupoids that we shall not define here. Let $q : G_0 \to Q$ be the quotient map, q (or G) is said to be *effective* if G_1 can be reconstructed from q as $G_1 = G_0 \times_Q G_0$. Since the whole set of arrows G_1 of the groupoid G can be reconstructed from the quotient, this means that the quotient contains the information about the multiplicity of identification between elements of G_0. In the category of sets, quotients of equivalences relations are always effective, but not quotients of groupoids. However quotients of groupoids are effective when

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damental computational tool is the long exact sequence of homotopy groups associated to any fibration sequence. And finally, they possess nice numerical invariants: when it is defined, the Euler characteristic provide a nice generalization of cardinality. For example, the Euler characteristic of $E//G$ is always (for *any* action of a finite group on a finite set) the rational number $\#E/\#G$, a formula which is false for classical quotients.[38]

The morale about homotopy quotients is that when no canonical identification exists between two elements to be identified, it is best to keep track of all identifications (which is the only canonical thing to do) by mean of a homotopy type. This extra structure not only satisfies a stronger universal property but gives rise to an object with more regular properties. Projecting the identification data onto a mere equivalence relation truncates the structure and kill the nice computational regularities.

These facts actually make sense, not only for sets, but in a large variety of mathematical contexts, as we shall explain below in linear algebra, commutative algebra, and geometry.

3.2.2 Quotients of vector spaces

The issues with quotients of sets are inherited by quotients of any structure based on sets, although the technologies to take care of them may differ. For example, in linear algebra, chain complexes and quasi-isomorphisms turn out to be handier than simplicial objets and homotopy equivalences. In consequence, homotopy colimits of diagrams of vector spaces are rather described as chain complexes.[39]

Let us consider the example of a map of vector spaces $d : E_1 \to E_0$. The classical quotient is the vector space $E_0/\mathrm{im}(d)$ which can be defined as the colimit of the diagram

$$E_1 \underset{0}{\overset{d}{\rightrightarrows}} E_0$$

where 0 is the zero map. The homotopy quotient of d, which is the homotopy colimit of this diagram, is the construction called the *mapping cone* (see [2, 17, 66] for details). In our example, the mapping cone is simply $E_1 \to E_0$ viewed as a chain complex E_*, concentrated in (homological) degrees 1 and 0. In particular, $H_0(E_*)$ is the classical quotient and $H_1(E_*) = \ker d$ remembers the multiple identifications between elements of E_0. The space $H_1(E_*)$ plays the role of the π_1 of homotopy quotients of sets, it classifies non-trivial syzygies.[40]

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computed in homotopy types. The precise statement of the property of effectivity of groupoid quotients demands a definition of a groupoid object that we shall not give here (but see [62, 39]).

[38] This last point may seem anecdotic, but for objects more sophisticated than sets, this kind of invariants is related to the so-called virtual dimensions, virtual K-theory classes, etc.

[39] It is possible to use simplicial methods in linear algebra but the Dold-Puppe-Kan equivalence [30] proves that the two langages are in fact equivalent (provided we consider only complexes in non-negative homological degrees).

[40] Again, this is an analogy with what will be told in 3.2.3.

In particular, we can understand now that the computation of the tangent complex of a quotient in 2.1.2 is nothing but the homotopy quotient of the map $T_eG \to T_xX$ between the tangent spaces. The non-trivial syzygies are essentially given by the stabiliser of the point. We shall see in 3.4.1 that the quotient stack $X//G$ is nothing but the homotopy quotient of the group action, so space and tangent are computed the same way.

Homotopy colimits of more complicated diagrams give rise to chain complexes with larger homology amplitude. In fact, any chain complex in homological non-negative degree can be obtained as the homotopy colimit of a diagram of vector spaces and the homology groups H_i have exactly the same interpretation in terms of non-trivial syzygies as the homotopy groups π_i for homotopy quotients of sets.[41] In the same way that homotopy types can be understood as homotopy colimits of sets, chain complexes can be (and should be) understood as homotopy colimits of vector spaces.

The reason to consider homotopy quotients in the context of linear algebra is classically the computational power of long exact sequences and their obstruction theory. Another reason is the effectivity of groupoid quotients, which, in this context, is reformulated into the property that the mapping cocone of a mapping cone is the identity.[42] We shall not expand on the homotopy theory of chain complexes. It is enough for this text to know that homotopy colimits of vector spaces are given by chain complexes up to quasi-isomorphism.

3.2.3   Quotients of rings

Quotients of rings raise the same issues as with sets and vector spaces. The classical theory characterizes a quotient $A \to A'$ of a ring A by the data of an ideal $I \subset A$. In practice though, quotients of A are not so much given this way but by systems of equations which are then interpreted as generators for an ideal and a quotient. The replacement of equations by an ideal is an operation of the same nature than truncating a graph or a group action into an equivalence relation and bears the same defects.

Let us consider the case of a system of equations $E = \{a_1 = 0, \ldots, a_n = 0\}$ for some $a_i \in A$. The quotient of A by the system E can be presented as the colimit, in the category of rings, of the diagram

$$A[x_1, \ldots, x_n] \underset{x_i \mapsto 0}{\overset{x_i \mapsto a_i}{\rightrightarrows}} A.$$

---

[41] The chain complexes of non-positive degree can be understood as generated by considering the dual notion of *homotopy limits*. The full category of unbounded chain complexes has a more subtle definition, it is generated by considering both homotopy colimits and limits of vector spaces, but with a constraint of commutation called *stability* imposing that finite homotopy limits and finite homotopy colimits should commute, see [40] and note 86.

[42] More precisely, for a map $f : E_1 \to E_0$ of (unbounded) chain complexes, let $F(f)$ be the mapping cone of f, there is a canonical map $g : F(f) \to E_1$ whose mapping cocone $C(g)$ is quasi-isomorphic to E_0. In other words, homotopy cartesian squares are also cocartesian squares in chain complexes. The same property is true in the category of non-negative chain complexes but the map f needs to be surjective.

It is classical that the quotient behaves well if the family of a_i is a *regular sequence* (neither element is a zero divisor relative to the previous ones). Let us recall why. An ideal I of a ring A which is generated by more than one element a_i is never free as an A-module, since there always exists the relation $a_i a_j - a_j a_i = 0$. Relations between generators of an A-module are called *syzygies*. Let us call *trivial syzygies* the relations that can be derived from the $a_i a_j - a_j a_i = 0$. The sequence of a_i is regular iff there is no non-trivial syzygies. Such syzygies exist when, for example, some a_{i_0} could be a zero-divisor, or equal to another a_{i_1} (repetition of equations) and this phenomenon can be understood as creating an ambiguity in the description of the elements of the ideal generated by E in terms of linear combinaisons of generators a_i.

For an element $a \in A$, let $K(a) : A \to A$ be the chain complex concentrated in (homological) degree 0 and 1, where the differential is given by the multiplication by a. The complex $K(a)$ is a dg-algebra, called the *Koszul dg-algebra* of a (the multiplication of elements of degree 1 is nilpotent and the other components are given by the multiplication of A, see [14, 17] for details). The group $H_0(K(a))$ is the classical quotient $A/(a)$ and the group $H_1(K(a))$ consists of all $b \in A$ such that $ba = 0$, it is non-trivial iff a is a zero-divisor. Hence, the dg-algebra $K(a)$, considered up to quasi-isomorphism, encodes the quotient but also keep track of the regularity of the element a.[43]

More generally, the *Koszul dg-algebra* of a family of elements $E = \{a_i\}$ is defined to be the dg-algebra $K(E) = K(a_1) \otimes \cdots \otimes K(a_n)$. The full combinatorics of trivial relations between the a_i is encoded by the differential of $K(E)$: trivial syzygies (in a given degree) are defined as the image of this differential, and general syzygies as the kernel, the (classes of) *non-trivial syzygies* are defined as the homology of $K(E)$. The sequence E is regular iff $K(E)$ has only homology in degree 0, iff $K(E)$ is quasi-isomorphic to (or is a resolution of) the classical quotient $A/(a_1, \ldots, a_n)$.

Although this was not at all clear (nor obvious) when it was invented, the construction $K(E)$ is now understood to be the homotopy quotient of A by the equations E. There exists a canonical map $A \to K(E)$ which image by H_0 is the classical quotient map $A \to H_0(K(E))$. The regularity of the sequence is the hypothesis under which the classical and the homotopy quotients agree. As for non-free group actions, irregular sequences produce higher homotopical structure in the quotient, and the higher homology groups $H_i(K(E))$ have the same interpretation in terms of syzygies or ambiguity as the homotopy groups π_i for homotopy quotients of sets. Also, this notion of homotopy quotient in dg-algebras does satisfies the property of effectivity of groupoid quotients. We shall come back to this in 3.2.5.

The theory of dg-algebras provides also the good setting where to define and compute cotangent complexes. Coming back to example 2.1.1, let $A = k[x_1, \ldots, x_n]$ and E be the system of m functions determined by f. Then, the

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[43]Geometrically, a function a on a space X is a zero-divisor (or irregular) essentially when the subset $Z = \{a = 0\}$ has a non-empty interior. In such a case, the dimension of Z would not be $\dim X - 1$ as expected, hence the term irregular.

derived fiber Z is encoded by $K(E)$, which is freely generated (as a graded algebra) by n generators x_i in degree 0 and m generators y_j in degree 1. Recall that the module of Kähler differential of a free ring is a free module on the same generators. The situation is the same for $K(E)$, the cotangent complex at some point $x : K(E) \to k$ is a free module on the same generators (renamed dx_i and dy_j), the only difference is a differential which can be computed to be the Jacobian matrix df of the function f, giving back the result of 2.1.1.

Finally, a word should be said about how homotopy colimits are related to better numerical invariants. The previous construction explained in terms of quotients can also be understood in terms of tensor products of rings (geometrically this correspond to describe a subspace as an intersection): $K(E)$ is also model for the derived tensor product $A/a_1 \otimes_A^{\mathbb{L}} \cdots \otimes_A^{\mathbb{L}} A/a_n$ and $H_0(K(E))$ is the underived tensor product.[44] From this point of view, the higher homology of $K(E)$ is nothing than some Tor modules. In the situation where A is local of dimension n, the Serre intersection formula says that the correct multiplicity of the intersection is the Euler characteristic of the complex $K(E)$ and not the dimension of the crude $H_0(K(E))$.

Serre formula is commonly used but totally ad hoc in classical Algebraic Geometry. It is only when one consider that the theory of algebras have to be enhanced into the homotopy theory of dg-algebras, which implies that colimits have to be replaced by homotopy colimits, that this formula finds its natural context.

3.2.4   Derivation and ∞-categories

Let us organize what we have said so far. In the three contexts of sets, vector spaces and rings, we have described a new operation of quotient: the homotopy colimit. It turns out that the proper setting to understand homotopy colimits (and in fact all homotopical phenomena) is Higher Category Theory. Intuitively, if a category is thought as an enhancement of a set by allowing morphisms between elements, then an ∞-category is the enhancement of a category where morphisms (renamed 1-morphisms) have morphisms between them (named 2-morphisms), and 2-morphisms have 3-morphisms between them, ad infinitum. We shall not need much of the theory of ∞-categories, it will sufficient to know a few things: 1) any category is an ∞-category; 2) the difference between the two notions is essentially that the arrows between two objects of an ∞-categories do form a homotopy types and not mere sets;[45] and 3) that the theory of ∞-categories is essentially the same as the theory of categories: all the notions of adjoint functors, diagrams, colimits, etc. make sense in this new context, behave

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[44]The tensor product of commutative rings is a particular colimit (a pushout), the derived tensor product is nothing but the corresponding notion of *homotopy* pushout of commutative dg-algebras.

[45]The easiest way to picture an ∞-categories is to think of a category enriched in topological spaces and then to consider those spaces only through their homotopy types. 2-arrows are given by homotopies, 3-arrows by homotopies between homotopies, etc. All higher arrows are invertible, we shall not consider here higher categories with non-invertible higher arrows.

the same way and restrict to the classical ones when applied to a category.[46]

There exists an ∞-category $\mathcal{S}$ of homotopy types, this is essentially because morphisms between two homotopy types do form a homotopy type and not a set (the 2-arrows are given by homotopies of maps, the 3-arrows by homotopies between homotopies, etc.). Let Set be the category of sets, there exists an embedding $Set \to \mathcal{S}$ by viewing sets as a discrete homotopy types. With this in mind, we can explain that the homotopy colimit of a diagram of sets is nothing but the colimit of this diagram computed in the ∞-category $\mathcal{S}$.

The same thing happens for vector spaces and rings. The homotopy theory of chain complexes and dg-algebras can be understood (and should) as the fact that they are naturally objects of ∞-categories rather than ordinary categories. Let $Vect$ and $Ring$ be the categories of vector spaces and rings, let $\mathcal{C}plx_{\geq 0}$ and $dg\mathcal{A}lg_{\geq 0}$ the ∞-categories of chains complexes and dg-algebras in homological non-negative degrees,[47] there exists embeddings

$$Vect \to \mathcal{C}plx_{\geq 0} \qquad \text{and} \qquad Ring \to dg\mathcal{A}lg_{\geq 0};$$

the notion of homotopy colimit in $Vect$ and $Ring$ is simply the natural notion colimit in the ∞-categories $\mathcal{C}plx_{\geq 0}$ and $dg\mathcal{A}lg_{\geq 0}$.

This kind of embedding of an ordinary category into an ∞-category, transforming automatically the way colimits (and also limits) are computed, can be taken as the formal meaning of the term "derivation" used in a homotopical/homological context. From this point of view, there is something arbitrary in the choice of the ∞-category used to derive a given category and we could as well have considered the ∞-categories $\mathcal{C}plx$ and $dg\mathcal{A}lg$ of unbounded chain complexes and dg-algebras. We could also have considered, as in [63], the notion of simplicial algebras instead of dg-algebras, or, as in [41], the notion of E_∞-ring spectra.

The purpose of derivation (as I see it) is to embed a category C into a ∞-category $\mathcal{D}$ having better colimits and limits properties. The choice of $\mathcal{D}$ depends on the kind of properties we are interested in. We already mentioned the property of effectivity of groupoid quotients, i.e. the possibility to work on quotients of groupoids by equivariant methods. This property is also what allows the construction of classifying objects (e.g. for any kind of bundles in geometry, or for extensions in ring theory, see 3.2.5).[48] [49]

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[46]The references for the formal definitions of ∞-categories theory are [29, 39]. The introduction of [39] explain the relations of ∞-category theory and classical homotopical features. Another good introduction to these ideas is [4].

[47]If ∞-categories are pictured as categories enriched in spaces, the ∞-categories $\mathcal{C}plx_{\geq 0}$ and $dg\mathcal{A}lg_{\geq 0}$ are classically constructed as Dwyer-Kan simplicial localisations of model categories.

[48]The weaker property of *effectivity of equivalence relations* only (and not all groupoids, see note 37) is one of the axioms of Grothendieck toposes, hence it has many instances within ordinary category theory. The stronger effectivity axiom becomes important once realized that most of equivalence relations considered in practice are built by truncation of groupoids. Effectivity of groupoids quotients is one of the axioms of ∞-toposes. The strength of the effectivity axiom is precisely where is the difference between ordinary toposes and ∞-toposes, [39, 52].

[49]In relation with Homotopy Type Theory, the property of effectivity of groupoid quotients

Another nice property for $\mathcal{D}$ is to be generated by homotopy colimits from C. This explain the hypothesis of support for chain complexes: for example $\mathcal{C}plx_{\geq 0}$ is generated by homotopy colimits by *Vect* but not $\mathcal{C}plx$, $dg\mathcal{A}lg_{\geq 0}$ is also generated as such from *Ring* but not $dg\mathcal{A}lg$. Moreover these examples of completion by colimits are free in a certain sense (see note 54).[50]

3.2.5 Classifying objects and cotangent complexes

In this section, we explain how the property of effectivity of groupoid quotients in the ∞-category of dg-algebras brings a nice feature to the theory of commutative rings.

Let us recall first that the category of homotopy types (and this would also be true in stacks), has classifying spaces and universal bundles. Let $p : P \to X$ be a principal G-bundle, then there exists a map $X \to BG$ (unique up to homotopy) and a (homotopy) cartesian square

$$
\begin{array}{ccc}
P & \longrightarrow & * \\
\downarrow & \ulcorner & \downarrow \\
X & \longrightarrow & BG.
\end{array}
$$

where the map $* \to BG$ is the one of note 32. The interpretation of this result is that $* \to BG$ is the universal G-bundle and BG is the classifying space for G-bundles.[51] The important remark for the sequel is that such a construction cannot exist in sets: even if G, X and P are sets the classifying space BG is a genuine homotopy type. The possibility to construct such classifying objects is the main application of the property of effectivity of groupoid quotients.

Something similar exists in rings. Let A be a ring and M an A-module, then $A_M := A \oplus M$ has the structure of a ring where the product of two elements of M is zero. Such a ring is called a *trivial first order extension* of A. It has a surjective map $A_M \to A$ with kernel M. More generally, any quotient of commutative rings $A' \to A$ with kernel M is called a *first order extension* of A by M if the product of two elements of M is zero.[52] It is classical that such extensions can be classified by cocycles of A with values in M.

The work of Quillen on cohomology of commutative rings brought a nice reformulation of these notions in more "geometrical" terms. He showed that trivial first order extensions of A are exactly the bundles of abelian groups over

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can be thought as a stronger form of the Univalence Axiom (it implies univalence since it implies the existence of classifying objects). However, since no good theory of quotients exists in HoTT, the effectivity property cannot be stated in this langage.

[50] A natural question about derivation is the necessity of ∞-categories. Why is it necessary to go beyond the notion of ordinary category? We shall come back to this in our conclusion.

[51] Classically, this map is denoted $EG \to BG$ and built in topological spaces, with EG a contractible space, but since we are working with homotopy types and not spaces, there is no difference between EG and the contractible homotopy type $*$.

[52] Recall that, geometrically, first order and, more generally any nilpotent extensions, $B \to A$ corresponds to infinitesimal thickenings.

A (defined in the category of rings) and that the first order extensions of A by M are precisely the torsors over the group bundle $A_M \to A$.

Now, within dg-algebras with their effectivity of groupoid quotients, it is possible to build classifying objects and universal fibrations for these situations. if A and M are fixed, the classifying dg-algebra for A_M-torsors is the trivial first order extension $A \oplus M[1]$ but where M is put in degree 1, and the universal extension by M is the canonical map $A \to A \oplus M[1]$. More precisely, a map of dg-algebras (up to quasi-isomorphisms) $C \to A \oplus M[1]$ classifies an extension of C by M (where M is view as a C-module through the induced map $C \to A$) given by the (homotopy) cartesian square.

$$
\begin{array}{ccc}
D & \longrightarrow & A \\
\downarrow & \ulcorner & \downarrow \\
C & \longrightarrow & A \oplus M[1].
\end{array}
$$

In other words, cocycles of rings can be represented by maps to some classifying object, but this object exists only in dg-algebras.[53]

This property, once interpreted geometrically, is the heart of the obstruction theory used in deformation theory which was one of the motivations to define Derived Geometry. We shall not expand on this in this text, we refer to [59, 65] for a more detailed account.

The definition of first order extensions of rings generalizes in the obvious way to for dg-algebras. In particular, it is possible, even for a ring to consider extensions where M is a complex of A-modules. In particular, it is possible to prove the following result: any dg-algebra A can be built from $H_0(A)$ as the limit of a tower of first order extensions (called the Postnikov tower of A)

$$A \to \cdots \to A_{\leq n} \to A_{\leq n-1} \to \cdots \to A_{\leq 1} \to H_0(A)$$

where the extension $A_{\leq n} \to A_{\leq n-1}$ is by the module $H_n(A)[n]$.

The work of Quillen actually went a bit further. If $A \oplus M$ is a trivial first order extension, then a map $C \to A \oplus M$ is simply a derivation of $C \to M$ (where M is view as a C-module through the induced map $C \to A$). Classically such maps are in bijection with maps of C-module $\Omega_C \to M$ where Ω_C is the module of differentials of C. In the context of dg-algebras where M can be a chain complex, Ω_C is not enough and derivations $C \to M$ are now represented by a complex of C-module $\mathbb{L}_C$, called the *cotangent complex*, such that $H_0(\mathbb{L}_C) = \Omega_C$. This proves that the cotangent complex is nothing but the module of differentials but in the context of dg-algebras. In particular, the maps $C \to A \oplus M[1]$ classifying extensions by M are equivalent to maps of chain complexes $\mathbb{L}_C \to M[1]$, so $\mathbb{L}_C$ "controls" the theory of extensions.

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[53]In comparison with the situation of homotopy types, the role of A is played by the contractible type $*$ and the fact that M is in degree 1 in $A \oplus M[1]$ is similar to BG where G is the π_1.

Geometrically, this says something new: it says that infinitesimal thickenings of a derived affine scheme are controled by its tangent bundle. We refer again to [59, 65] for more details.

3.2.6 Other contexts of derived rings

We have presented the ∞-category of dg-algebras as a convenient context to enhance the theory of rings. However, this is not the only context that is used to derive the theory of commutative rings and the corresponding geometry. We briefly review a list of other contexts.

In order to derive Algebraic Geometry, simplicial rings can also be used. They should be thought as a commutative ring with an underlying homotopy type instead of a set. This is the original setting for Derived Algebraic Geometry presented in [63]. The theory of simplicial rings is equivalent to that of dg-algebras in characteristic zero, but better suited in positive characteristic. One can also use *unbounded dg-algebras*, but the theory turns out to be a bit awkward (essentially the infinitesimal theory is more subtle) some details are given in [63].

Lurie is using yet another model, E_∞-ring spectra (i.e. multiplicative cohomologies theories), which can be thought as commutative rings with an underlying spectrum (in the sense of stable homotopy theory). These objects are what become commutative rings when the full structure of associativity and commutativity of both the product and the addition are relaxed homotopically (in comparison, simplicial rings have the addition and commutation of the product are kept strict). This context, which was an original motivation for the whole theory, is well suited to transform into geometry the results of stable homotopy theory. This compares to the way arithmetic properties have been translated into geometry with the theory of schemes (see the introduction of [41]).[54]

The fruitful (and elegant) algebraization of the geometry of polynomial and rational functions has always inspired attempts to turn the geometries of differentiable and holomorphic functions into algebra. An important problem to do so has been the extra-structure that the rings of those functions should have. The first idea to do so is probably to add some topology to these rings (in order to take care of convergence questions). But Lurie, in his development of Derived Geometry, has advertised another way to do so.

For the purposes of differential geometry, the theory of simplicial C^∞-rings seems well suited.[55] [56] Some foundational work has been done in [56], but the

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[54] The plurality of these generalizations of commutative rings can be explained with the notion of algebraic theory (in the sense of Lawvere). The problem to define "homotopy rings" can be set into the problem of finding an algebraic theory such that the models in the category of sets are commutative rings, then the models in the category of homotopy types will be "homotopy rings". Dg-algebras, simplicial rings, E_∞-ring spectra are different such theories. Moreover, if the algebraic theory is encoded by an ordinary category, the category of models in homotopy types is always a colimit completion of the category of models in sets.

[55] And probably the "dg-" version also, although I don't know if anybody has looked into this.

[56] The idea of a C^∞-rings is the following: in a classical ring A, the operations of sum and

theory is not much developed yet. An obstruction is certainly the importance in the field of geometrical and analytical methods (in particular problems of convergence) over algebraic methods. Another might be the natural focus of the theory on smooth objects rather than singular ones, whereas scheme theory (derived or not) is intended for a study of singular objects. However, the recent development of Derived Symplectic Geometry, and the potential application to Floer homology could bring some motivations for the development of the theory.

Finally, Lurie proposes also a similar theory for the purposes of complex geometry. The idea is the same: try to avoid topological rings and find an algebraic structure encoding holomorphic functions, but the definition is a bit more subtle than for C^∞-rings.[57] However, the theory seems to work nicely enough to have results such as a GAGA theorem [50].

3.3 Geometry of derived affine schemes

We have explained how the theory of rings could benefit from its enlargement into the theory of dg-algebras. Although it should be clear enough that most results of commutative algebra have analogs for dg-algebras, it is not immediately clear that dg-algebras have also a geometrical side. The geometric objects corresponding to dg-algebras are named *derived affine schemes*, but what do they look like? and what are their new features with respect to ordinary affine schemes?

As we mentioned, classical commutative rings are related to geometry because it is possible to interpret the langage and structure of geometry (points, open subsets, fibrations... and their expected structural relations) in the opposite category of commutative rings. We shall sketch how it is possible to do the same for dg-algebras.

3.3.1 Comparison with ordinary schemes and infinitesimal structure

If A is dg-algebra (in non-negative homological degrees), there always exists a quotient map $A \to H_0(A)$ (killing higher degree elements). This suggests that the derived affine scheme X corresponding to A is related to the classical affine

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product can be combined to give more elaborated n-ary operations $A^n \to A$ indexed by any polynomial in $\mathbb{Z}[x_1, \ldots x_n]$. With this in mind, C^∞-rings are ring with n-ary operations given not only by polynomial functions but by any C^∞ function of n variables, for example, in a C^∞-ring it make sense to compute $\cos(e^x + y)$ for any elements x and y. An important difference with classical rings is that these operations cannot be generated from the sum and product only. The reader interested to learn more should look at [48] and learn about Lawvere theories.

In contrast to topological rings, C^∞-rings, by enhancing the addition and product of the ring with operations indexed by C^∞ functions provide a *purely algebraic* setting where methods from Algebraic Geometry can be adapted. In particular, the tensor product of C^∞-rings is the good one with no need for completion.

[57]The trouble is essentially the fact that not every open subset is the complement of zeros of a holomorphic function. This forces to encode within the ring of holomorphic functions the data of these extra open subsets. For this reason, the definition of an *holomorphic ring* is a bit more involved than that of C^∞-rings and ordinary rings (see [44, 50]).

scheme X_0 corresponding to $H_0(A)$ by a map $X_0 \to X$. X_0 is called the *classical part* or *truncation* of X.

It is reasonable to think that the points of this geometry will still be in correspondance with fields in dg-algebras. But, for reason of degree, such fields have to be concentrated in degree 0, so there are no new points in the new geometry. Actually more is true: a map $A \to k$ from a dg-algebra A to a field k has to factor through $H_0(A)$. This says that the map $X_0 \to X$ induces a bijection on points.

It is also reasonable to expect that the correspondance between ring quotients and closed immersions stays true in the new context. So $X_0 \to X$ would be a closed immersion inducing a bijection on points. Classically, this would say that X is an infinitesimal thickening of X_0. And this is the way it turns out to be: a derived affine scheme is an ordinary affine scheme endowed with an infinitesimal thickening, although this thickening can be of a new kind.[58]

3.3.2   Self-intersections

One of the interest of schemes is that they give rise to spaces of infinitesimally closed points. For example, the ring $\mathbb{C}[x]/x^n$ is the ring of function on the subspace of the affine line $\mathbb{A}^1$ formed by $n{+}1$ points infinitesimally close. Derived Geometry also gives rise to new spaces, for example spaces of self-intersections.

Let us consider the simplest case of the self-intersection of a point in the line. The dg-algebra corresponding to this is the (derived) tensor product $\mathbb{C} \otimes_{\mathbb{C}[x]} \mathbb{C}$ which can also be written as a quotient of $\mathbb{C}[x]$ by the irregular system of equations $\{x = 0, x = 0\}$. With our previous notations, if $A = \mathbb{C}[x]$ and $E = \{x, x\}$, the computation of $K(E)$ gives the dg-algebra $\mathbb{C} \oplus \mathbb{C}[1]$ (with trivial differential). This algebra has to be understood as such: it is essentially $\mathbb{C}$ but with a non-trivial loop at any number (given the the generator of $\mathbb{C}[1]$).

Let X be the geometric object corresponding to $K(E)$, its classical part X_0 is a point x since $H_0(\mathbb{C} \oplus \mathbb{C}[1]) = \mathbb{C}$. What is the thickening of the point giving X? It is difficult to say: the structure is "purely derived" and not easy to describe geometrically, particularly in classical terms. However, a few things can be said. First, recall from 2.2.2 for the pointed scheme $(\mathbb{A}^1, 0)$, the homotopy fiber product $* \times_{\mathbb{A}^1} *$ is nothing but the loop space of $\mathbb{A}^1$ at 0. Since $* \times_{\mathbb{A}^1} *$ is also the self-intersection X, the non-triviality of X means that the affine line has a non-trivial loop space at 0!

This situation is not specific to $\mathbb{A}^1$: unless the point is isolated, the self-intersection $* \times_X *$ of a point x in any derived affine scheme X will be non-trivial. Then since a loop space is a group, we can understand its Lie algebra and its actions. The Lie algebra of $* \times_X *$ can be proven to be $\mathbb{T}_x X[1]$ the tangent

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[58] Formally, such a thickening is still encoded by nilpotent extensions, as mentioned in 3.2.5, but they are harder to interpret geometrically when the extension module has positive homology. With classical infinitesimal thickenings, the limit of two points converging to each other is understood as two points infinitesimally closed, i.e. a vector. But derived infinitesimal thickenings encompasses also the limit of the intersection of two lines (say in the plane) when the lines become the same. What then become of the intersection point at the limit ? I don't know how to understand this in geometrical terms.

complex of X at x with degree shifted by -1 (see 2.2.2). As for the action of $* \times_X *$, let us say only that in the same way the loop group of a homotopy type acts on the fiber of any covering space, $* \times_X *$ acts on the fiber at x of any A-module (where A is the dg-algebra corresponding to X) [31, 23].

Finally, it can be proven that the classifying space (or better the classifying scheme) of this group is the formal completion of X at x (see [59] for more on the matter).[59]

3.3.3   Intrinsic geometry

The definition of geometric notions in Derived Algebraic Geometry is done essentially the same way as in Algebraic Geometry. We refer to [40, 41, 63] for details.

Points corresponds to fields (viewed as dg-algebras concentrated in degree zero) and the set of points of a derived scheme X is defined as in Algebraic Geometry by equivalence classes of maps from points. It is possible to define a notion of localisation for dg-algebras and a corresponding notion of *Zariski open immersion*. It is also possible to define the notion of *smooth* and *etale* maps for dg-algebras (either by lifting properties of infinitesimal thickening or by using the cotangent complex). Moreover, all these notions satisfy the expected properties (stability by composition, base change, locality...).

Once accepted the idea that the extra-structure of a derived affine scheme is infinitesimal, it is easy to reduce certain features of X/A to features of $X_0/H_0(A)$. For example, since open subsets contain the infinitesimal neighborhood of their points, the Zariski open subsets of X are in bijection with Zariski open subsets of its truncation X_0 in the classical sense. Also, using the infinitesimal lifting property, the category of etale maps over X can be shown to be equivalent to that of etale maps over X_0. As a consequence, a dg-algebra A is local (resp. henselian) iff $H_0(A)$ is local (resp. henselian). Another consequence is that the extra derived structure do not create more Zariski or etale topology, with the consequence that the Zariski and etale spectra of A and $H_0(A)$ will coincide.

There exists also a notion of *closed immersion*, corresponding to (homotopy) quotients of dg-algebras and this is a notion with quite a different behavior than the classical one. Essentially, the main difference could be summarized by saying that closed immersions $Y \to X$ are not immersions.[60] If $Y \to X$ is a closed immersion, then it is not a monomorphism, i.e. the canonical diagonal map $Y \to Y \times_X Y$ is not an isomorphism, as we saw in 3.3.2.

It is possible to define also notions of *maps of finite presentation*, *flat maps*, *separated maps* and *proper maps* (although this notion is more interesting for non-affine scheme). Altogether, the whole formal structure of Algebraic Geome-

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[59]This relationship between the formal neighborhood of a point and its loop stack, called *infinitesimal descent* in [59], is still a mysterious feature of Derived Geometry for me.

[60]Algebraically, this is somehow related with the fact that the notion of ideal is no longer suited to described homotopy quotients of dg-algebras: ideals only encode quotients by equivalence relations, but more is needed to describe all homotopy quotients.

try (in particular, all the aforementionned classes of maps) has been successfully transposed into Derived Algebraic Geometry.

3.3.4 Geometrization via spectra

Derived schemes are presented in [41] and [59] as particular ringed spaces. I have voluntarily chosen to delay this presentation because it is not intrinsic. Indeed, such a presentation using topological spaces is implicitly based on the notion of Zariski spectrum of a dg-algebra (generalizing the same notion for rings) but other notions of spectra exists (etale, Nisnevich...) and there is no reason to prefer Zariski to the other ones (in fact, there are good reasons not to do so).

From the class of Zariski open immersions into a derived scheme X, it is possible to construct a topological space $Spec_{Zar}(X)$ called the *Zariski spectrum of X* (or of A if X is dual to A). Similarly, from the etale maps with target X, it is possible to construct a topos[61] $Spec_{et}(X)$ called the *etale spectrum of X*. Both these spectra functors are not faithful (a lot of dg-algebras have the same spectrum) but it is possible to enhance them into faithful functors by adding a structure sheaf. Moreover, together with their structure sheaf, the spectra can be interpreted as a moduli spaces such that the structure sheaf become the universal family: the Zariski spectrum of A is the moduli space of localisations of A and the etale spectrum of A is the moduli space of strict henselisations of A.[62] This nice point of view, inherited from classical considerations on spectra (since at least [22]), is fully developed for Derived Geometry by Lurie in [41, 43].

Spectra are nice because they are genuine spaces and not the abstract objects of the opposite category of rings or dg-algebras. They are definitely useful in order to get a geometric intuition: for example, the Zariski spectrum of a ring can convince anybody immediately that rings have indeed something to do with geometry. However, the representation of the spatial nature of a ring via its Zariski spectrum (or any spectrum in fact) can also be misleading. The reason is that some geometrical features can be well understood in terms of the underlying space of the spectrum, but other still hide within the structure sheaf. For example, a scheme can have many closed subschemes but its Zariski spectrum can be a single point. Also, the underlying space of the Zariski spectrum does not commute with products (there are more points in $\mathbb{A}^2$ than pairs of points in $\mathbb{A}^1$). Hence, the Zariski topological space reflect quite poorly the intrinsic geometry of affine schemes (already in the un-derived setting). The etale spectrum is

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[61]Toposes are what become topological spaces if the poset of specialization maps between points is allowed to be replaced by a category. They provide a notion of highly unseparated spaces (way below T_0-spaces!) well suited to study spaces such as bad quotients of group actions and moduli spaces. Toposes provide a context of spaces where the notion of open subset is not enough to capture the whole of the topology of these spaces, and one has to look at the category of local homeomorphisms over the space instead of the category of open immersions only. It turns out that some unseparated spaces, such as quotients by group action, can have few open subsets but many local homeomorphism, in which case they can be well described by a topos.

[62]This point of view suits also Nisnevich spectra which is the space enlarging the etale spectra by classifying all henselianisation of A and not only strict ones.

better[63] but also suffer the problems listed above. Another drawback of both Zariski and etale spectra, specific to Derived Geometry, is that they do not reflect the non-trivial nature of loop spaces at a point described in 3.3.2, and one is forced to hide this bit of geometry within the structure sheaf.[64]

In fact, each spectra should be taken as a reflection, or a projection, of the true, or intrinsic, geometric nature of affine schemes (derived of not); and these projections are preferred one to another, as Poincaré would say, for sake of convenience (typically to define or to control cohomology theories). In my opinion, the most convincing fact that dg-algebras (or already rings) have a geometrical side is not so much their spectral theories, but the successful interpretation of the whole langage and expected structure of geometry they provide. To be able to talk about points, open subsets, etale maps, proper maps, fiber bundles and other geometrical features, is more important in order to do geometry than to be able to faithfully describe a ring or a dg-algebra as a genuine space with a structure sheaf.[65]

3.4 Derived stacks

3.4.1 Quotients of spaces and stacks

The construction of non-affine spaces in Algebraic Geometry is done by taking affine schemes as elementary building blocks and by defining the new objects to be formal pastings, i.e. formal colimits, of these blocks. Let *Aff* be the category of affine schemes, the construction of all formal colimits from objects of *Aff* is the meaning of the category $\mathrm{Pr}(Aff)$ of presheaves of sets over *Aff*.[66] However, this construction is "too free" because it does not paste the pieces of a Zariski (or etale) atlas of an affine scheme X to X itself.[67] This

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[63]An important drawback of Zariski spectrum is that it does not reflect etale maps of rings into etale maps of topological spaces, which forbids to use it to define cohomology theories with etale descent. This problem was the motivation to introduce the etale spectrum in order to define etale cohomology, but the toll was to work with spaces even more unseparated than Zariski spectra. These spaces which cannot be defined as topological spaces, where themselves the motivation for introducing toposes. The situation is the same in Derived Geometry where ∞-toposes have to be used.

[64]Is there a notion of spectrum that could reflect this "infinitesimal descent" (as called by Toën in [59]) in terms of classical descent in toposes ? See also 3.3.2 and note 59.

[65]Such an interpretation of the langage of geometry is precisely what is missing for non-commutative rings, essentially because there is no good notion of locality for non-commutative rings (which leads Kontsevich to joke about "non-commutative non-geometry"). In consequence, there exists no non-commutative geometry in the sense of building a spectral theory for non-commutative rings. There is, however, a number of features of geometrical objects that can still be defined for non-commutative objects (properness, smoothness...)

[66]It is a classical result of category theory that the Yoneda embedding $C \to \mathrm{Pr}(C)$ of a category C into its presheaves of sets, sending an object to its so-called *functor of points*, is the free completion of C for the existence of colimits.

[67] Recall that an etale atlas for an affine scheme X is a family of etale maps $U_i \to X$ covering X. In the category *Aff*, X can be recovered as the quotient of an equivalence relation on the disjoint union $\coprod_i U_i$ (subcanonicity of etale topology). However, the quotient $|U_\bullet|$ of this equivalence relation in the category $\mathrm{Pr}(Aff)$ of presheaves (of sets) will not be (the image by Yoneda embedding of) X. Indeed, because colimits are "freely" added in $\mathrm{Pr}(Aff)$, the Yoneda

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leads to distinguish certain pastings of affine schemes (Zariski, etale or any other class of atlases) and to consider rather the completion $\mathrm{Sh}(Aff)$ of Aff which preserves these pastings. Technically, the distinguished atlases define a Grothendieck topology and the completion is the category of sheaves of sets for this topology.[68]

Now, following the philosophy presented in 3.2.1, in order to have a better behavior of colimits, we should replace them by homotopy colimits in the construction of sheaves. This is precisely the definition of stacks: *stacks are to sheaves what homotopy types are to sets.* Or, in other terms, the category of stacks is the free homotopy colimit completion of the category of affine schemes (preserving atlases).[69] Technically, presheaves of sets are replaced by presheaves of homotopy types, called prestacks, and the definition of stacks copy that of sheaves by imposing that the homotopy colimits of atlases of X be X itself.[70] Intuitively, this means that we need to change our idea of a space with a underlying set of points to a notion of space with an underlying homotopy type (or ∞-groupoid) of points. The resulting morphisms between the points are of the same nature as the specialization morphisms in classical topology.

As with sets, complexes and dg-algebras, a fundamental property of the category of stacks is the effectivity of groupoid quotients. This property is actually the essential reason to consider stacks and stacky quotients: if a group G acts on space X, then anything defined over the quotient stack $X//G$ (function, bundle, sheaf...) can equivalently be described by something over X which is equivariant for the action of G.

Finally, *derived stacks* are defined exactly the same way starting from the

embedding $Aff \to \mathrm{Pr}(Aff)$ does not preserve the colimits existing on Aff. All we get is a canonical morphism $|U_\bullet| \to X$. The category of sheaves is then defined by imposing that the maps $|U_\bullet| \to X$, for all etale coverings of all affine schemes, be isomorphisms.

Technically, such an operation is a localisation of categories and corresponds to a quotient of $\mathrm{Pr}(Aff)$, but it can also be described as a full subcategory of $\mathcal{P}(Aff)$ (this is similar to the fact that a quotient can be described as a subset by imposing a gauge condition). Precisely, a presheaf F is called a *sheaf* (for the etale topology) if it "sees" all the maps $|U_\bullet| \to X$ as isomorphisms, i.e. if all the maps $\mathrm{Hom}(X,F) \to \mathrm{Hom}(|U_\bullet|,F)$ are isomorphisms. This condition is often called the *descent condition* [13, 55, 58].

[68] For the reader unfamiliar with sheaves, this completion is analogous to the more classical completion of $\mathbb{Q}$ into $\mathbb{R}$: a sheaf is very close to a Dedekind cut and sheaves are defined as "formal colimits" in the same way real numbers are defined as formal limits of sequences.

[69]Stacks were invented and used before this universal property was understood. This elegant description of the category of stacks (which only is true if stacks means higher stacks or ∞-stacks) is the result of a long interaction between Algebraic Geometry (where stacks were defined first), Algebraic Topology (were all the necessary homotopical techniques were invented) and the philosophy of Higher Category Theory (where the notion of homotopy colimit completion was conceived).

[70]This is done as in note 67. Recall the inclusion $Set \to \mathcal{S}$ of the category of sets into the ∞-category of homotopy types from 3.2.4. Let $\mathcal{P}(Aff)$ be the category of functors $Aff^{op} \to \mathcal{S}$ (called *prestacks*). The inclusion $Set \to \mathcal{S}$ induce an inclusion $\mathrm{Pr}(Aff) \to \mathcal{P}(Aff)$ and a Yoneda embedding $Aff \to \mathcal{P}(Aff)$. The category of stacks (for the etale topology) is defined from $\mathcal{P}(Aff)$ by inverting the maps $|U_\bullet| \to X$, for all etale coverings of all affine schemes. It can be described as sub-category of prestacks: a prestack F is called a *stack* if the maps $\mathrm{Hom}(X,F) \to \mathrm{Hom}(|U_\bullet|,F)$ are all equivalences of homotopy types [13, 58, 55].

∞-category $d\mathcal{A}ff$ of derived affine spaces instead of $\mathcal{A}ff$, relatively to one of the Grothendieck topology existing on them (usually the etale topology) [41, 43, 63].

3.4.2 Geometric stacks

Derived stacks are formal, or free, homotopy colimits of derived affines schemes in a special sense. The category of stacks is useful because of this completion property, but, as often with completions, it turns out that the general object of this category is quite wild and improper to the purposes of geometry. The problem is the local structure of these objects: for example, the tangent spaces may not have an addition.[71]. Concretely, in order to have a tamed, or *geometric*, local structure (with tangent spaces, infinitesimal neighborhoods, and everything) we would like to describe the neighborhood of a point in a stack, as for affine schemes, by means of a local ring (maybe henselian or more).[72] [73] However, this is not possible in general. Any stack Y can be described as the homotopy colimit of some diagram of derived affine schemes X_i, and the local structure of Y can be tamed only if the neighborhoods of points of Y can be lifted to neighborhoods of points in some X_i for some diagram presenting Y, i.e. if the quotient map $\coprod_i X_i \to Y$ has a lifting property for neighborhoods.

A map $f : X \to Y$ is said to have the *lifting property for etale neighborhoods* if for any scheme Z, corresponding to a strict henselian local ring A, with unique closed point named z (such a Z is an etale neighborhood of z) and for any commutative square as follows, there exists a diagonal lift

$$\begin{array}{ccc} z & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ Z & \longrightarrow & Y. \end{array}$$

A sufficient condition for a map to admit a such lifting property is to be smooth (submersive), moreover the lift is unique if the map is etale.[74] This leads to

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[71]It is only possible to define a tangent cone. The situation is the same with diffeologies. This is also related to the *microlinearity* condition in Synthetic Differential Geometry

[72]This problem is already there with sheaves, it has nothing to do with the homotopical nature of stacks.

[73]In the analogy of note 68 between sheaves (or stacks) and real numbers, the wild nature of general sheaves and stacks compares well to that of general real numbers. In order to work with a given real number, it is better to give it more structure, like the property to be algebraic. Algebraic numbers compare well with "tamed", or "geometric", sheaves and stacks.

[74]Several kinds of neighborhoods exists in (classical or derived) Algebraic Geometry. Let X be an affine scheme, A its ring of functions, k a field, and x a point of X given by a map $A \to k$. The most common neighborhoods of x in X are the Zariski, Nisnevich, Etale (related to the corresponding Grothendieck topology) and formal neighborhoods. They are defined respectively as the local ring of A at x, the henselian local ring of A at x, the strict henselian local ring of A at x, and the formal completion of A at x. Depending the kind of local structure we are interested in, the lifting property for maps can be defined with respect to any of these notions. The most common choices are the *smooth* and *formally smooth* maps defined with the etale and formal neighborhoods respectively. All this can be transcribed in analytic and differential geometry, where there is even more choices of neighborhood notions (e.g. Gevrey classes).

consider those diagrams such that the quotient map $\coprod_i X_i \to Y$ is smooth or etale. This is done by restricting to specific diagrams, called smooth or etale *internal groupoid objects*, and by considering only colimits which are quotients of those groupoid objects. If X is the quotient of a groupoid G, then G is called an *atlas* for X. The choice of smooth or etale maps distinguishes two classes of geometric stacks called respectively *Artin stacks* and *Deligne-Mumford stacks*. For such stacks, it is possible to define local features such as tangent complexes, and we get back all computations of section 2. We shall not say more here and refer to [57, 59] for more explanations and to [41, 51, 63] for the details.

Finally we have the following inclusions of geometrical objects:

	Basic objects	Objects with Zariski atlas	Objects with etale atlas	Objects with smooth atlas
Sheaves (ordinary AG)	Affine schemes	Schemes	Algebraic spaces	
Stacks (AG + derived colimits)			Deligne-Mumford stacks	Artin stacks
Derived stacks (AG + derived limits & colimits)	Derived affine scheme	Derived schemes	Derived D-M stacks	Derived Artin stacks
Tangent complex	in d° ≤ 0	in d° ≤ 0	in d° ≤ 0	unbounded

3.4.3   Geometry of derived stacks

We have finally arrived at the notion of derived geometric stack which are the spaces of derived Algebraic Geometry.[75] It is convenient to compare them with classical schemes or sheaves by mean of a diagram (which I stole to Toën and Vezzosi).

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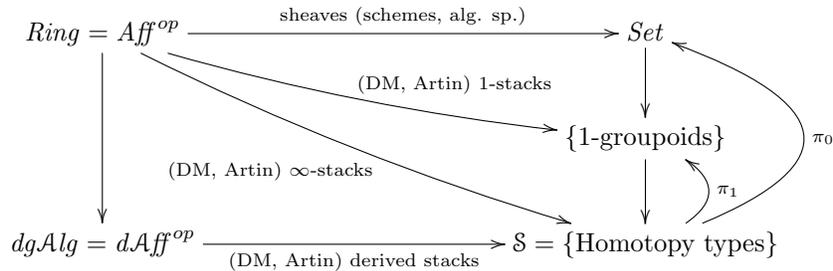
[75]Their definition can be transposed in differential or complex geometry with no trouble.

The arrows from the left to the right side are the *functor of points* of the differents objects of the theory. On the left side are the elementary blocks, or affine objects, and on the right are the categories of values for the *points* of the objects. It is convenient to read this as: the points of a scheme (or an algebraic space) form a set, the points of a 1-stack form a groupoid, the points of a derived stack form a homotopy type (or ∞-groupoid).

We mentioned in the introduction that derived stacks could be truncated like chain complexes, this is done by mean of the previous diagram. If $X : d\mathcal{A}ff \to \mathcal{S}$ is a derived stack, its *derived truncation* is the classical ∞-stack $Z_{\geq 0}X = Aff \to d\mathcal{A}ff \to \mathcal{S}$ and its *coarse space* is the sheaf $\pi_0 Z_{\geq 0}X = Aff \to Set$. At the level of the tangent complexe $\mathbb{T}_x X$, for some point $x \in X$, these operations correspond to the positive truncation $Z_{\geq 0}\mathbb{T}_x X$ and to $H_0(\mathbb{T}_x X)$.

Finally, let us mentioned that all the notions developed in 3.3.3 for derived schemes (points, open immersion, etale, smooth and proper maps...) can be generalized for derived stacks, so the langage of geometry finds yet another model. In particular, it is possible to define Zariski and etale spectra of stacks. A very nice theorem of Lurie [43] state that Deligne-Mumford derived stacks can be described faithfully by their etale spectra (which is an ∞-topos) together with the canonical structure sheaf. For Artin stacks, however, no description as structured space is possible. We refer to [41, 63] for the definitions of geometric features of derived stacks.

3.5 Derived Geometry and the BRST construction

Applications of Derived Geometry are largely explained in [57, 59], we shall say only a few things about it related to the connection with symplectic geometry and BRST construction.

We mentioned in the introduction that the first formalisms of derived geometry were algebraic, using dg-algebras (commutative or Lie) as opposed to geometric. Perhaps the origin of this is actually to be looked in Physics during the '70s, where the BRST construction was developped. We shall briefly tell how to interpret the algebraic construction of [36] in terms of derived geometry.

Recall first that a symplectic reduction is constructed in two steps: from a symplectic manifold, we extract a subspace (possibly with intersection singularities) and then take a quotient by a group action (possibly creating further quotient singularities). From everything that we have tried to explain in this text, it should be clear that derived geometry is the perfect setting where to compute such a reduction. And indeed, such a reduction is one of the main operations of the new field of *Derived Symplectic Geometry* where it is proven that the resulting object, however singular it may be, is still symplectic, see [6, 49] for more details.

Now, in order to make the connection with [36], we are going to construct a functor $\mathcal{O} : d\mathcal{S}tack \to dg\mathcal{A}lg$ which send a derived stack X to an unbounded dg-algebra $\mathcal{O}(X)$ playing the role of the algebra of (globally defined) functions in X.

Let $d\mathcal{S}tack$ be the ∞-category of derived stacks, recall that the Yoneda embedding $d\mathcal{A}ff \to d\mathcal{S}tack$ is a homotopy colimit completion, then the construction of $\mathcal{O}$ is done using this universal property: if X is a derived affine scheme corresponding to a dg-algebra A, we put $\mathcal{O}(X) = A$ and for a general derived stack X, we write it as a colimit of derived affine schemes $X = \text{colim} X_i$, and we put $\mathcal{O}(X) = \lim A_i$ where the (homotopy) limit is computed in the ∞-category $dg\mathcal{A}lg$ of unbounded dg-algebras. The resulting functor $\mathcal{O} : d\mathcal{S}tack \to dg\mathcal{A}lg$ is far from being faithful, so it is not true that Derived Geometry can be replaced by a study of unbounded dg-algebras. This setting is more the poor man's approximation to Derived Geometry.

Coming back to [36], let us start with a symplectic algebraic variety[76] M with a group action $G \times M \to M$ and a moment map $\mu : M \to \mathfrak{g}^*$. The system of equations $\mu = 0$ (usually irregular) can be used to define the level set $\mu^{-1}(0)$ as a derived affine scheme X, encoded by a Koszul dg-algebra $K(\mu = 0)$ (see 3.2.3) which is $\wedge\mathfrak{g} \otimes \mathcal{O}(M)$ with $\mathfrak{g}$ in (homological) degree 1 and the Koszul differential d. Now recall that, by design, $\mathcal{O}$ send colimits of derived stacks to limits of dg-algebras, in particular, if M_0 is the quotient of X by the canonical action of G, $\mathcal{O}(M_0)$ will be described by as a limit construction involving $K(\mu = 0)$ and functions on G. This limit is the meaning of the Chevalley complex construction built from $\mathfrak{g}^*$ on $K(\mu = 0)$, which is $\wedge\mathfrak{g} \otimes \mathcal{O}(M) \otimes \wedge\mathfrak{g}^*$ (where $\mathfrak{g}^*$ in degree -1) with total differential $d + \delta$, where the Koszul differential d is twisted by the Chevalley differential δ. This corresponds exactly to the two steps of the construction of [36] and interpret the resulting dg-algebra as the algebra of functions on the derived symplectic reduction M_0.

4 Conclusion

4.1 Success and prospects

The background for the development of Derived Geometry was a number of techniques used in geometry (algebraic, complex or differential) but not fully justified by geometrical reasons, they were ad hoc computational tools, efficient but somehow mysterious. Here is a short list:[77]

- homological features in commutative algebra (derived tensor products of rings, Serre formula, Koszul resolutions, cotangent complexes, virtual classes...);

- equivariant techniques in geometry to work on bad quotients;

- cohomology with coefficients in bundles (or sheaves), derived categories and functors;

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[76]The setting for symplectic manifolds has not been developed yet.

[77]To which one could add the use of geometric methods to understand some features of Stable Homotopy Theory (the stack of formal groups for cobordisms, the stack of elliptic curves for topological modular forms) which was also ill-justified by the theory. But I'm going to forget this aspect to restrict to more classical geometry. See [41] for more on the matter.

- the chain complexes with their Lie structure controlling deformation problems.

We have tried to explain how Derived Geometry has successfully found *geometrical principles* behind these constructions.

- Once commutative rings (affines schemes) are enhanced into dg-algebras[78] (derived affine schemes), derived tensor products, Serre formula, Koszul resolutions, cotangent complexes, virtual classes becomes the proper notions of tensor product, intersection formula, quotient, Kähler differential and fundamental classes (see 3.2.3 and 3.2.5). Moreover, this enhancement is of geometric nature (albeit surprising) because the full langage and structure of geometry make sense in derived affine schemes (see 3.3.3).

- Equivariant techniques correspond indeed to work on the quotient if groupoid quotients are effective, which forces to compute it in the higher categories of stacks (see 3.4.1). Moreover, as above, this extension of affine objects by stacks is of geometric nature because the full langage and structure of geometry extends to stacks (see 3.4.3).

- Once accepted the context of dg-algebras, the natural categories of modules are formed by chain complexes and not only modules, even for ordinary rings. From this point of view, the old abelian categories of modules are not regarded as a fundamental object anymore and the problem of derivation of functors that goes along somehow disappear. Derived functors are replaced by the natural functors between ∞-categories of chain complexes.[79] By extension, the categories of modules on a derived stack are also ∞-categories of chain complexes.[80]

- The chain complexes of deformation problems are simply the tangent spaces of Derived Geometry. The Lie structure is explained by the loop stacks (see 2.2.2), the obstruction theory by the structure theory of dg-algebras (see 3.2.5).

Altogether, this presents Derived Geometry as a new theory of geometry, not only encompassing the classical geometry, but having much better computational properties with respect to the whole homological/homotopical apparatus. The heart of this new geometry is to keep track of the inherent ambiguity to identify things when computing quotients (whether they be quotients of rings or quotients of spaces) by considering always homotopy quotients.[81] This is

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[78]Or simplicial algebras, or E_∞-ring spectra.

[79]This approach to the derivation of functors does even precise the role played by abelian categories where they become *hearts* of *t-structures*. Ironically, in Derived Geometry, it is the notion of abelian category that is *derived*.

[80]Strictly speaking, when presented this way, only non-negative chain complexes should be considered. The consideration of unbounded chain complexes is motivated because they have the nice extra property of *stability* (see note 86).

[81]Instead of Derived Geometry, I think this should be more appropriately called the *Geometry of Ambiguity*, hence the title of this text.

incredibly well suited for the study of moduli spaces as can be checked by the list of examples in [57, 59].

Having said so, the above tools and techniques certainly do not exhaust the methods existing in geometry and Derived Geometry is not the answer to all problems.[82] As we have tried to explain, it is only an enhancement taking better care of intersection and quotient singularities. But since singular spaces appear also in Differential and Complex Geometries, Derived Geometry do have something to say in these contexts. For example, because of the more regular tangential structure at singularities given by the tangent complexes and not only the tangent spaces, it is possible to defined on the whole of singular spaces (an not only on the smooth locus) a proper notion of differential forms and of symplectic or Poisson structures. These definitions have led to a huge extension of the notion of symplectic variety and to a very nice algebra of operations producing symplectic spaces from other ones (symplectic reductions of course, but also intersection of lagrangians maps and mapping spaces) [6, 7, 49, 60]. So far this extension has been done only in the algebraic setting, but no doubt something similar could be done in Differential and Complex Geometries

4.2   Higher categorical mathematics

Some ideas subsuming Derived Geometry are not actually specific to geometry, they are ideas of Higher Category Theory. The cross-breading in the '90s-'00s of Algebraic Geometry, Algebraic Topology and Category Theory, that gave birth at the same time to Higher Stacks Theory and Higher Category Theory may very well be one of the most fruitful of mathematics because it lays the ground of a deep revolution that we would like to advertise.[83]

We left pending in 3.2.4 and note 50, the question of the necessity of ∞-categories in the derivation process. It is indeed a natural question to ask why ∞-categories have become so important. The answer, that we can only sketch here, is simple and deep: ∞-categories provide computational properties inaccessible to ordinary categories. For example, we have underlined several times in this text the role played by the important property of *effectivity of groupoid quotients*, i.e. the ability to works on quotient of groupoids by equivariant methods (see 3.2.4). This property of a category has the most remarkable property: if it is true in an ordinary category, then this category is the trivial category with one object.[84] This means than this property can only have non-trivial models in proper ∞-categories! (and it does, we have seen a few).[85]   Another very

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[82]Notably, Derived Geometry as little to say about spaces of infinite dimension.

[83]Actually, Derived Geometry may very well be the first field born out of higher categorical ideas. Other fields could be Stable Category Theory [40], Homotopy Type Theory, Derived Symplectic Geometry.

[84]More precisely, $(n-1)$-groupoids can be effective in n-categories, but the effectivity of n-groupoids in an n-category forces it to be trivial. Only when $n = \infty$ can n-groupoids be effective in an n-category.

[85]This is in particular the case of all the so-called ∞-toposes and of all the ∞-categories of models of algebraic theories taken in an ∞-topos. For example, chain complexes or dg-algebras are models of algebraic theories in the topos $\mathbb{S}$ if homotopy types, hence their effectivity

important such property, although outside the scope of this text, is the axiom of *stability*, which enhances and simplifies the theory of triangulated categories [40].[86] [87]

A second quite natural question about ∞-categories is: why are homotopy types (or ∞-groupoids) so important? Here again there exists a simple and deep answer: because they can be seen as a notion more primitive than sets. At first, this might seems silly because all mathematical objects, including homotopy types, are classically defined using sets as a primitive notion. But, assuming the notion of homotopy types, it is also possible to define sets as *discrete homotopy types* i.e. as homotopy types with a specific property.

Following this line of thought proposes to redefine the whole of mathematical structures with an underlying homotopy type instead of an underlying set. The first motivation for such a bold idea is the fact that examples of such notions of "structures with underlying homotopy type" exists (as we have tried to advertise in this text). But the main motivation is the fact that homotopy types provide a notion of identification for elements (through paths) more suited to talk about certain structures. We already mentioned how homotopy types could be seen as an enhancement of sets incorporating the ambiguity of identifications (see 3.2.1 and note 36). Another situation is the following, of a more logical flavor. With respect to the manipulation of vector spaces, the set of all vector spaces is less natural than the groupoid of all vector spaces: since all constructions on vector spaces are expected to be invariant by isomorphisms, they should be defined with respect to the latter and not only the former. Developing a langage based on homotopy types instead of sets would provide automatically that any construction be invariant by isomorphism.[88]

Since the '60s, the best practical approach is Quillen's model categories (and their variations) which is underlying all approaches to higher categories. However, model categories are a way to reduce Higher Category Theory to ordinary Category Theory and the feeling is that there should be a proper theory for higher categories. However no such approach as been successfully developed yet. Recently, Homotopy Type Theory [28] has proposed an interesting syntactic approach to homotopy types/∞-groupoids that could provide a foundational langage for mathematical alternative to Set Theory axioms. This is well suited to get some aspects of homotopy types but largely insufficient yet for the purposes of the working mathematician. History will show if a better langage can be

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property.

[86] *Stability* is the property of a pointed category that any commutative square is a pushout iff it is a pullback, or equivalently that finite colimits commute with finite limits. Again, only the trivial ordinary category with one object is stable, but there are many non-trivial stable ∞-categories: ∞-categories of (unbounded) chain complexes are stable, so are ∞-categories of spectra and parametrized spectra. See [40] for details and examples.

[87] From a logical point of view, one would say that there exists logical theories (or syntaxes) that have non-trivial semantics only in higher categories. For example, an object X such that $\Omega\Sigma X \simeq X$ where Ω and Σ are the loop space and suspension functors.

[88] This would be a strong version of Leibniz's principle of indiscernables. This strengthening is false with ZFC or axioms of the like. However, this issue is one of the motivations for Martin-Löf to have introduced its identity types [28], within this syntax all propositions about, say, groups are automatically stable by isomorphism.

found for higher categories.

4.3 Toward a new axiomatisation of geometry?

The objects of Derived Geometry are more complex than ordinary manifolds or even schemes but they enjoy better properties than their classical counterparts. So what should we prefer? This is actually an illustration of a tension that exists within mathematics about whether to define its objects individually by some intrinsic structure (the affine plane as $\mathbb{R}^2$) or in family by an algebra of operations on the family (planar geometry with its figures and incidence rules). Algebraic Geometry, particularly under the influence of A. Grothendieck, has rather favored the simplicity of properties over the simplicity of its objects: points at infinity (projective geometry), points with complex coordinates (complex geometry), multiple points (schemes) were in particular motivated by the regularization of the intersection properties of the plane. Derived geometry, with its treatment of singularities, is but the latest step in the same direction. Its objects might be strange in their individual nature, they may have new properties, but, as a whole they behave more regularly and, remarkably, the langage to talk about them is still the same.

We have insisted several times on the fact that the geometric nature of objects such rings, dg-algebras, sheaves or stacks was in the successful interpretation of the langage of geometry (together with some expected structural properties) such as points, etale and smooth maps, proper maps... (see 3.3.3, 3.3.4, 3.4.3). Actually, in front of the many settings where this langage makes sense (topological spaces, toposes, manifolds, schemes... derived stacks being the latest), it is tempting to try to axiomatize an abstract setting for *geometry* as a category together with classes of maps corresponding to all the aforementioned classes. The notion of geometry of Lurie [43] is a first attempt in this direction, so is Schreiber's setting for "differential cohesive homotopy theory" [53]. Such an axiomatisation of geometry, emphasizing the structure of the relations between the objects of the geometry rather than the structure of the objects themselves, would be the XXI century version of Euclid's axioms.

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