Goodwillie and Weiss Towers

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October 9, 2024

Abstract

Unfinished draft! We revisit Goodwillie's calculus and Weiss' Orthogonal Calculus, by taking advantage of the natural Day convolution products existing in both examples. We then show that these defines acyclic towers of congruences in the sense of [ABFJ24b].

Contents

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1 Introduction

Our goal is to develop a general setting for Goodwillie and Weiss calculi. We are using the theory of ∞-categories systematically, except that every ∞-category in this paper is simply said to be a category. There is no confusion, since an ordinary category is said to be 1-category if necessary. For example, our category of spaces $\mathcal{S}$ is Lurie's ∞-category of ∞-groupoids []. For a discussion on our convention and related matter, see the end of the present introduction.

Here we only consider the two basic examples of calculi:

1. the Goodwillie calculus of functors $\mathsf{Fin} \to \mathcal{S}$, where $\mathcal{S}$ (resp. Fin) is the category of spaces (resp. finite spaces)
2. the Weiss calculus of functors $\mathbb{W} \to \mathcal{S}$, where $\mathbb{W}$ is the category finite dimensional real euclidian vector spaces.

More cases will be presented in a second paper [ABFJ25]; it includes for example the Goodwillie calculus of functors $\mathbb{A} \to \mathcal{S}$, where $\mathbb{A}$ is any small category with finite colimits.

1. The Goodwillie calculus of functors $\mathsf{Fin} \to \mathcal{S}$. Let $\mathcal{S}$ (resp. Fin) be the category of spaces (resp. finite spaces). Recall that a functor $F : \mathsf{Fin} \to \mathcal{S}$ is said to be *n-excisive* (Goodwillie) it it takes every completely cocartesian $(n+1)$-cube $\chi : \mathcal{P}(n+1) \to \mathscr{S}$ to a cartesian $(n+1)$-cube $F \circ \chi$. Goodwillie showed that the sub-category of n-excisive functors $[\mathsf{Fin}, \mathcal{S}]^{n-ex}$ is a reflexive (in the category of all functors $[\mathsf{Fin}, \mathcal{S}]$) by constructing a reflector

$$P_n : [\mathsf{Fin}, \mathcal{S}] \to [\mathsf{Fin}, \mathcal{S}]^{n-ex}$$

A 0-excisive functor is constant and the reflector P_0 takes a functor $F : \mathsf{Fin} \to \mathcal{S}$ to its value $P_0(F) = F(1)$. We shall prove that the localization P_n is the $(n+1)$-fold acyclic power of P_0. More explicitly, this means a functor $F \in [\mathsf{Fin}, \mathcal{S}]$ is n-excisive if and only if the functor $\mathrm{map}(-, F) : [\mathsf{Fin}, \mathcal{S}]^{op} \to \mathcal{S}$ takes every *completely cartesian* $(n+1)$-cube of

P_0-equivalences to a cartesian $(n+1)$-cube. Our proof is using the symmetric monoidal structure on the category Fin defined by the join operation $(A, B) \mapsto A \star B$, and whose unit object is empty space $\varnothing$. From the symmetric monoidal structure $(\mathsf{Fin}, \star, \varnothing)$ we obtain by Day convolution a symmetric monoidal closed structure on the category $[\mathsf{Fin}, \mathcal{S}]$. By construction, the tensor product of two functors $F, G : \mathsf{Fin} \to \mathcal{S}$ is the functor $F \otimes G$ defined by letting

$$(F \otimes G)(K) = \int^{A \in \mathsf{Fin}} \int^{B \in \mathsf{Fin}} F(A) \times G(B) \times \mathrm{map}(A \star B, K)$$

for every every $K \in \mathsf{Fin}$. The unit object for the Day convolution product is the functor $R^{\varnothing} := \mathrm{map}(\varnothing, -) = 1$. The internal hom $[F, G]$ is constructed by the formula $[F, G](K) = \mathrm{map}(F, G(K \star -))$ for every $K \in \mathsf{Fin}$. The symmetric monoidal category $[\mathsf{Fin}, \mathcal{S}]$ is ω-presentable and *confined*, which means that the tensor product of ω-compact objects is ω-compact and that its unit object is ω-compact. Let Z_n is the $(n+1)$-join power of the object $R^1 = \mathrm{map}(1, -)$ of the category $[\mathsf{Fin}, \mathcal{S}]$. Then we have

$$[Z_n, F](X) = \lim_{U \in \mathcal{P}_0(n+1)} F(X \star U)$$

for every $F : \mathsf{Fin} \to \mathcal{S}$ and $X \in \mathsf{Fin}$, where $\mathcal{P}_0(n+1)$ is the poset of non-empty subsets of the set $\{1, \dots, n+1\}$. We shall prove that Z_n is *perfect*; this notion introduced by Weiss in context of Weiss calculus means that Z_n is ω-compact and if $T_n := [Z_n, -]$ is the pointed endo-functor of $[\mathsf{Fin}, \mathcal{S}]$ defined by the map $Z_n \to R^{\varnothing} = 1$ and if $P_n := \mathrm{colim}_{k \geqslant 0} T_n^k$, then $P_n(Z_n) = 1$. The perfectness of Z_n implies that P_n is a reflector and that a functor $F : \mathsf{Fin} \to \mathcal{S}$ is P_n-local if and only if it is T_n-local if and only if it is n-excisive. We also prove that the category $[\mathsf{Fin}, \mathscr{S}]^{n-ex}$ is symmetric monoidal closed and confined.

2. The Weiss calculus of functors $\mathbb{W} \to \mathcal{S}$. Let $\mathbb{W}$ be the category of finite dimensional real euclidian vectors space and isometric embeddings. The orthogonal sum $(U, V) \mapsto U \oplus V$ is a symmetric monoidal structure on the category $\mathbb{W}$, with unit object the nul space 0. Recall that a functor $F : \mathbb{W} \to \mathcal{S}$ is said to be *n-polynomial* (Weiss) if the canonical map $F \to T_n(F)$ is invertible, where

$$T_n(F)(V) = \lim_{0 < U \subseteq \mathbb{R}^{n+1}} F(V \oplus U)$$

for every $V \in \mathbb{W}$. Weiss showed that the sub-category of n-polynomial functors $[\mathbb{W}, \mathcal{S}]^{n-pol}$ is reflexive (in the category of all functors $[\mathbb{W}, \mathcal{S}]$) by constructing a reflector

$$P_n : [\mathbb{W}, \mathcal{S}] \to [\mathbb{W}, \mathcal{S}]^{n-pol}$$

A 0-excisive functor is constant and the functor P_0 takes a functor $F : \mathbb{W} \to \mathcal{S}$ to its colimit $\mathrm{colim}_{\mathbb{W}} F \in \mathcal{S}$ (the ∞-category $\mathbb{W}$ is directed). We shall prove that the localization P_n is the $(n+1)$-fold acyclic power of the localization P_0. More explicitly, this means that a functor $F \in [\mathbb{W}, \mathcal{S}]$ is n-polynomial if and only if the functor $\mathrm{map}(-, F) : [\mathbb{W}, \mathcal{S}]^{op} \to \mathcal{S}$ takes every *completely cartesian* $(n+1)$-cube of P_0-equivalences to a cartesian $(n+1)$-cube. By construction, $P_n = \mathrm{colim}_{k \geqslant 0} T_n^k$, where $Id \to T_n$ is the pointed endo-functor of $[\mathbb{W}, \mathcal{S}]$ defined above. The endo-functor T_n can be described by using the symmetric monoidal closed structure defined by Day convolution on the functor category $[\mathbb{W}, \mathcal{S}]$. By construction, the tensor product of two functors $F, G : \mathbb{W} \to \mathcal{S}$ is the functor $F \otimes G$

defined by letting

$$(F \otimes G)(U) = \int^{A \in \mathbb{W}} \int^{B \in \mathbb{W}} F(A) \times G(B) \times \mathrm{map}(A \oplus B, U)$$

for every $U \in \mathbb{W}$. The unit object of this monoidal structure is the functor $R^0 := \mathrm{map}(0, -) = 1$. The internal hom $[F, G]$ is constructed by the formula $[F, G](U) = \mathrm{map}(F, G(U \oplus -))$ for every $U \in \mathbb{W}$. The symmetric monoidal category $[\mathbb{W}, \mathcal{S}]$ is ω-presentable and confined. If Z_n denotes the $(n+1)$-join power of the functor $\mathrm{map}(\mathbb{R}, -) : \mathbb{W} \to \mathcal{S}$, then we have $T_n(F) = [Z_n, F]$ for every $F : \mathbb{W} \to \mathcal{S}$. Weiss has proved that the object Z_n is perfect. It follows that the functor P_n is a reflector and that a functor $F : \mathbb{W} \to \mathcal{S}$ is P_n-local if and only if it is T_n-local if and only if it is n-polynomial. We also prove that the category $[\mathbb{W}, \mathcal{S}]^{n-pol}$ is symmetric monoidal closed and confined.

We now describe some of the key tools introduced in the paper

Definition 1.0.1. (3.2.4) We will say that a smc (=symmetric monoidal closed) category $\mathcal{V} = (\mathcal{V}, \otimes, I)$ is *confined* if its underling category $\mathcal{V}_o$ is ω-presentable, if the unit object I is compact, and if the tensor product of two compact objects is compact.

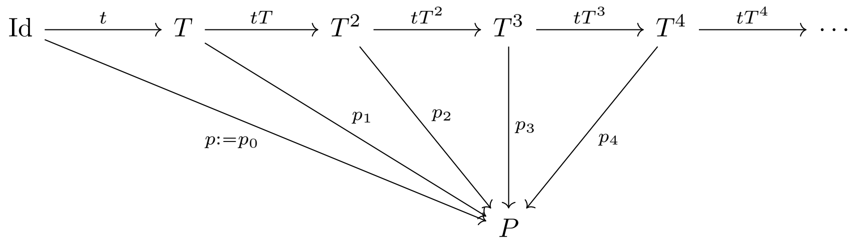
Let $\mathcal{V}$ be a confined symmetric monoidal closed category. From a map $z : Z \to I$ in $\mathcal{V}$, we obtain a $\mathcal{V}$-functor

$$T := [Z, -] : \mathcal{V} \to \mathcal{V}$$

and an enriched natural transformation $t := [z, -] : \mathrm{Id} \to T$. Let $P : \mathcal{V} \to \mathcal{V}$ be the colimit of the sequence of endofunctors

$$P := \mathrm{colim}\big(\mathrm{Id} \xrightarrow{t} T \xrightarrow{tT} T^2 \xrightarrow{tT^2} T^3 \xrightarrow{tT^3} T^4 \to \ldots\big).$$

By definition, we have a colimit cone



(1.0.2)

with conical maps $p_n : T^n \to P$. Let us put $p := p_0 : \mathrm{Id} \to P$. Both P and p are $\mathcal{V}$-natural transformations.

Definition 1.0.3. (3.4.6) Let $\mathcal{V}$ a confined symmetric monoidal closed category. We say that a map $z : Z \to I$ in $\mathcal{V}$ is *perfect* if the object Z is compact and the map $P(z) : P(Z) \to P(I)$ is invertible.

Theorem 1.0.4. (3.4.19) *If the map $z : Z \to I$ is perfect , then the natural transformations $pP : P \to P^2$ and $Pp : P \to P^2$ are equal and invertible.*

Definition 1.0.5. (3.3.6) We say that an object $X \in \mathcal{V}$ is T-closed (resp. P-closed) if the map $tX : X \to TX$ (resp. $pX : X \to PX$) is invertible.

Theorem 1.0.6. (3.4.32) *Suppose that the symmetric monoidal closed category $\mathcal{V}$ is confined and that the map $z : Z \to I$ is perfect. Define $T := [Z, -]$, $t := [z, -] : \mathrm{Id} \to T$, $P = \mathrm{colim}_n T^n$ and $p : Id \to P$. Let $\mathcal{V}^P$ be the subcategory P-closed objects of $\mathcal{V}$. Then,*

1. *an object $X \in \mathcal{V}$ is T-closed if and only if it is P-closed;*
2. *$X \in \mathcal{V}^P \Rightarrow [A, X] \in \mathcal{V}^P$ for every $A \in \mathcal{V}$;*
3. *the subcategory $\mathcal{V}^P$ is $\mathcal{V}$-reflective, the reflector $P : \mathcal{V} \to \mathcal{V}^P$ is left exact, and the map $pX : X \to PX$ is $\mathcal{V}$-reflecting into $\mathcal{V}^P$ for every of $X \in \mathcal{V}$;*
4. *the category $\mathcal{V}^P$ is symmetric monoidal closed; its tensor product $\otimes_P P$ is defined by $X \otimes_P Y := P(X \otimes Y)$ for every $X, Y \in \mathcal{V}^P$ and its unit object is $P(I)$; the localization functor $P : \mathcal{V} \to \mathcal{V}^P$ is symmetric monoidal;*
5. *the smc category $(\mathcal{V}^P, \otimes_P, P(I))$ is confined and the localization functor $P : \mathcal{V} \to \mathcal{V}^P$ is confined.*
6. *every compact object of $\mathcal{V}^P$ is a retract of an object in $P(\mathsf{c}(\mathcal{V}))$*

2 Preliminary

3 Localizations of symmetric monoidal closed categories

3.1 Compact objects, ω-presentable categories and confined functors

Recall that a category $\mathcal{C}$ is *finitely complete* if it has pullbacks and a terminal object. Recall that a functor between finitely complete categories $F : \mathcal{C} \to \mathcal{D}$ is said to preserves *finite limits*, or to be *lex*, if it preserves pullbacks and terminal objects. Dually, a category $\mathcal{C}$ is said to be *finitely cocomplete* if the opposite category $\mathcal{C}^{op}$ is finitely complete, and a functor $F : \mathcal{C} \to \mathcal{D}$ is said to preserves *finite colimits*, or to be *rex*, if the opposite functor $F^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$ preserves finite limits.

The category of spaces (=small groupoids) $\mathcal{S}$ is complete and cocomplete. If $\mathbb{C}$ is a small category, then the category $\mathscr{P}(\mathbb{C}) := ([\mathbb{C}^{op}, \mathcal{S}]$ of pre-sheaves on $\mathbb{C}$ is complete and cocomplete. If $y : \mathbb{C} \to \mathscr{P}(\mathbb{C})$ is the Yoneda functor, we shall denote by $Fin\mathscr{P}(\mathbb{C})$ the smallest full subcategory of $\mathscr{P}(\mathcal{C})$ which contains the representables $y(C) = \mathrm{map}(C, -)$ and is closed under finite colimits. We will say that a presheaf in $Fin\mathscr{P}(\mathbb{C})$ is *finitely presentable*. The notion of finitely presentable co-presheaf $F : \mathbb{C} \to \mathcal{S}$ is defined similarly by using $Fin\mathscr{P}(\mathbb{C}^{op})$.

Recall that the category of small categories Cat is complete and cocomplete. Let $Fin(\mathsf{Cat})$. be the smallest full subcategory of Cat closed under finite colimits and which contains the n-chain $[n] = \{0 < 1 < \cdots < n\}$ for every $n \geqslant 0$. We shall say that a category in $Fin(\mathsf{Cat})$ is *finitely presentable.*

Recall that if $\mathcal{J}$ is a small category, then a functor $F : \mathcal{J} \to \mathcal{C}$ is often said to be a *diagram* with values in the category $\mathcal{C}$. We say that the diagram $F : \mathcal{J} \to \mathcal{C}$ is *finitely presentable* if the category $\mathcal{J}$ is finitely presentable.

Recall that a small category $\mathcal{J}$ is said to be *filtered* if the colimit functor colim : $\mathcal{S}^{\mathcal{J}} \to \mathcal{S}$ preserves finite limits (equivalently, if it preserves the limit of finitely presentable diagrams).

A diagram $F : \mathcal{J} \to \mathcal{E}$ is said to be *filtered* if the category $\mathcal{J}$ is filtered. A category $\mathcal{E}$ is said to have *filtered colimits* if every filtered diagram $F : \mathcal{J} \to \mathcal{E}$ has a colimit colim $F \in \mathcal{E}$.

Let $\mathcal{E}$ be a category with filtered colimits. Recall that an object $K \in \mathcal{E}$ is to be *compact* (or *ω-compact*) if the functor $Map(K, -) : \mathcal{E} \to \mathcal{S}$ preserves filtered colimits. A finite colimit of compact objects is compact, and a retract of a compact object is compact. We small denote by $\mathsf{c}(\mathcal{E})$ the full subcategory of compact objects of $\mathcal{E}$.

If $\mathbb{C}$ is a small category, then a presheaf $F : \mathbb{C}^{op} \to \mathcal{S}$ is compact in the category $\mathscr{P}(\mathbb{C})$ if and only if it is a retract of a finitely presentable presheaf. Dually, a co-presheaf $F : \mathbb{C} \to \mathcal{S}$ is compact if and only if it is a retract of a finitely presentable co-presheaf.

Definition 3.1.1. Recall that a small full subcategory $\mathbb{C}$ of a category $\mathcal{E}$ is said to be *dense* if the (restricted) Yoneda functor $\mathcal{E} \to [\mathbb{C}^{op}, \mathcal{S}]$ is fully faithful. Equivalently, $\mathbb{C}$ is dense if every object in $\mathcal{E}$ is the colimit of a diagram $D : \mathcal{J} \to \mathbb{C}$

Lemma 3.1.2. *Let $\mathbb{C} \subseteq \mathcal{E}$ be a small dense full subcategory of a category $\mathcal{E}$. Then a morphism $f : X \to Y$ in $\mathcal{E}$ is invertible if and only if the map $Map(C, f) : Map(C, X) \to Map(C, Y)$ is invertible for every object $C \in \mathbb{C}$.*

Proof. The (restricted) Yoneda functor $\mathcal{E} \to Fun(\mathsf{c}(\mathbb{C})^{op}, \mathcal{S})$ is fully faithful, since the subcategory $\mathbb{C}$ is dense. The result follows, since a fully faithful functor is conservative. □

Definition 3.1.3. Recall that a category $\mathcal{E}$ is said to be *ω-presentable* if it is cocomplete and its full subcategory of compact objects $\mathsf{c}(\mathcal{E}) \subseteq \mathcal{E}$ is essentially small and dense in $\mathcal{E}$.

For example, the category of presheaves $\mathscr{P}(\mathbb{C})$ on a small category $\mathbb{C}$ is ω-presentable, since every representable presheaf is compact and every presheaf is a colimit of representable presheaves.

If $\mathcal{E}$ is ω-presentable, then the subcategory of compact objects $\mathsf{c}(\mathcal{E})$ is closed under finite colimits and retracts.

Recall that every category $\mathcal{C}$ has a free cocompletion under *filtered colimits* called the *ind completion* of $\mathcal{C}$ and denoted $Ind(\mathcal{C})$. An object of $Ind(\mathcal{C})$ is compact if and only if it is a retract of an object of $\mathcal{C}$. If $\mathbb{C}$ is a small category, then the category $Ind(\mathbb{C})$ is a full subcategory of the presheaf category $Fun(\mathbb{C}^{op}, \mathcal{S})$. A presheaf $F : \mathbb{C}^{op} \to \mathcal{S}$ belongs to $Ind(\mathbb{C})$ if and only if its category of elements $el(F) = \mathbb{C}/F$ is filtered. When the category $\mathbb{C}$ has finite colimits, the opposite category $\mathbb{C}^{op}$ is an essentially algebraic theory, since it has finite limits; in that case a presheaf $F : \mathbb{C}^{op} \to \mathcal{S}$ belongs to $Ind(\mathbb{C})$ iff the functor F preserves finite limits. In other words, if $\mathbb{C}$ has finite colimits, then $Ind(\mathbb{C})$ is the category of models $Mod(\mathbb{C}^{op})$ of the algebraic theory $\mathbb{C}^{op}$. Conversely, if a category $\mathcal{E}$ is ω-presentable, then its full subcategory of compact objects $\mathsf{c}(\mathscr{E})$ is small, it has finite colimits and every idempotent splits. Moreover,

$$\mathcal{E} \simeq Ind(\mathsf{c}(\mathcal{E})) = Mod(\mathsf{c}(\mathcal{E})^{op}) \tag{3.1.4}$$

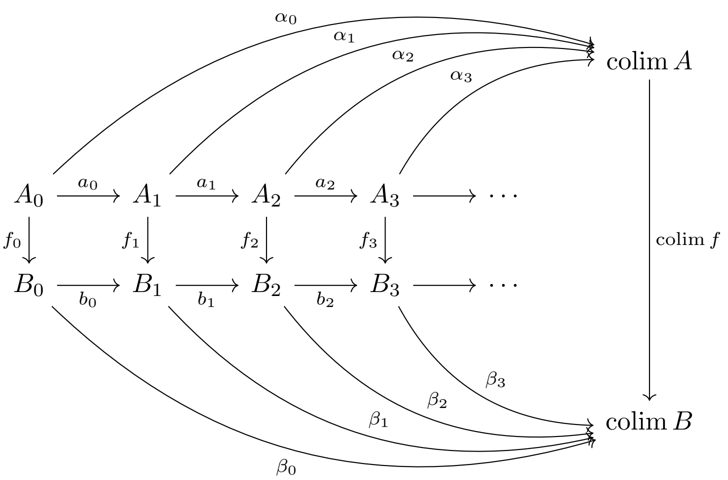
Let $\mathbb{N} = (\mathbb{N}, \leqslant)$ be the poset of natural numbers. If $\mathcal{E}$ is a category, then a functor $A : \mathbb{N} \to \mathcal{E}$ is an increasing sequence of objects in $\mathcal{E}$,

$$A_0 \xrightarrow{a_0} A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} A_3 \longrightarrow \cdots$$

and a natural transformation $f : A \to B$ between two functors $A, B : \mathbb{N} \to \mathcal{E}$ is a ladder of commutative squares,

$$\begin{array}{ccccccccc} A_0 & \xrightarrow{a_0} & A_1 & \xrightarrow{a_1} & A_2 & \xrightarrow{a_3} & A_3 & \longrightarrow & \cdots \\ {\scriptstyle f_0}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & \\ B_0 & \xrightarrow[b_0]{} & B_1 & \xrightarrow[b_1]{} & B_2 & \xrightarrow[b_2]{} & B_3 & \longrightarrow & \cdots \end{array}$$

Suppose that $\mathcal{E}$ is cocomplete and write $\alpha_n : A_n \to \operatorname{colim} A$ and $\beta_n : B_n \to \operatorname{colim} B$ for the conical maps. Then the following diagram commutes:



(3.1.5)

Lemma 3.1.6. *Let $\mathcal{E}$ be an ω-presentable category and let $f : A \to B$ be a natural transformation between two diagrams $A, B : \mathbb{N} \to \mathcal{E}$. If the square*

$$\begin{array}{ccc} A_n & \xrightarrow{\alpha_n} & \operatorname{colim}(A) \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle \operatorname{colim}(f)} \\ B_n & \xrightarrow{\beta_n} & \operatorname{colim}(B) \end{array} \tag{3.1.7}$$

has a diagonal filler for every $n \geqslant 0$, then the map $\operatorname{colim}(f)$ *is invertible.*

Proof. We first consider the case where $\mathcal{E} = \mathcal{S}$. Let S^k be the k-sphere ($k \geqslant -1$). By the Whitehead Theorem, the map $\mathrm{colim}(f)$ is invertible, i.e. a homotopy equivalence, if and only if every commutative square

$$\begin{array}{ccc} S^k & \xrightarrow{x} & \mathrm{colim}(A) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{colim}(f)} \\ 1 & \xrightarrow{y} & \mathrm{colim}(B) \end{array} \tag{3.1.8}$$

has a diagonal filler. The square 3.1.8 is the composite of two commutative squares

$$\begin{array}{ccccc} S^k & \xrightarrow{x'} & A_n & \xrightarrow{\alpha_n} & \mathrm{colim}(A) \\ \downarrow & & {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle \mathrm{colim}(f)} \\ 1 & \xrightarrow{y'} & B_n & \xrightarrow{\beta_n} & \mathrm{colim}(B) \end{array} \tag{3.1.9}$$

for some $n \geqslant 0$, since $S^k \to 1$ is a map between finite spaces (we implicitly using the fact that the map $S^k \to 1$ is a compact object of the category $\mathcal{S}^{[1]}$). Hence the square 3.1.8 has a diagonal filler, since the square 3.1.7 has a diagonal filler. This shows that the map $\mathrm{colim}(f)$ is invertible. The proposition is proved in the case where $\mathcal{E} = \mathcal{S}$.

Let us now return to the general case of an ω-presentable category $\mathcal{E}$. By Lemma 3.1.2 it suffices to show that $Map(K, \mathrm{colim}(f))$ is invertible for every compact object K in $\mathcal{E}$. But the functor $Map(K, -) : \mathcal{E} \to \mathcal{S}$ preserves filtered colimits, since K is compact. Hence the map $Map(K, \mathrm{colim}(f))$ is the colimit of the natural transformation $Map(K, f) : Map(K, A) \to Map(K, B)$,

$$\begin{array}{ccc} Map(K, A_n) & \xrightarrow{Map(K,\alpha_n)} & Map(K, \mathrm{colim}(A)) \\ {\scriptstyle Map(K,f_n)}\downarrow & & \downarrow{\scriptstyle Map(K,\mathrm{colim}(f))} \\ Map(K, B_n) & \xrightarrow{Map(K,\beta_n)} & Map(K, \mathrm{colim}(B)) \end{array} \tag{3.1.10}$$

For every $n \geqslant 0$, the square 3.1.10 has a diagonal filler since the square (3.1.7) has a diagonal filler. Hence the map $Map(K, \mathrm{colim}(f)) = \mathrm{colim}\, Map(K, f)$ is invertible by the first part of the proof. It follows by 3.1.2 that the map $\mathrm{colim}(f)$ is invertible. □

Definition 3.1.11. We shall say that a cocontinuous functor between ω-presentable categories $\phi : \mathcal{E} \to \mathcal{F}$ is *confined* if it takes compact objects to compact objects.

Recall Lurie HTT (31/7/2008) [Corollary 5.5.2.9] that every cocontinuous functor between presentable categories $\phi : \mathcal{E} \to \mathcal{F}$ has a right adjoint $\phi_\star : \mathcal{F} \to \mathcal{E}$.

Lemma 3.1.12. *A cocontinuous functor between ω-presentable categories $\phi : \mathcal{E} \to \mathcal{F}$ is confined if and only its right adjoint $\phi_\star : \mathcal{F} \to \mathcal{E}$ preserves filtered colimits.*

Proof. If the functor $\phi_\star$ preserves filtered colimits, let us show that the functor ϕ is confined. If $K \in \mathcal{E}$ is compact, let us show that $\phi(K) \in \mathcal{F}$ is compact. By the adjunction $\phi \vdash \phi_\star$, the functor $\mathrm{map}(\phi(K), -) : \mathcal{F} \to \mathcal{S}$ is isomorphic to the functor $\mathrm{map}(K, \phi_\star(-)) : \mathcal{F} \to \mathcal{S}$. The functor $\mathrm{map}(K, -)$ preserves filtered colimits, since K is compact, hence also the functor $\mathrm{map}(K, \phi_\star(-))$, since the functor $\phi_\star$ preserves filtered colimits by hypothesis.

It follows that the functor $\mathrm{map}(\phi(K), -) : \mathcal{F} \to \mathcal{S}$ preserves filtered colimits, and hence that $\phi(K)$ is compact. Conversely, if ϕ is confined, let us show that $\phi_\star$ preserves filtered colimits. The subcategory $\mathbf{c}(\mathbb{E})$ of compact objects of $\mathbb{E}$ is dense, since $\mathbb{E}$ is ω-presentable. Hence it suffices to show that the functor $\mathrm{map}(K, \phi_\star(-)) : \mathcal{F} \to \mathcal{S}$ preserves filtered colimits for every $K \in \mathbf{c}(\mathbb{E})$, since the functor $\mathrm{map}(K, -) : \mathcal{E} \to \mathcal{S}$ preserves filtered colimits for every $K \in \mathbf{c}(\mathbb{E})$. But the functor $\mathrm{map}(K, \phi_\star(-)) : \mathcal{F} \to \mathcal{S}$ is isomorphic to the functor $\mathrm{map}(\phi(K), -) : \mathcal{F} \to \mathcal{S}$. The object $\phi(K) \in \mathcal{F}$ is compact, since ϕ is confined and $K \in \mathbf{c}(\mathbb{E})$. Hence the functor $\mathrm{map}(\phi(K), -)$ preserves directed colimits for every $K \in \mathbf{c}(\mathbb{E})$. This proves that the functor $\phi_\star$ preserves filtered colimits. □

Proposition 3.1.13. *Let $\phi : \mathcal{E} \to \mathcal{F}$ be cocontinuous functor between ω-presentable categories, and let $\mathbb{C} \subseteq \mathcal{E}$ be a dense subcategory of compact objects in $\mathcal{E}$. If the functor ϕ takes every object of $\mathbb{C}$ to a compact object of $\mathcal{E}$, then ϕ is confined.*

Proof. The functor $\phi : \mathcal{E} \to \mathcal{F}$ has a right adjoint $\phi_\star : \mathcal{F} \to \mathcal{E}$ by 3.1.12. Let us show that the functor $\phi_\star$ preserves filtered colimits. For this, it suffices to show that the functor $\mathrm{map}(K, \phi_\star(-)) : \mathcal{F} \to \mathcal{S}$ preserves filtered colimits for every $K \in \mathbb{C}$, since the functor $\mathrm{map}(K, -) : \mathcal{E} \to \mathcal{S}$ preserves filtered colimits for every $K \in \mathbb{C}$ and the subcategory $\mathbb{C}$ is dense. But the functor $\mathrm{map}(K, \phi_\star(-)) : \mathcal{F} \to \mathcal{S}$ is isomorphic to the functor $\mathrm{map}(\phi(K), -) : \mathcal{F} \to \mathcal{S}$, since $\phi \dashv \phi_\star$. Moreover, the functor $\mathrm{map}(\phi(K), -)$ preserves directed colimits, since $\phi(K)$ is compact for every $K \in \mathbb{C}$. We have proved that the functor $\phi_\star$ preserves filtered colimits. It then follows from 3.1.12 that the functor ϕ is confined. □

Let $\mathcal{E}$ be an ω-presentable category. If $\mathbb{A}$ is a small category, then every functor $\phi : \mathbb{A} \to \mathcal{E}$ has a left Kan extension $\phi_! : \mathscr{P}\mathbb{A} \to \mathcal{E}$.

Corollary 3.1.14. *If $\phi(\mathbb{A}) \subseteq \mathbf{c}(\mathcal{E})$, then the functor $\phi_! : \mathscr{P}\mathbb{A} \to \mathcal{E}$ is confined.*

Proof. If $y : \mathbb{A} \to \mathscr{P}\mathbb{A}$ is the Yoneda functor, then the subcategory of representables presheaves $y(\mathbb{A}) \subset \mathscr{P}\mathbb{A}$ is dense and every representable is compact. Moreover the functor $\phi_!$ takes every object in $y(\mathbb{A})$ to a compact object in $\mathcal{E}$, since $\phi_!(y(a)) = \phi(a)$ is compact for every $a \in \mathbb{A}$. It then follows from 3.1.13 that the functor $\phi_!$ is confined. □

Example 3.1.15. If $\phi : \mathbb{A} \to \mathbb{B}$ is functor between small categories. Then the functor $\phi^\star : Fun(\mathbb{B}, \mathcal{S}) \to Fun(\mathbb{A}, \mathcal{S})$ has a left adjoint $\phi_! : Fun(\mathbb{A}, \mathcal{S}) \to Fun(\mathbb{B}, \mathcal{S})$ and $\phi_!$ is confined.

3.2 Confined symmetric monoidal categories

For a precise definition of the notion of symmetric monoidal ∞-category, see Lurie [HA, Definition 2.0.0.7]. See also remarks 2.1.2.18 and 2.1.2.19 in HA.

Recall that a symmetric monoidal category $\mathcal{V} = (\mathcal{V}, \otimes, I)$ is said to be *closed* if the functor $A \otimes (-) : \mathcal{V} \to \mathcal{V}$ has a right adjoint $[A, -] : \mathcal{V} \to \mathcal{V}$ for every object $A \in \mathcal{V}$. The object $[A, B]$ is called the *internal hom* between the objects $A, B \in \mathcal{V}$.

Examples 3.2.1. (Day symmetric monoidal closed category)[HA, Example 2.2.6.17, Corollary 4.8.1.12, Remark 4.8.1.13] If $\mathbb{V} = (\mathbb{V}, \oplus, 0)$ is a symmetric monoidal category, then the functor category $\mathcal{V} := Fun(\mathbb{V}, \mathcal{S})$ equipped with the Day convolution product is symmetric monoidal closed. Recall that the *convolution product* $F \otimes G$ of $F \in \mathcal{V}$ and $G \in \mathcal{V}$ is given by the formula

$$(F \otimes G)(c) = \int^{a \in \mathbb{V}} \int^{b \in \mathbb{V}} F(a) \times G(b) \times Map(a \oplus b, c)$$

for every object $c \in \mathbb{V}$. The unit object for the convolution product is the corepresentable functor $R^0 := Map(0, -) : \mathbb{V} \to \mathcal{S}$. If $R^a := Map(a, -) : \mathbb{V} \to \mathcal{S}$ for every object $a \in \mathbb{V}$, then $R^a \otimes R^b = R^{a \oplus b}$ for every $a, b \in \mathbb{V}$. The *internal hom* $[F, G]$ of $F \in \mathcal{V}$ and $G \in \mathcal{V}$ is calculated by the formula

$$[F, G](a) = Nat(F, G(a \oplus -)) \tag{3.2.2}$$

for every $a \in \mathbb{V}$. In particular, $[R^a, G] = G(a \oplus -)$.

Proposition 3.2.3. *Let* $\mathbb{C} = (\mathbb{C}, \oplus, 0)$ *be a small symmetric monoidal category with finite colimits. If the tensor product* $\oplus : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ *preserves finite colimits in each variable, then the category* $Ind(\mathbb{C}) = Mod(\mathbb{C}^{op})$ *is a symmetric monoidal closed sub-category of the symmetric monoidal closed category* $Fun(\mathbb{C}^{op}, \mathcal{S})$ *equipped with the convolution product. In fact, if* $G \in Ind(\mathbb{C})$*, then* $[F, G] \in Ind(\mathbb{C})$ *for every* $F \in Fun(\mathbb{C}^{op}, \mathcal{S})$*.*

Proof. If F and G belongs to $Ind(\mathbb{C})$, let us show that their convolution product $F \otimes G$ in $Fun(\mathbb{C}^{op}, \mathcal{S})$ belongs to $Ind(\mathbb{C})$. The categories of elements $el(F)$ and $el(G)$ are filtered, since F and G belongs to $Ind(\mathbb{C})$. Moreover,

$$F = \operatorname*{colim}_{(A,a) \in el(F)} R^A \quad \text{and} \quad G = \operatorname*{colim}_{(B,b) \in el(G)} R^B$$

by Yoneda. Thus,

$$\begin{aligned} F \otimes G &= \operatorname*{colim}_{(A,a) \in el(F)} R^A \otimes \operatorname*{colim}_{(B,b) \in el(F)} R^B \\ &= \operatorname*{colim}_{(A,a) \in el(F)} \operatorname*{colim}_{(B,b) \in el(F)} R^A \otimes R^B \\ &= \operatorname*{colim}_{((A,a),(B,b)) \in el(F) \times el(G)} R^{A \oplus B} \end{aligned}$$

The category $el(F) \times el(G)$ is filtered, since the categories $el(F)$ and $el(G)$ are filtered. This shows that $F \otimes G$ is a filtered colimit of representables and hence that $F \otimes G$ belongs to $Ind(\mathbb{C})$. If $G \in Ind(\mathbb{C})$, let us show that $[F, G] \in Ind(\mathbb{C})$ for every $F \in Fun(\mathbb{C}^{op}, \mathcal{S})$. For this, it suffices to show that the contravariant functor $c \mapsto [F, G](c)$ takes finite colimits to finite limits. But we have $[F, G](c) = Nat(F, G(c \oplus -))$ by 3.2.2. The functor $G(c \oplus -)$ takes finite colimits to finite limits, since the functor $c \oplus - : \mathbb{C} \to \mathbb{C}$ preserves finite colimits and the functor G takes finite colimits to finite limits. It follows that the functor $Nat(F, G(c \oplus -))$ takes finite colimits to finite limits, and hence that $[F, G] \in Ind(\mathbb{C})$. ☐

Definition 3.2.4. We will say that a smc category (=symmetric monoidal closed category) $\mathcal{V} = (\mathcal{V}, \otimes, I)$ is *confined* if $\mathcal{V}$ is ω-presentable, if the tensor product of two compact objects is compact, and if the unit object I is compact.

Examples 3.2.5. Examples of confined symmetric monoidal closed categories:

1. the category of spaces $\mathcal{S}$ equipped with the cartesian monoidal structure is confined;

2. the category of pointed spaces $\mathcal{S}_\bullet$ equipped with the smash product is confined;

3. the category of spectra Spec equipped with the smash product is confined.

4. If $\mathbb{V} = (\mathbb{V}, \oplus, 0)$ is a small symmetric monoidal category, then the category $\mathcal{V} := Fun(\mathbb{V}, \mathcal{S})$ equipped with the Day convolution product of 3.2.1 is symmetric monoidal closed and confined.

Proposition 3.2.6. *Let $\mathcal{V} = (\mathcal{V}, \otimes, I)$ be a symmetric monoidal closed category. Suppose that $\mathcal{V}$ is ω-presentable, and that $\mathbb{C} \subseteq \mathcal{V}$ is a dense subcategory of compact objects of $\mathcal{V}$. If I is compact and $X \otimes Y$ is compact for every $X, Y \in \mathbb{C}$, then the smc $\mathcal{V}$ is confined.*

Proof. Let us show that $\mathbf{c}(\mathcal{V}) \otimes \mathbf{c}(\mathcal{V}) \subseteq \mathbf{c}(\mathcal{V})$. If $A \in \mathbb{C}$, let us show that the functor $A \otimes - : \mathcal{V} \to \mathcal{V}$ is confined. The functor $A \otimes -$ is cocontinuous, since it has a right adjoint $[A, -]$. Moreover, $A \otimes \mathbb{C} \subseteq \mathbf{c}(\mathcal{V})$, since $\mathbb{C} \otimes \mathbb{C} \subseteq \mathbf{c}(\mathcal{V})$ by the hypothesis. It then follows from 3.1.13 that $A \otimes \mathbf{c}(\mathcal{V}) \subseteq \mathbf{c}(\mathcal{V})$, since the subcategory $\mathbb{C}$ is dense. This proves that $\mathbb{C} \otimes \mathbf{c}(\mathcal{V}) \subseteq \mathbf{c}(\mathcal{V})$. If $B \in \mathbf{c}(\mathcal{V})$, let us show that the functor $- \otimes B : \mathcal{V} \to \mathcal{V}$ is confined. The functor $- \otimes B$ is cocontinuous, since it has a right adjoint $[B, -]$. Moreover, $\mathbb{C} \otimes B \subseteq \mathbf{c}(\mathcal{V})$, since $\mathbb{C} \otimes \mathbf{c}(\mathcal{V}) \subseteq \mathbf{c}(\mathcal{V})$ by the above. It then follows from 3.1.13 that $\mathbf{c}(\mathcal{V}) \otimes B \subseteq \mathbf{c}(\mathcal{V})$, since the subcategory $\mathbb{C}$ is dense. Thus, $\mathbf{c}(\mathcal{V}) \otimes \mathbf{c}(\mathcal{V}) \subseteq \mathbf{c}(\mathcal{V})$. □

For example, the category of simplicial spaces $\mathscr{P}(\Delta)$ is ω-presentable and cartesian closed. Let us show that it is confined. The subcategory of representables $y(\Delta)$ is obviously dense. The unit object for the cartesian product is the terminal object $1 = \Delta[0]$ which is compact since it is representable. The cartesian product $\Delta[m] \times \Delta[n]$ is finitely presentable for every $mn \geqslant 0$ by a classical result. Hence the subcategory $\mathbb{C} = y(\Delta)$ satisfies the conditions of 3.2.6. This shows that the cartesian closed category $\mathscr{P}(\Delta)$ is confined.

Remark 3.2.7. The category of small categories Cat is cartesian closed and confined.

If a symmetric monoidal closed category $\mathcal{E} = (\mathcal{E}, \otimes, I)$ is confined, then the category $\mathbf{c}(\mathcal{E})$ of compact objects of $\mathcal{E}$ is small and symmetric monoidal. Moreover, $\mathbf{c}(\mathcal{E})$ has finite colimits and the induced product $\otimes : \mathbf{c}(\mathcal{E}) \times \mathbf{c}(\mathcal{E}) \to \mathbf{c}(\mathcal{E})$ preserves finite colimits in each variable. By 3.1.4 and 3.2.3, we have an equivalence of symmetric monoidal categories

$$\mathcal{E} \simeq Ind(\mathbf{c}(\mathcal{E})) = Mod(\mathbf{c}(\mathcal{E})^{op}) \tag{3.2.8}$$

3.3 Good functors

Let $\mathcal{V}$ be a symmetric monoidal closed category.

If $F : \mathcal{V} \to \mathcal{V}$ is a $\mathcal{V}$-functor, then for every pair of objects $A, X \in \mathcal{V}$ we have an assembly map $\sigma(A, X) : A \otimes F(X) \to F(A \otimes X)$ and a coassembly map $\gamma(A, X) : F[A, X] \to [A, FX]$.

Definition 3.3.1. We say that a $\mathcal{V}$-functor $F : \mathcal{V} \to \mathcal{V}$

1. preserves *tensors* if the assembly map $\sigma(A, X) : A \otimes F(X) \to F(A \otimes X)$ is invertible for every objects X and $A \in \mathcal{V}$

2. preserves *cotensors* if the coassembly map $\gamma(A, X) : F[A, X] \to [A, FX]$. is invertible for every objects X and $A \in \mathcal{V}$

Lemma 3.3.2. *The $\mathcal{V}$-functor $[B, -] : \mathcal{V} \to \mathcal{V}$ preserves contensors for every object $B \in \mathcal{V}$.*

Proof. The coassembly map $\gamma(A, X) : [B, [A, X]] \to [A, [B, X]]$ is the composite of the natural isomorphisms

$$[B, [A, X]] \xrightarrow{\cong} [A \otimes B, X] \xrightarrow{\cong} [A, [B, X]]. \tag{3.3.3}$$

Hence the map $\gamma(A, X)$ is invertible. □

Definition 3.3.4. Suppose that the smc category $\mathcal{V}$ is confined. We say that a $\mathcal{V}$-functor $F : \mathcal{V} \to \mathcal{V}$ preserves *compact cotensors* if the coassembly map $\gamma(A, X) : F[A, X] \to [A, FX]$ is invertible for every compact $A \in \mathcal{V}$.

From a map $z : Z \to I$ in $\mathcal{V}$, we obtain a $\mathcal{V}$-functor

$$T := [Z, -] : \mathcal{V} \to \mathcal{V}$$

and a $\mathcal{V}$-natural transformation $t := [z, -] : \mathrm{Id} \to T$.

Lemma 3.3.5. *Let $z : Z \to I$ be a map in a smc category $\mathcal{V}$. If a $\mathcal{V}$-functor $F : \mathcal{V} \to \mathcal{V}$ preserves compact cotensors and $Z \in \mathcal{V}$ is compact then we have a commutative diagram of $\mathcal{V}$-natural transformations,*

$$\begin{array}{ccccc} & & F & & \\ & \swarrow^{Ft} & & \searrow^{tF} & \\ FT & & \xrightarrow[\cong]{\gamma} & & TF \end{array}$$

where $\gamma = \gamma(Z, X) : F[Z, X] \to [Z, FX]$ is a co-assembly map of the functor F.

Proof. From the map $z : Z \to I$ we obtain a commutative square of $\mathcal{V}$-natural transformations in the variable $X \in \mathcal{V}$.

$$\begin{array}{ccc} F[I, X] & \xrightarrow{\gamma(I,X)} & [I, FX] \\ {\scriptstyle F[z,X]}\downarrow & & \downarrow{\scriptstyle [z,FX]} \\ F[Z, X] & \xrightarrow{\gamma(Z,X)} & [Z, FX] \end{array}$$

The top horizontal map is isomorphic to the identity of FX. The co-assembly map $\gamma(Z, X) : F[Z, X] \to [Z, FX]$ is invertible, since the functor F preserves compact cotensors and Z is compact. □

Definition 3.3.6. Let $L : \mathcal{C} \to \mathcal{C}$ be an endo-functor of a category $\mathcal{C}$ with a coaugmentation $\ell : \mathrm{Id} \to L$. We say that an object $X \in \mathcal{C}$ is *L-closed* if the map $\ell X : X \to LX$ is invertible. We say that a map $f : X \to Y$ in $\mathcal{C}$ is *L-closed* if the naturality square

$$\begin{array}{ccc} X & \xrightarrow{\ell X} & LX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Lf} \\ Y & \xrightarrow{\ell Y} & LY \end{array}$$

is cartesian. We denote by $\mathcal{V}^L$ the full subcategory of $\mathcal{C}$ formed by L-closed objects.

Corollary 3.3.7. *Let $\mathcal{V}$ a smc category. Let $z : Z \to I$ be map in $\mathcal{V}$, let $T = [Z, -]$ and $t = [z, -] : Id \to T$. If $X \in \mathcal{V}$ is T-closed, then so is the object $[A, X]$ for every $A \in \mathcal{V}$.*

Proof. The horizontal maps of the following naturality square are invertible, since the functor $[A, -]$ preserves cotensors by 3.3.2.

$$\begin{array}{ccc} [I,[A,X]] & \xrightarrow{\gamma(I,X)} & [A,[I,X]] \\ {\scriptstyle [z,[A,X]]}\downarrow & & \downarrow{\scriptstyle [A,[z,X]]} \\ [Z,[A,X]] & \xrightarrow{\gamma(Z,X)} & [A,[Z,X]] \end{array}$$

Hence the maps $t[A, X] = [z, [A, X]$ and $[A, tX] = [A, [z, X]]$ are isomorphic. The map $[A, tX]$ is invertible, since the map tX is invertible by assumption. It follows that the map $t[A, X]$ is invertible. □

Definition 3.3.8. Suppose that the smc category $\mathcal{V}$ is confined. We say that a $\mathcal{V}$-functor $F : \mathcal{V} \to \mathcal{V}$ is *good* if it preserves filtered colimits, finite limits and compact cotensors.

Lemma 3.3.9. *Suppose that the smc category $\mathcal{V}$ is confined. Then the functor $[A, -] : \mathcal{V} \to \mathcal{V}$ is good for every compact object $A \in \mathcal{V}$.*

Proof. (1) The functor $[A, -] : \mathcal{V} \to \mathcal{V}$ preserves compact cotensors, since it preserves all cotensors by 3.3.2; it preserves finite limits since it has a left adjoint $A \otimes (-)$; it preserve filtered colimits by Lemma 3.1.12 since its left adjoint $A \otimes (-)$ preserves compact objects. □

Let us denote by $[\mathcal{V}, \mathcal{V}]$ the category of $\mathcal{V}$-functors $\mathcal{V} \to \mathcal{V}$ and $\mathcal{V}$-natural transformations. The category $[\mathcal{V}, \mathcal{V}]$ has limits and colimits, and they are computed pointwise by **??**.

Lemma 3.3.10. *Suppose that the smc category $\mathcal{V}$ is confined. Then the composite of two good functors is good. The sub-category of good functors $Good(\mathcal{V}, \mathcal{V}) \subseteq [\mathcal{V}, \mathcal{V}]$ is closed under filtered colimits.*

Proof. ? □

3.4 Perfect localizations

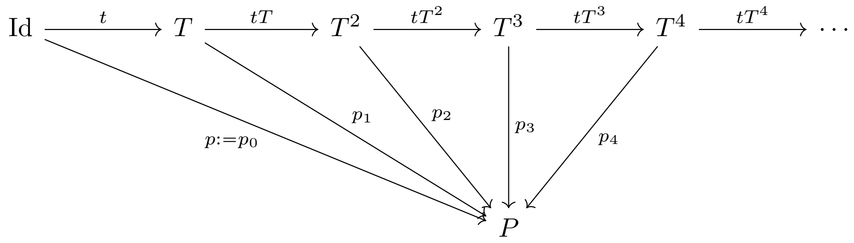
From a map $z : Z \to I$ in $\mathcal{V}$, we obtain a $\mathcal{V}$-functor $T := [Z, -] : \mathcal{V} \to \mathcal{V}$ and a $\mathcal{V}$-natural transformation $t := [z, -] : \mathrm{Id} \to T$.

Definition 3.4.1. Suppose that the smc category $\mathcal{V}$ is confined. If $z : Z \to I$ is a map in $\mathcal{V}$, let us put $T := [Z, -] : \mathcal{V} \to \mathcal{V}$ and $t := [z, -] : \mathrm{Id} \to T$. Let $P : \mathcal{V} \to \mathcal{V}$ be the colimit of the sequence of endofunctors

$$P := \mathrm{colim}\big(\mathrm{Id} \xrightarrow{t} T \xrightarrow{tT} T^2 \xrightarrow{tT^2} T^3 \xrightarrow{tT^3} T^4 \to \ldots\big).$$

and let $p : Id \to P$ be the canonical $\mathcal{V}$-natural transformation.

By construction, we have a colimit cone



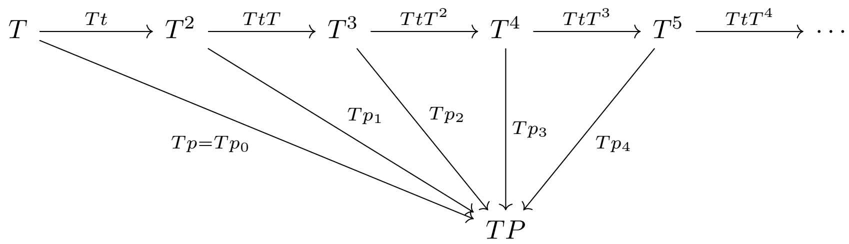
(3.4.2)

with conical maps $p_n : T^n \to P$. By construction, P is a $\mathcal{V}$-functor and p_n is a $\mathcal{V}$-natural transformation for every $n \geqslant 0$; in particular, $p := p_0 : \mathrm{Id} \to P$ is a $\mathcal{V}$-natural transformation (the enrichment of a colimit like P is constructed explicitely in Proposition A.1.1.)

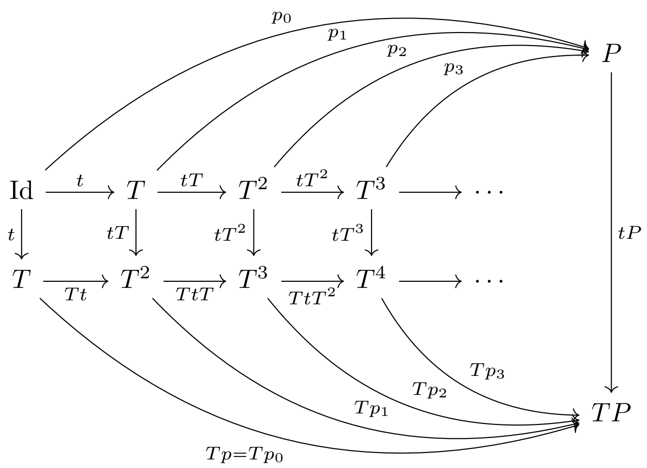
Lemma 3.4.3. *Suppose that the smc category $\mathcal{V}$ is confined. If the object $Z \in \mathcal{V}$ is compact, then the endofunctors T and P are good.*

Proof. The endofunctor T is good by Lemma 3.3.9, since Z is compact. Hence also the endo-functor T^n for every $n \geqslant 0$ by 3.3.10. The endofunctor P is also good by 3.3.10 since it is a filtered colimit of good endofunctors. □

In preparation for the proof of Theorem 3.4.32 we postcompose Diagram (3.4.2) by T.



Putting the previous diagram back to back with Diagram (3.4.2) we obtain the following diagram that commutes by naturality of the map $t : \mathrm{Id} \to T$.



(3.4.4)

Lemma 3.4.5. *If Z is compact in $\mathcal{V}$, then the natural transformation $tP : P \to TP$ is the filtered colimit of the natural transformations $tT^n : T^n \to T^{n+1}$.*

Proof. The top cone of the diagram is a colimit cone by definition of P. The bottom cone is obtained by applying the functor T to the top cone. But the functor T preserves filtered colimits by Lemma 3.3.9, since Z is compact. It follows that the bottom cone is also a colimit cone. Hence the map tP is the colimit of the sequence of maps $tT^n : T^n \to T^{n+1}$. □

Definition 3.4.6. Suppose that the smc category $\mathcal{V}$ is confined. We will say that a map $z : Z \to I$ in $\mathcal{V}$ is *perfect* if the object Z is compact and the map $P(z) : P(Z) \to P(I)$ is invertible.

We shall see in **??** that if $z : Z \to I$ is a perfect map, then the natural transformation $p : Id \to P$ defined in 3.4.1 is reflecting the category $\mathcal{V}$ into the sub-category of P-closed objects $\mathcal{V}^P$ of 3.3.6.

The following lemma was used by Weiss in his construction of orthogonal calculus [**?**].

Lemma 3.4.7 (Weiss)**.** *Let $\mathcal{V}$ a symmetric monoidal closed category, $P : \mathcal{V} \to \mathcal{V}$ a $\mathcal{V}$-functor and $p : \mathrm{Id} \to P$ a $\mathcal{V}$-natural transformation. If $f : A \to B$ be a fixed map in $\mathcal{V}$, then the following naturality square*

$$\begin{array}{ccc} [B,X] & \xrightarrow{[B,pX]} & [B,PX] \\ {\scriptstyle [f,X]}\downarrow & & \downarrow{\scriptstyle [f,PX]} \\ [A,X] & \xrightarrow[{[A,pX]}]{} & [A,PX] \end{array} \tag{3.4.8}$$

commutes for every object $X \in \mathcal{V}$. If the map $P(f) : PA \to PB$ is invertible, then the square has a diagonal filler $\delta(X) : [A,X] \to [B,PX]$ which is $\mathcal{V}$-natural in $X \in \mathcal{V}$.

Proof. The square commutes since p and P are enriched. If θ is the enrichment of P constructed in Lemma A.1, then the following naturality square commutes,

$$\begin{array}{ccc} [B,X] & \xrightarrow{\theta(B,X)} & [PB,PX] \\ =\downarrow & & \downarrow{\scriptstyle [pB,PX]} \\ [B,X] & \xrightarrow[{[B,pX]}]{} & [B,PX] \end{array}$$

Thus, $[B,PX] = [pB,PX]\theta(B,X)$, and similarly $[A,PX] = [pA,PX]\theta(A,X)$. Hence the square 3.4.8 is the composite of the squares of the following diagram :

$$\begin{array}{ccccc} [B,X] & \xrightarrow{\theta} & [PB,PX] & \xrightarrow{[pB,PX]} & [B,PX] \\ {\scriptstyle [f,X]}\downarrow & & {\scriptstyle [Pf,PX]}\downarrow & & \downarrow{\scriptstyle [f,PX]} \\ [A,X] & \xrightarrow[\theta]{} & [PA,PX] & \xrightarrow[{[pA,PX]}]{} & [A,PX] \end{array}$$

But the map $[Pf, PX]$ is invertible, since the map $P(f)$ is invertible by hypothesis. It follows that the composite square has a diagonal filler. More precisely, if $g := P(f)^{-1}$, then $[g, PX] = [Pf, PX]^{-1}$ and the map

$$\delta(X) = [pB, PX][g, PX]\theta(A, X) : [A, X] \to [PA, PX] \to [PB, PX] \to [B, PX]$$

is a diagonal filler of the square 3.4.8. Moreover, the map $\delta(X)$ is a $\mathcal{V}$-natural transformation in $X \in \mathcal{V}$, since it is a composite of $\mathcal{V}$-natural transformations. □

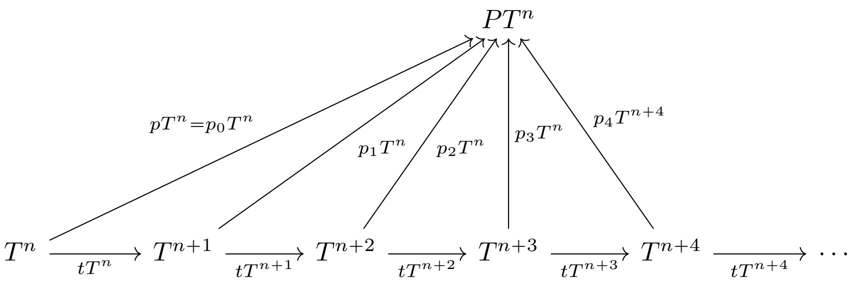
Lemma 3.4.9. *If $\mathcal{V}$ is confined and $z : Z \to I$ is perfect, then the following square of $\mathcal{V}$-natural transformations*

$$\begin{array}{ccc} \mathrm{Id} & \xrightarrow{p} & P \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle tP} \\ T & \xrightarrow{Tp} & TP \end{array}$$

has a diagonal filler $T \to P$.

Proof. Apply Corollary 3.4.7 to the map $z : Z \to I$. □

In preparation for the proof of Theorem 3.4.12 we precompose Diagram (3.4.2) with T^n to obtain a new colimit diagram:

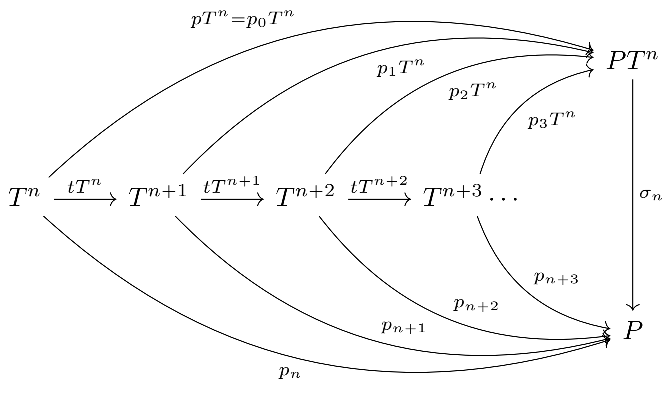


(3.4.10)

Observe that the bottom line of Diagram (3.4.10) is a cofinal sequence of the top line of Diagram (3.4.2). It follows that there is a unique isomorphism $\sigma_n : PT^n \to P$ such that

$$\sigma_n p_k T^n = p_{n+k} \tag{3.4.11}$$

for every $k \geqslant 0$. This is depicted in the following diagram:



Lemma 3.4.12. *If $\mathcal{V}$ is confined and $z : Z \to I$ is perfect, then the $\mathcal{V}$-natural transformation*

$$tP : P \to TP : \mathcal{V} \to \mathcal{V}$$

is invertible

Proof. Let us first show that $tP : P \to TP$ is invertible. By Lemma 3.4.5, this map is the filtered colimit of the Diagram (3.4.4) of maps $tT^n : T^n \to T^{n+1}$. By Lemma 3.1.6 we may prove that the map $tP : P \to TP$ is invertible by showing that the square

$$\begin{array}{ccc} T^n & \xrightarrow{p_n} & P \\ {\scriptstyle tT^n}\downarrow & & \downarrow{\scriptstyle tP} \\ T^{n+1} & \xrightarrow{Tp_n} & TP \end{array} \tag{3.4.13}$$

has a diagonal filler for every $n \geqslant 0$. If $\sigma_n : PT^n \to P$ is the isomorphism defined in Equation (3.4.11), then we have $p_n = \sigma_n(pT^n)$. Hence the Square (3.4.13) is the composite of the following two commutative squares:

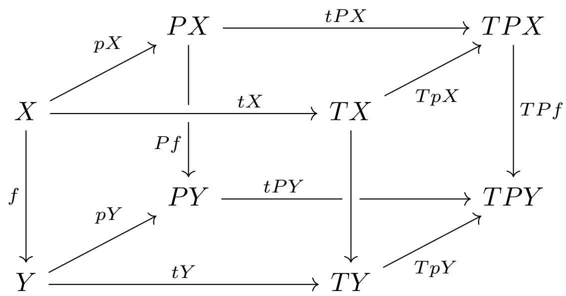
$$\begin{array}{ccccc} T^n & \xrightarrow{pT^n} & PT^n & \xrightarrow{\sigma_n} & P \\ {\scriptstyle tT^n}\downarrow & & \downarrow{\scriptstyle tPT^n} & & \downarrow{\scriptstyle tP} \\ T^{n+1} & \xrightarrow{TpT^n} & TPT^n & \xrightarrow{T\sigma_n} & TP \end{array}$$

But the left hand square of this diagram is obtained by precomposing the square in Lemma 3.4.9 with T^n. Hence the left hand square has a diagonal filler, since the square in Lemma 3.4.9 has a diagonal filler. It follows that the composite square has a diagonal filler for every $n \geqslant 0$ proving that $tP : P \to TP$ is invertible. □

Lemma 3.4.14. *Suppose that $\mathcal{V}$ is confined and $z : Z \to I$ is perfect. Then*

1. *a map $f : X \to Y$ in $\mathcal{V}$ is T-closed if and only if it is P-closed;*
2. *an object in $\mathcal{V}$ is T-closed if and only if it is P-closed;*
3. *the object PX is P-closed for every $X \in \mathcal{V}$.*

Proof. Let us prove (1). Let us show that a P-closed map $f : X \to Y$ in $\mathcal{V}$ is T-closed. Consider the following commutative cube:



The left hand face of the cube is cartesian, since f is P-closed. The right hand face is also cartesian, since the functor T preserves limits. But the horizontal maps of the back face, tPX and tPY, are isomorphism by Lemma 3.4.12. It follows that the front face is cartesian. Thus, f is T-closed.

Conversely, let us show that every T-closed map $f : X \to Y$ is P-closed. For this, we need to show that the following square is cartesian

$$\begin{array}{ccc} X & \xrightarrow{pX} & PX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Pf} \\ Y & \xrightarrow{pY} & PY \end{array} \tag{3.4.15}$$

But the square is the "infinite composition" of the squares in the following sequence:

$$\begin{array}{ccccccccc} X & \xrightarrow{tX} & TX & \xrightarrow{tTX} & T^2X & \xrightarrow{tT^2X} & T^3X & \longrightarrow & \cdots \\ {\scriptstyle f}\downarrow & & {\scriptstyle Tf}\downarrow & & {\scriptstyle T^2f}\downarrow & & {\scriptstyle T^3f}\downarrow & & \\ Y & \xrightarrow[tY]{} & TY & \xrightarrow[tTY]{} & T^2Y & \xrightarrow[tT^2Y]{} & T^3Y & \longrightarrow & \cdots \end{array} \tag{3.4.16}$$

It is enough to show that every square in the sequence is cartesian, since filtered colimits preserves finite limits in $\mathcal{V}$ by **??**. We need to show that the following square is cartesian for every $n \geqslant 0$.

$$\begin{array}{ccc} T^nX & \xrightarrow{tT^nX} & T(T^nX) \\ \downarrow{\scriptstyle T^nf} & & \downarrow{\scriptstyle T(T^nf)} \\ T^nT & \xrightarrow{tT^nY} & T(T^nY) \end{array} \tag{3.4.17}$$

The case $n = 0$ is clear, since the map f is T-closed by the hypothesis of f. The square 3.4.17 is the composite of the following two squares

$$\begin{array}{ccccc} T^nX & \xrightarrow{T^ntX} & T^n(TX) & \xrightarrow{\gamma} & T(T^nX) \\ \downarrow{\scriptstyle T^nf} & & \downarrow{\scriptstyle T^n(Tf)} & & \downarrow{\scriptstyle T(T^nf)} \\ T^nY & \xrightarrow{T^ntY} & T^n(TY) & \xrightarrow{\gamma} & T(T^nY) \end{array} \tag{3.4.18}$$

where γ is the natural isomorphism in Lemma 3.3.5 with $F := T^n$. The left hand square of diagram 3.4.18 is the image by T^n of the case $n = 0$ considered before; hence the left hand square is cartesian, since the functor T^n preserves limits. The right hand square of 3.4.18 is also cartesian, since its horizontal maps are invertible. it follows by composition that the square (3.4.17) is cartesian. We have proved that every square of the sequence 3.4.16 is cartesian; it follows that the square 3.4. We have proved that the map f is P-closed.

Let us prove (2). Note that an object X in $\mathcal{V}$ is T-closed (resp. P-closed) if and only if the map $X \to 1$ is T-closed (resp. P-closed). Thus, (1) $\Rightarrow$ (2).

Let us prove (3). The object PX is T-closed by Theorem 3.4.12. Thus, PX is P-closed by (2).

□

Theorem 3.4.19. *If $\mathcal{V}$ is confined and $z : Z \to I$ is perfect, then the natural transformations*

$$pP : P \to P^2 \text{ and } Pp : P \to P^2$$

are equal and invertible.

Proof. The proof has three parts.

Part 1: let us show that the map (=natural transformation) $pP : P \to P^2$ is invertible. If we precompose the colimit cone 3.4.2 with P, we obtain a colimit cone

$$\begin{tikzcd} P \arrow[r, "tP"] \arrow[drrr, "pP:=p_0P"'] & TP \arrow[r, "tTP"] \arrow[drr, "p_1P"] & T^2P \arrow[r, "tT^2P"] \arrow[dr, "p_2P"] & T^3P \arrow[r, "tT^3P"] \arrow[d, "p_3P"] & T^4P \arrow[r, "tT^4P"] \arrow[dl, "p_4P"] & \cdots \\ & & & P^2 & & \end{tikzcd} \tag{3.4.20}$$

The map tT^n is isomorphic to the map $T^n t$, since the map $z \otimes Z^{\otimes n}$ is isomorphic to the map $Z^{\otimes n} \otimes z$. Hence the map tT^nP is isomorphic to the map $T^n tP$. But the map $T^n tP$ is invertible, since the map tP is invertible by 3.4.12. This shows that the map $tT^nP : T^nP \to T^{n+1}P$ is invertible for every $n \geqslant 0$. It follows that the conical map pP is invertible, since the colimit of an increasing sequence of isomorphisms is isomorphic to every object of that sequence.

Part 2: let us show that the map $Pp : P \to P^2$ is invertible. The functor P preserves filtered colimits since it is good by Lemma 3.4.3. If we compose the colimit cone 3.4.2 with P, we obtain a colimit cone

$$\begin{tikzcd} P \arrow[r, "Pt"] \arrow[drrr, "Pp:=Pp_0"'] & PT \arrow[r, "PtT"] \arrow[drr, "Pp_1"] & PT^2 \arrow[r, "PtT^2"] \arrow[dr, "Pp_2"] & PT^3 \arrow[r, "PtT^3"] \arrow[d, "Pp_3"] & PT^4 \arrow[r, "PtT^4"] \arrow[dl, "Pp_4"] & \cdots \\ & & & P^2 & & \end{tikzcd} \tag{3.4.21}$$

with conical maps $Pp_n : PT^n \to P^2$. The map $tP : P \to TP$ is invertible by Theorem 3.4.12, since the map $z : Z \to I$ is perfect. It follows that the map $Pt : P \to PT$ is invertible by Lemma 3.3.5, since the functor P is good by Lemma 3.4.3. Hence the map $PtT^n : PT^n \to PT^{n+1}$ is invertible for every $n \geqslant 0$. It follows that P^2 is the colimit of an increasing sequence of isomorphisms. Hence the conical map Pp is invertible, since the colimit of an increasing sequence of isomorphisms is isomorphic to every object of that sequence.

Part 3: Let us show that $pP = Pp$. The following three squares generated by the map $p : I \to P$ in the monoidal category of endo-functors of $\mathcal{V}$ commute:

$$\begin{array}{ccc} P & \xrightarrow{pP} & P^2 \\ {\scriptstyle pP}\downarrow & & \downarrow{\scriptstyle pP^2} \\ P^2 & \xrightarrow{PpP} & P^3 \end{array} \qquad \begin{array}{ccc} P & \xrightarrow{Pp} & P^2 \\ {\scriptstyle pP}\downarrow & & \downarrow{\scriptstyle pP^2} \\ P^2 & \xrightarrow{P^2p} & P^3 \end{array} \qquad \begin{array}{ccc} P & \xrightarrow{Pp} & P^2 \\ {\scriptstyle Pp}\downarrow & & \downarrow{\scriptstyle PpP} \\ P^2 & \xrightarrow{P^2p} & P^3 \end{array} \tag{3.4.22}$$

We have $pP^2 = PpP$ by the first square (on the left hand side), since the map pP is invertible by the first part of the proof. The map $P^2p = P(Pp)$ is invertible, since the map Pp is invertible by the second part of the proof. It follows that

$$pP = (P^2p)^{-1}(pP^2)(Pp) \quad \text{and} \quad Pp = (P^2p)^{-1}(pPp)(Pp)$$

by the second and third squares respectively . Thus, $pP = pP$, since $pP^2 = PpP$. □

Recall that $\mathcal{V}^P$ denotes the full subcategory of P-closed objects of $\mathcal{V}$.

Definition 3.4.23. We say that a map $\rho : X \to Y$ in $\mathcal{V}$ is a *$\mathcal{V}$-reflection into $\mathcal{V}^P$* if Y belongs to $\mathcal{V}^P$ and the map $[\rho, W] : [Y, W] \to [X, W]$ is invertible for every $W \in \mathcal{V}^P$

Lemma 3.4.24. *Suppose that $\mathcal{V}$ is confined and that $z : Z \to I$ is perfect. Then the map $pX : X \to PX$ is a $\mathcal{V}$-reflection into $\mathcal{V}^P$ for every object $X \in \mathcal{V}$.*

.

Proof. By Corollary 3.4.14, $PX \in \mathcal{V}^P$. Let us show that the map $[pX, W] : [PX, W] \to [X, W]$ is invertible for every $W \in \mathcal{V}^P$. We shall use Weiss Lemma 3.4.7. The following square commutes by the double functoriality of the internal hom $[-, -]$ applied to the maps $pX : X \to PX$ and $pW : X \to PW$.

$$\begin{array}{ccc} [PX, W] & \xrightarrow{[PX,pW]} & [PX, PW] \\ {\scriptstyle [pX,W]}\downarrow & & \downarrow{\scriptstyle [pX,PW]} \\ [X, W] & \xrightarrow{[X,pW]} & [X, PW] \end{array} \tag{3.4.25}$$

The map $P(pX)$ is invertible by 3.4.19. Hence the square 3.4.25 has a diagonal filler d by Weiss Lemma 3.4.7.

$$\begin{array}{ccc} [PX, W] & \xrightarrow{[PX,pW]} & [PX, PW] \\ {\scriptstyle [pX,W]}\downarrow & \nearrow^{d} & \downarrow{\scriptstyle [pX,PW]} \\ [X, W] & \xrightarrow[{[X,pW]}]{} & [X, PW] \end{array} \tag{3.4.26}$$

But pW is invertible, since $W \in \mathcal{V}^P$. Hence the horizontal maps of the diagram 3.4.26 are invertible. It follows that every map in this diagram is invertible. Hence the map $[pX, W]$ is invertible. □

Definition 3.4.27. We will say that a map $f : X \to Y$ in $\mathcal{V}$ is a *P-equivalence* if the map $P(f) : PX \to PY$ is invertible.

Obviously, every isomorphism is a P-equivalence, the composite of two P-equivalences is a P-equivalence. More generally, the class of P-equivalences has the 3-for-2 property.

Lemma 3.4.28. *If $\mathcal{V}$ is confined and $z : Z \to I$ is perfect, then a map $f : X \to Y$ in $\mathcal{V}$ is a P-equivalence if and only if the map $[f, W] : [Y, W] \to [X, W]$ is invertible for every $W \in \mathcal{V}^P$.*

Proof. by definition, $f : X \to Y$ is a P-equivalence if and only if the map $P(f) : PX \to PY$ is invertible. The image of the commutative square

$$\begin{array}{ccc} X & \xrightarrow{pX} & PX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle P(f)} \\ Y & \xrightarrow{pY} & PY \end{array} \tag{3.4.29}$$

by the contravariant functor $[-, W]$ is the following commutative square.

$$\begin{array}{ccc} [X, W] & \xleftarrow{[pX,W]} & [PX, W] \\ {\scriptstyle [f,W]}\uparrow & & \uparrow{\scriptstyle [P(f),W]} \\ [Y, W] & \xleftarrow[{[pY,W]}]{} & [PY, W] \end{array} \tag{3.4.30}$$

By Yoneda, the map $P(f) \in \mathcal{V}^P$ is invertible if and only if the map $[P(f), W] \in \mathcal{V}$ is invertible for every object $W \in \mathcal{V}^P$. But the horizontal maps of the square 3.4.30 are invertible, since the maps $pX : X \to PX$ and $pY : Y \to PY$ are $\mathcal{V}$-reflecting into $\mathcal{V}^P$ by 3.4.24. Hence the map $P(f)$ is invertible if and only if the map $[f, W]$ is invertible for every object $W \in \mathcal{V}^P$. We have prove that $f : X \to Y$ is a P-equivalence if and only if the map $[f, W] : [Y, W] \to [X, W]$ is invertible for every $W \in \mathcal{V}^P$. □

Lemma 3.4.31. *Suppose that $\mathcal{V}$ is confined and that $z : Z \to I$ is perfect. The tensor product $f \otimes f'$ of two P-equivalences $f, f' \in \mathcal{V}$ is a P-equivalence.*

Proof. If $f : X \to Y$ is a P-equivalence and $K \in \mathcal{V}$, let show that the map $K \otimes f$ is a P-equivalence. The map $[f, W]$ is invertible for every object $W \in \mathcal{V}^P$ by 3.4.28. Hence the map $[K \otimes f, W] = [K, [f, W]]$ is invertible for every object $W \in \mathcal{V}^P$. This shows by 3.4.28 that the map $K \otimes f$ is a P-equivalence. In general, if $f : X \to Y$ and $f' : X' \to Y'$ are P-equivalences, then the map $f \otimes f' = (f \otimes Y')(X \otimes f')$ is a P-equivalence, since the composite of two P-equivalences is a P-equivalence. □

Theorem 3.4.32. *Suppose that the symmetric monoidal closed category $\mathcal{V}$ is confined and that the map $z : Z \to I$ is perfect. Define $T := [Z, -]$, $t := [z, -] : \mathrm{Id} \to T$, $P = \mathrm{colim}_n T^n$ and $p : Id \to P$. Let $\mathcal{V}^P$ be the subcategory P-closed objects of $\mathcal{V}$ and let $\mathsf{c}(\mathcal{V})$ be the sub-category of compact objects of $\mathcal{V}$. Then,*

1. *an object $X \in \mathcal{V}$ is T-closed if and only if it is P-closed;*
2. *$X \in \mathcal{V}^P \Rightarrow [A, X] \in \mathcal{V}^P$ for every $A \in \mathcal{V}$;*
3. *the subcategory $\mathcal{V}^P$ is $\mathcal{V}$-reflective, the reflector $P : \mathcal{V} \to \mathcal{V}^P$ is left exact, and the map $pX : X \to PX$ is $\mathcal{V}$-reflecting into $\mathcal{V}^P$ for every of $X \in \mathcal{V}$;*
4. *the category $\mathcal{V}^P$ is symmetric monoidal closed; its tensor product $\otimes_P P$ is defined by $X \otimes_P Y := P(X \otimes Y)$ for every $X, Y \in \mathcal{V}^P$ and its unit object is $P(I)$; the localization functor $P : \mathcal{V} \to \mathcal{V}^P$ is symmetric monoidal;*
5. *the smc category $(\mathcal{V}^P, \otimes_P, P(I))$ is confined and the localization functor $P : \mathcal{V} \to \mathcal{V}^P$ is confined.*
6. *every compact object of $\mathcal{V}^P$ is a retract of an object in $P(\mathsf{c}(\mathcal{V}))$*

Proof. (1) This follows from 3.4.14.

(2) Recall from 3.4.14 that an object of $\mathcal{V}$ is T-closed if and only if it is P-closed. Hence it suffices to show that if an object $X \in \mathcal{V}$ is T-closed then the object $[A, X]$ is T-closed for every $A \in \mathcal{V}$. But this was proved in 3.3.7.

(3) The first and last statements were proved in Lemma 3.4.24. The localization functor $P : \mathcal{V} \to \mathcal{V}^P$ preserves finite limits since the functor $P : \mathcal{V} \to \mathcal{V}$ is good by Lemma 3.4.3.

(4) The tensor product of two P-equivalences is a P-equivalence by Lemma 3.4.31. It follows from [**?**][Prop 4.1.7.4] that the functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ induces a functor $\otimes^P : \mathcal{V}^P \times \mathcal{V}^P \to \mathcal{V}^P$ and the following square commutes:

$$\begin{array}{ccc} \mathcal{V} \times \mathcal{V} & \xrightarrow{\quad\otimes\quad} & \mathcal{V} \\ {\scriptstyle P\times P}\downarrow & & \downarrow{\scriptstyle P} \\ \mathcal{V}^P \times \mathcal{V}^P & \xrightarrow{\quad\otimes_P\quad} & \mathcal{V}^P. \end{array}$$

We then have $P(X \otimes Y) = P(X) \otimes_P P(Y)$ for every $X, Y \in \mathcal{V}$. It follows that $P(X \otimes Y) = X \otimes_P Y$ for every $X, Y \in \mathcal{V}^P$. Hence the functor $- \otimes_P -$ is the tensor product of a symmetric monoidal structure on $\mathcal{V}^P$, with unit object $P(I)$. Let us show that the symmetric monoidal category $(\mathcal{V}^P, \otimes_P, P(I))$ is closed. If $W \in \mathcal{V}^P$, then $[X, W] \in \mathcal{V}^P$ for every $X \in \mathcal{V}$ by (2). Moreover, $[P(X \otimes Y), W] = [X \otimes Y, W]$ for every $X, Y \in \mathcal{V}$ by 3.4.24. Thus, $[X \otimes_P Y, W] \simeq [X \otimes Y, W] \simeq [Y, [X, W]]$ for every $X, Y, W \in \mathcal{V}^P$. This shows that the functor $X \otimes_P - : \mathcal{V}^P \to \mathcal{V}^P$ is left adjoint to the functor $[X, -] : \mathcal{V}^P \to \mathcal{V}^P$.

(5) Let us show that the symmetric monoidal category $(\mathcal{V}^P, \otimes_P, P(I))$ is ω-presentable. The category $\mathcal{V}^P$ is cocomplete and the localization functor $P : \mathcal{V} \to \mathcal{V}^P$ preserves all colimits by a general categorical argument. The functor $P : \mathcal{V} \to \mathcal{V}$ preserves filtered colimits by Lemma 3.4.3; it follows that the inclusion functor $i : \mathcal{V}^P \subseteq \mathcal{V}$ preserves filtered colimits. It then follows from the adjointness $P \vdash i$ that the functor $P : \mathcal{V} \to \mathcal{V}^P$ takes compact objects to compact objects (beware that a compact object in $\mathcal{V}^P$ may not be compact in $\mathcal{V}$). Hence we have $P(\mathsf{c}(\mathcal{V})) \subseteq \mathsf{c}(\mathcal{V}^P)$. Every object $X \in \mathsf{c}(\mathcal{V}^P)$ is the colimit in $\mathcal{V}$ of a diagram $F : \mathcal{J} \to \mathsf{c}(\mathcal{V})$, since the category $\mathcal{V}$ is ω-presentable. The object $X = PX$ is then the colimit in $\mathcal{V}^P$ of the diagram $PF : \mathcal{J} \to P(\mathsf{c}(\mathcal{V})) \subseteq \mathsf{c}(\mathcal{V}^P)$. This shows that the category $\mathcal{V}^P$ is ω-presentable. Let us show that the smc category $\mathcal{V}^P$ is confined. We have $I \in \mathsf{c}(\mathcal{V})$ and $\mathsf{c}(\mathcal{V}) \otimes \mathsf{c}(\mathcal{V})) \subseteq \mathsf{c}(\mathcal{V})$, since the smc $\mathcal{V}$ is confined. The object $P(I) \in \mathcal{V}^P$ is compact since $P(\mathsf{c}(\mathcal{V})) \subseteq \mathsf{c}(\mathcal{V}^P)$. Moreover, we have

$$P(\mathsf{c}(\mathcal{V})) \otimes_P P(\mathsf{c}(\mathcal{V})) = P(\mathsf{c}(\mathcal{V}) \otimes \mathsf{c}(\mathcal{V})) \subseteq P(\mathsf{c}(\mathcal{V})) \subseteq \mathsf{c}(\mathcal{V}^P)$$

by (4). It follows by 3.2.6 that the smc category $\mathcal{V}^P$ is confined. Moreover, the functor $P : \mathcal{V} \to \mathcal{V}^P$ is confined, since $P(\mathsf{c}(\mathcal{V})) \subseteq \mathsf{c}(\mathcal{V}^P)$.

(6) Let us show that every compact object of $\mathcal{V}^P$ is a retract of an object in $P(\mathsf{c}(\mathcal{V}))$. Every object $X \in \mathsf{c}(\mathcal{V}^P)$ is the colimit in $\mathcal{V}$ of a filtered diagram $F : \mathcal{J} \to \mathsf{c}(\mathcal{V})$, since the category $\mathcal{V}$ is ω-presentable. For each object $j \in \mathcal{J}$, let $\alpha_j : F(j) \to X$ be the conical map of the colimit cone. The object $X = PX$ is the colimit in $\mathcal{V}^P$ of the diagram $PF : \mathcal{J} \to P(\mathsf{c}(\mathcal{V}))$, since the functor $P : \mathcal{V} \to \mathcal{V}^P$ preserves colimits. Let $P(\alpha_j) : PF(j) \to PX = X$ be the conical map for each $j \in \mathcal{J}$. The functor $\mathrm{map}(X, -) : \mathcal{V}^P \to \mathcal{S}$ preserves directed colimits, since the object X is compact in the category $\mathcal{V}^P$. The conical maps $\mathrm{map}(X, P(\alpha_j)) : \mathrm{map}(X, PF(j)) \to \mathrm{map}(X, X)$ are collectively surjective, since the cone is a colimit cone. Hence there exists an object $j \in \mathcal{J}$ together with a map $s : X \to PF(j)$ such that $P(\alpha_j)s = 1_X$. Hence the object X is a retract of $PF(j) \in P(\mathsf{c}(\mathcal{V}))$. □

Notice that $\mathcal{V}$-functor $P : \mathcal{V} \to \mathcal{V}$ is $\mathcal{V}$-*lex* in the following sense: it preserves finite limits and compact cotensors.

Recall that a factorisaton system $(\mathcal{L}, \mathcal{R})$ in category with finite limits $\mathcal{E}$ is said to be a *modality* if the class $\mathcal{L}$ is closed under base changes. The modality $(\mathcal{L}, \mathcal{R})$ is said to be *left exact* if the class $\mathcal{L}$ satisfies 3-for-2.

Lemma 3.4.33. *Suppose that the symmetric monoidal closed category $\mathcal{V}$ is confined and that the map $z : Z \to I$ is perfect. Let $\mathcal{L} \subseteq \mathcal{V}$ be the class of P-equivalences and $\mathcal{R} \subseteq \mathcal{V}$ be the class of P-closed maps. Then the pair $(\mathcal{L}, \mathcal{R})$ is a left exact modality in $\mathcal{V}$. Moreover, if Σ is the set of maps $z \otimes A : Z \otimes A \to A$ for $A \in \mathsf{c}(\mathcal{V})$, then*

$$\Sigma^{\perp} = \mathcal{R} \quad \text{and} \quad \mathcal{L} = {}^{\perp}(\Sigma^{\perp})$$

Proof. The functor P is a left exact reflector by Theorem 3.4.32. Now the first statement is proved in [ABFJ22, Proposition 4.1.6] or [ABFJ18, Lemma 2.6.4].

Let us prove the second statement. A map $f : X \to Y$ in $\mathcal{V}$ is P-closed if an only if it is T-closed by 3.4.14. By definition, $f : X \to Y$ is T-closed if and only if the following naturality square is cartesian

$$\begin{array}{ccc} X & \xrightarrow{[z,X]} & [Z,X] \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle [Z,f]} \\ Y & \xrightarrow{[z,Y]} & [Z,Y] \end{array}$$

is cartesian. By 3.1.2, this square is cartesian if and only if the following square is cartesian for every compact object $A \in \mathcal{V}$, since the subcategory of compact objects $\mathsf{c}(\mathcal{V}) \subseteq \mathcal{V}$ is dense.

$$\begin{array}{ccc} \mathrm{map}(A,X) & \xrightarrow{\mathrm{map}(A,[z,X])} & \mathrm{map}(A,[Z,X]) \\ {\scriptstyle \mathrm{map}(A,f)}\downarrow & & \downarrow{\scriptstyle \mathrm{map}(A,[Z,f])} \\ \mathrm{map}(A,Y) & \xrightarrow{\mathrm{map}(A,[z,Y])} & \mathrm{map}(A,[Z,Y]) \end{array}$$

But the square is isomorphic to the square

$$\begin{array}{ccc} \mathrm{map}(A,X) & \xrightarrow{\mathrm{map}(z\otimes A,X)} & \mathrm{map}(Z\otimes A,X) \\ {\scriptstyle \mathrm{map}(A,f)}\downarrow & & \downarrow{\scriptstyle \mathrm{map}(Z\otimes A,f)} \\ \mathrm{map}(A,Y) & \xrightarrow{\mathrm{map}(z\otimes A,Y)} & \mathrm{map}(Z\otimes A,Y) \end{array}$$

And the latter square is cartesian if and only if the map $z \otimes A : Z \otimes A \to A$ is left orthogonal to the map $f : X \to Y$. This can be restated as $\Sigma^{\perp} = \mathcal{R}$. Now we already know by (i) that $(\mathcal{L}, \mathcal{R})$ form a factorization system. Thus, $\mathcal{L} = {}^{\perp}\mathcal{R} = {}^{\perp}(\Sigma^{\perp})$. See e.g. [**?**, Proposition 5.5.5.7]. ☐

4 Goodwillie calculus revisited

The goal in this section is to show that the methods of Chapter 4 can be applied to Goodwillie's calculus of finitary functors $\mathscr{S} \to \mathcal{S}$.

(1) In the unpointed case, the monoidal category $\mathbb{V}$ is the category of finite spaces Fin with the symmetric monoidal structure given by the join operation $(A, B) \mapsto A \star B$ and the unit object given by the empty space $\varnothing$. The functor category $\mathcal{V} := \mathscr{S}^{\mathsf{Fin}}$ is then equipped with the a symmetric monoidal closed structure $(\otimes, I)$ defined by Day convolution from the join product. If $R^A = \mathrm{map}(A, -)$ denotes a representable functor, then $R^A \otimes R^B = R^{A\star B}$ for every $A, B \in \mathsf{Fin}$, and the unit object $I = R^{\varnothing}$ is the terminal presheaf 1. The functor $R = R^1 = \mathrm{map}(1, -)$ is the forgetful functor $\mathsf{Fin} \to \mathscr{S}$. We will prove if $R^{\star(n+1)}$ is the $(n+1)$-join power of the object R in $\mathcal{V}$, then the map $R^{\star(n+1)} \to 1$ is perfect, and that the monoidal localization $P_n : \mathcal{V} \to \mathcal{V}$ generated by the endofunctor $T_n := [R^{\star(n+1)}, -]$ is the n-excisive reflector of Goodwillie. The proof has two parts: (1) the map $R^{\star(n+1)} \to 1$ is perfect; (2) a functor F in $\mathcal{V}$ is P_n-closed if and only if it is n-excisive.

4.1 The unpointed case

Our gool here is to show that the methods of Chapter 4 can be applied to Goodwillie's calculus for finitary functors $\mathscr{S} \to \mathscr{S}$. Recall that a finitary functor $F : \mathscr{S} \to \mathscr{S}$ is said to be *n-excisive* it it takes every completely cocartesian $(n+1)$-cube $\chi : \mathcal{P}(n+1) \to \mathscr{S}$ to a cartesian $(n+1)$-cube $F \circ \chi$. The category $[\mathscr{S}, \mathscr{S}]^{n-ex}$ of n-excisive finitary functors $\mathscr{S} \to \mathscr{S}$ is reflexive in the category of all finitary functors $[\mathscr{S}, \mathscr{S}]^f$ and Goodwillie constructs a reflector

$$P_n : [\mathscr{S}, \mathscr{S}]^f \to [\mathscr{S}, \mathscr{S}]^{n-ex}$$

By construction, $P_n = \mathrm{colim}_{k\geqslant 0}\, T_n^k$, where $Id \to T_k$ is a pointed endo-functor of $[\mathscr{S}, \mathscr{S}]^f$. The endo-functor T_k can be descibed by using a symmetric monoidal structure in the category $[\mathscr{S}, \mathscr{S}]^f = [\mathsf{Fin}, \mathscr{S}]$ of all functors $\mathsf{Fin} \to \mathscr{S}$, where $\mathsf{Fin} \subset \mathscr{S}$ is the category of finite spaces. The join operation $(A, B) \mapsto A \star B$ is defining a symmetric monoidal structure, with unite object empty space $\varnothing$, on the category $\mathscr{S}$, hence also on the subcategory $\mathsf{Fin} \subseteq \mathscr{S}$. Hence the the functor category $[\mathsf{Fin}, \mathscr{S}]$ is symmetric monoidal closed, with the tensor product $F \otimes G$ given by Day's convolution product with respect to the join operation

$$(F \otimes G)(K) = \int^{A\in\mathsf{Fin}} \int^{B\in\mathsf{Fin}} F(A) \times G(B) \times \mathrm{map}(A \star B, K)$$

and with unit object the representavle functor $R^{\varnothing} = \mathrm{map}(\varnothing, -) = 1$. If $[F, G]$ denote the internal hom of this monoidal structure, then $T^n = [Z_n, -]$., where Z_n is the $(n+1)$ join power of the corepresentable functor $R = \mathrm{map}(1, -)$ then It follows from this description that the category of n-excisive functors $[\mathsf{Fin}, \mathscr{S}]^{n-ex} = [\mathscr{S}, \mathscr{S}]^{n-ex}$ is symmetric monoidal closed and ω-presentable. It follows that $[\mathsf{Fin}, \mathscr{S}]^{n-ex}$ is equivalent to the category of models of an an essentially algebraic theory EX_n.

If Fin $\subset \mathscr{S}$ denotes the subcategory of finite spaces, then the restriction functor $F \mapsto F|\mathrm{Fin}$ induces an equivalence between the category of finitary functors $\mathscr{S} \to \mathscr{S}$ and the category $\mathscr{S}^{\mathrm{Fin}}$ of all functors $\mathsf{Fin} \to \mathscr{S}$.

The category $\mathscr{S}^{\mathrm{Fin}}$ is a logos and the inclusion functor $R : \mathrm{Fin} \to \mathscr{S}$ is corepresentable by the terminal space $1 \in \mathrm{Fin}$, since $\mathrm{map}(1, K) = K$ for every $K \in \mathsf{Fin}$. Notice that $R^A(K) := R(K)^A = K^A = \mathrm{map}(A, K)$ for every $A \in \mathsf{Fin}$. The evaluation functor $ev_1 : \mathscr{S}^{\mathrm{Fin}} \to \mathscr{S}$ defined by letting $ev_1(F) = F(1)$ is a morphism of logoi. The class $\mathscr{J} \subset Fun(\mathrm{Fin}, \mathcal{S})$ of maps inverted by the functor ev_1 is a logos congruence generated by the diagonal maps $\delta(K) : R \to R^K$ for $K \in \mathsf{Fin}$. and it follows that its box power $\mathscr{J}^{\square(n+1)}$ is the congruence generated by the maps $\delta(K_1)\square \cdots \square\delta(K_{n+1})$ for every $(n+1)$-tuples $(K_1, \ldots, K_{n+1})$ of object of Fin. The join operation $(A, B) \mapsto A \star B$ is defining a symmetric monoidal structure on the category $\mathscr{S}$ with unite object empty space $\varnothing$, and also on the subcategory $\mathsf{Fin} \subseteq \mathscr{S}$. Hence the the functor category $\mathcal{V} = \mathscr{S}^{\mathsf{Fin}}$ is symmetric monoidal closed, with the tensor product $F \otimes G$ given by Day's convolution product with respect to the join operation

$$(F \otimes G)(K) = \int^{A \in \mathsf{Fin}} \int^{B \in \mathsf{Fin}} F(A) \times G(B) \times \mathrm{map}(A \star B, K)$$

with unit object the representavle functor $R^{\varnothing} = \mathrm{map}(\varnothing, -) = 1$. If Z_n is the $(n+1)$ join power of the corepresentable functor $R = \mathrm{map}(1, -)$ then $T^n = [Z_n, -]$. It follows from this description

with unit object the empty space $\varnothing$; it induces symmetric monoidal structure on the category Fin. Hence the functor category $\mathcal{V} = \mathscr{S}^{\mathsf{Fin}}$ is symmetric monoidal closed, with the tensor product $F \otimes G$ given by Day's convolution product with respect to the join operation

$$(F \otimes G)(K) = \int^{A \in \mathsf{Fin}} \int^{B \in \mathsf{Fin}} F(A) \times G(B) \times \mathrm{map}(A \star B, K)$$

The inclusion functor $R : \mathrm{Fin} \to \mathscr{S}$ is representable by the terminal space $1 \in \mathrm{Fin}$, since $\mathrm{map}(1, K) = K$ for every $K \in \mathsf{Fin}$. Notice that $R^A(K) := R(K)^A = K^A = \mathrm{map}(A, K)$ for every $A \in \mathsf{Fin}$.

The set Σ of diagonal maps $\delta(K) : R \to R^K$ for $K \in \mathrm{Fin}$ is a lex generator of the congruence $\mathscr{J}$ by **??**. It follows by **??** that the set $\Sigma^{\square(n+1)}$ is a lex generator of the congruence $\mathscr{J}^{(n+1)}$. But we have $\Sigma^{\square(n+1)} \subset \mathscr{P}_n$ by 4.1.10. It follows that $\mathscr{J}^{(n+1)} \subset \mathcal{P}_n$.

The join operation $(A, B) \mapsto A \star B$ is defining a symmetric monoidal structure on the category $\mathscr{S}$, with unit object the empty space $\varnothing$; it induces symmetric monoidal structure on the category Fin. Hence the functor category $\mathcal{V} = \mathscr{S}^{\mathsf{Fin}}$ is symmetric monoidal closed, with the tensor product $F \otimes G$ given by Day's convolution product with respect to the join operation

$$(F \otimes G)(K) = \int^{A \in \mathsf{Fin}} \int^{B \in \mathsf{Fin}} F(A) \times G(B) \times \mathrm{map}(A \star B, K)$$

Then $R^A \otimes R^B = R^{A \star B}$ for every $A, B \in \mathrm{Fin}$. The unit object for the convolution product is the terminal functor $1 = R^{\varnothing}$, since $\varnothing$ is the unit object for the join product. We will denote by $[F, G]$ the internal hom between F and G. We have $[F, G](A) = Nat(F, G(A \star -)$ for every $A \in \mathrm{Fin}$. In particular, $[R^A, G] = G(A \star -)$.

Let us now suppose that $\mathscr{E}$ is a logos. The poset $[1] = \{0, 1\}$ can be viewed category. The category $\mathscr{E}^{[1]} = Fun([1], \mathscr{E})$ is the category of arrows of $\mathscr{E}$. Recall that the *box product* $u\square v$ of two maps in $\mathscr{E}$ is defined to be the cocartesian gap map

$$u\square v : (A \times D) \sqcup_{A \times C} (B \times C) \to B \times C$$

of the following square

$$\begin{array}{ccc} A \times C & \xrightarrow{u \times C} & B \times C \\ {\scriptstyle A\times v}\big\downarrow & & \big\downarrow{\scriptstyle B \times v} \\ A \times D & \xrightarrow[u \times D]{} & B \times D \end{array}$$

The pushout product is the tensor product of a symmetric monoidal closed structure on the category $\mathscr{E}^{[1]}$ (it is actually the Day tensor product with respect to the monoidal structure on the poset [1] defined by the infimum operation $\wedge : [1] \times [1] \to [1]$ with unit element $1 \in [1]$). The unit object for the box product is the map $\varnothing \to 1$. The *join* of two objects $A, B \in \mathscr{E}$ is the object $A \star B$ is defined by the following pushout square

$$\begin{array}{ccc} A \times B & \xrightarrow{p_2} & B \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle i_2} \\ A & \xrightarrow{i_1} & A \star B \end{array}$$

It follows from this definition that the map $A \star B \to 1$ is the box product of the maps $A \to 1$ and $B \to 1$. The join operation is defining a symmetric monoidal structure on the category $\mathscr{E}$ with unit object $\varnothing \in \mathscr{E}$. The monoidal structure is not closed. However,

For every $n \geqslant 0$, the n-fold join power $Z^{\star n} : \mathrm{Fin} \to \mathcal{S}$ of the functor $Z := \mathrm{map}(1, -) : \mathsf{Fin} \to \mathcal{S}$ takes a space $A \in \mathrm{Fin}$ to its n-fold join power $A^{\star n}$.

Let $\langle n \rangle = \{1, \ldots, n\}$.

Lemma 4.1.1. *If* $R := \mathrm{map}(1, -) : \mathsf{Fin} \to \mathscr{S}$, *then*

$$R^{\star n} = \operatorname*{colim}_{\varnothing \neq U \subset \langle n \rangle} R^U$$

for every $n \geqslant 0$. *Moreover, for every* $F : \mathsf{Fin} \to \mathscr{S}$

$$[R^{\star n}, F] = \lim_{\varnothing \neq U \subset \langle n \rangle} F(U \star -)$$

The proof below uses the box product of maps and the external cartesian product of cubes. We first recall these notions.

If [1] is the poset $\{0 < 1\}$ and $\mathscr{C}$ is a category, then $\mathscr{C}^{[1]} := Fun([1], \mathscr{C})$ is the category of arrows of $\mathscr{C}$.

A n-cube in the category $\mathcal{S}$ is defined to be a functor $f : [1]^n \to \mathcal{S}$ or equivalently a functor $f : \mathcal{P}(\langle n \rangle) \to \mathcal{S}$. The *external cartesian product* of a cube $f : [1]^m \to \mathcal{S}$ with a cube $g : [1]^n \to \mathcal{S}$ is the cube $f \boxtimes g : [1]^{m+n} \to \mathcal{S}$ defined by putting $(f \boxtimes g)(a, b) = f(a) \times g(b)$ for $(a, b) \in [1]^m \times [1]^n$. The *cocartesian gap map* of a cube $f : [1]^n \to \mathcal{S}$ is defined to be the map

$$cog(f) : \operatorname*{colim}_{U \subset \langle n \rangle} f(U) \to f(1^n).$$

It is easy to check that $cog(f \boxtimes g) = cog(f) \square cog(g)$. It follows from this relation that the box product $f_1 \square \cdots \square f_n$ of a sequence of maps $(f_1, \ldots, f_n)$ in $\mathcal{S}$ is the cocartesian gap map of the n-cube $f_1 \boxtimes \cdots \boxtimes f_n$. The map $A^{\star n} \to 1$ is the n-fold box power $p(A)^{\square n}$ of the map $p(A) : A \to 1$; it is thus the cocartesian gap map of the n-cube $\chi := p(A)\boxtimes\cdots\boxtimes p(A)$. By construction, $\chi(U) = A^{\complement U}$ for every subset $U \subseteq \langle n\rangle$. It follows that

$$A^{\star n} = \operatorname*{colim}_{U\subset\langle n\rangle} \chi(U) = \operatorname*{colim}_{\varnothing\neq U\subseteq\langle n\rangle} \chi(\complement U) = \operatorname*{colim}_{\varnothing\neq U\subseteq\langle n\rangle} A^U \tag{4.1.2}$$

for every $n \geqslant 0$.

Proof. of Lemma 4.1.1. The first formula of the Lemma follows from 4.1.2 since $R^{\star n}(A) = A^{\star n}$ and $A^U = R^U(A)$ for every $A \in$ Fin and every $U \subseteq \langle n\rangle$. It follows that for every $F : \mathsf{Fin} \to \mathscr{S}$ we have

$$[R^{\star n}, F] = \lim_{\varnothing\neq U\subseteq\langle n\rangle} [R^U, F] = \lim_{\varnothing\neq U\subseteq\langle n\rangle} F(U \star -)$$

since $[R^U, F] = F(U \star -)$. □

Let us denote by T_n the endofunctor of $\mathscr{S}^{\mathsf{Fin}}$ defined by letting $T_n(F) := [R^{\star(n+1)}, F]$. From the canonical map $c_{n+1} : R^{\star(n+1)} \to 1$ we obtain a map $t_n(F) = [c_{n+1}, F] : F \to T_n(F)$.

Let P_n be the endofunctor of $\mathscr{S}^{\mathsf{Fin}}$ defined by

$$P_n := \operatorname{colim}\big(\mathrm{Id} \xrightarrow{t_n} T_n \xrightarrow{t_nT_n} T_n^2 \xrightarrow{t_nT_n^2} T_n^3 \xrightarrow{t_nT_n^3} T_n^4 \to \dots\big)$$

and let $p_n : \mathrm{Id} \to P_n$ be the canonical map. We will prove in 4.1.8 that the map $R^{\star(n+1)} \to 1$ in $\mathcal{S}^{\mathrm{Fin}}$ is perfect. By definition 3.4.6, we must verify two conditions: (1) the functor $R^{\star(n+1)}$ is compact in $\mathcal{S}^{\mathrm{Fin}}$; (2) $P_n(R^{\star(n+1)}) = 1$. But condition (1) holds since $R^{\star(n+1)}$ is a finite colimit of representables R^U by Lemma 4.1.1. It remains to show that $P_n(R^{\star(n+1)}) = 1$. Following Goodwillie we will prove this by estimating the connectivity of the space $R^{\star(n+1)}(K)$ for a finite space K and by applying Goodwillie's Proposition 4.1.7 below. We first prove two elementary statements on cubical diagrams taken from [**?**].

Warning: We are saying that a map $f : A \to B$ is *n-connected* if all its homotopy fibers are n-connected. This is the notion of connectivity for maps used in the ABFJ papers. Even though it differs by 1 from the classical notion of connectivity that Goodwillie uses, the statements in this section are formally the same.

Recall that the *cartesian gap map* of a cube $\mathcal{X} : [1]^n \to \mathcal{S}$ is defined to be the map

$$\mathcal{X}(\varnothing) \to \lim_{\varnothing\neq U\subseteq\langle n\rangle} \mathcal{X}(U).$$

Definition 4.1.3. We say that a cube $\mathcal{X} : [1]^n \to \mathcal{S}$ is *k-cartesian*, if its cartesian gap map is k-connected.

Notice that a morphism of n-cube $\alpha : \mathcal{X} \to \mathcal{Y}$ in a category $\mathscr{E}$ is defining a $(n+1)$-cube $[\alpha] : [1]^{n+1} \to \mathscr{E}$ since $Fun([1], Fun([1]^n, \mathscr{E})) = Fun([1]^{n+1}, \mathscr{E})$.

Lemma 4.1.4. *[**?**, Prop 1.6] Let $\alpha : \mathcal{X} \to \mathcal{Y}$ be morphism of n-cubes in $\mathcal{S}$.*

(i) *If the $(n+1)$-cube $[\alpha]$ is k-cartesian and $\mathcal{Y}$ is k-cartesian, then $\mathcal{X}$ is k-cartesian.*

(ii) *If $\mathcal{X}$ is k-cartesian and $\mathcal{Y}$ is $(k+1)$-cartesian, then $[\alpha]$ is k-cartesian.*

Proof. We only give a sketch. The cartesian gap map of $[\alpha]$ is the cartesian gap map of a square whose vertical sides are the cartesian gap maps of $\mathcal{X}$ and $\mathcal{Y}$. Then use the following two facts: the composite of two k-connected maps is k-connected; if the composite gf of two maps is k-connected and g is $(k+1)$-connected, then f is k-connected. ☐

The next lemma is a simplified version of [**?**, Thm 1.20] (in the case $T = \langle 1 \rangle$).

Lemma 4.1.5. *Let $\alpha : \mathcal{X} \to \mathcal{Y}$ be a morphism of n-cubes in $\mathscr{S}$. Suppose that the $(n+1)$-cube $[\alpha]$ is k-cartesian and that the map $\alpha(U)$ is $(k+|U|-1)$-connected for every non-empty subset $U \subseteq \langle n \rangle$. Then the map $\alpha(\varnothing)$ is k-connected.*

Proof. The case $n = 0$ is easy. The rest follows from Lemma 4.1.4 by induction on n. ☐

Definition 4.1.6. [**?**, Def 1.2] A map $\alpha : F \to G$ in $\mathcal{S}^{\mathrm{Fin}}$ is said to satisfy condition $O_n(c, \kappa)$ for some $c \in \mathbb{Z}$ and $\kappa \geqslant -2$ if the connectivity of the map $\alpha(K) : F(K) \to G(K)$ is $\geqslant (n+1)k - c$ for every finite space K of connectivity $k \geqslant \kappa$.

Note: the differing notions of connectivity mentioned above are absorbed into the constant. Notice also that the condition $O_n(c, \kappa)$ implies the condition $O_n(c', \kappa')$ for every $c' \geqslant c$ and $\kappa' \geqslant \kappa$. If a map $\alpha : F \to G$ satisfies condition $O_n(c, \kappa)$ for all $c \leqslant C$ for some constant $C \in \mathbb{Z}$, then α is ∞-connected, and hence invertible by Whitehead theorem.

Lemma 4.1.7. *[**?**, Prop 1.6] If a map $\alpha : F \to G$ in $Fun(\mathrm{Fin}, \mathcal{S})$ satisfies condition $O_n(c, \kappa)$ for some c, then the induced map $P_n\alpha : P_nF \to P_nG$ is an isomorphism.*

We reproduce Goodwillie's proof.

Proof. Suppose α satisfies $O_n(c, \kappa)$. Then the map $\alpha(K) : F(K) \to G(K)$ is $\big((n+1)k-c\big)$-connected whenever $K \in \mathrm{Fin}$ has connectivity $k \geqslant \kappa$. We will prove that the map $T_n(\alpha)$ is $\big((n+1)k-c+1\big)$-connected whenever K has connectivity $k \geqslant \kappa$. By **??**, the connectivity of $K \star U$ is then at least $k+1$ for every nonempty set U. Thus the connectivity of the map $F(K \star U) \to G(K \star U)$ is at least $(n+1)(k+1) - c$ for every nonempty $U \subseteq \langle n+1 \rangle$. But

$$(n+1)(k+1) - c \geqslant (n+1)k - c + |U| = \big((n+1)k - c + 1\big) + |U| - 1$$

since $|U| \leqslant n+1$. Consider the diagram $D : \mathcal{P}_0(\langle n+1 \rangle) \to \mathcal{S}$ defined by putting $D(U) := K \star U$ for every non-empty subset $U \subseteq \langle n+1 \rangle$. By Lemma 4.1.1, we have $T_n(F)(K) = \varprojlim F \circ D$. Hence the diagram $F \circ D : \mathcal{P}_0(\langle n+1 \rangle) \to \mathcal{S}$ can be uniquely extended as cartesian $(n+1)$-cube $F_C : \mathcal{P}(\langle n+1 \rangle) \to \mathcal{S}$ by putting $F_C(\varnothing) := T_n(F)(K)$. From the map $\alpha : F \to G$ we obtain a morphism of $(n+1)$-cube $\alpha_C : F_C \to G_C$. By construction, $\alpha_C(U) = \alpha(K \star U) : F(K \star U) \to G(K \star U)$ for every non-empty subset $U \subseteq \langle n+1 \rangle$ and $\alpha_C(\varnothing) = T_n(\alpha(K)) : T_nF(K) \to T_nG(K)$. The $(n+2)$-cube $[\alpha_C]$ is cartesian, since the cubes F_C and G_C are cartesian by construction. We can then apply Lemma 4.1.5 to $\alpha_C : F_C \to G_C$ since the maps $\alpha_C(U)$ for $U \neq \varnothing$ are sufficiently connected, as estimated above. We conclude that the map $(T_n\alpha)(K) = \alpha_C(\varnothing) : T_nF(K) \to T_nG(K)$ is $\big((n+1)k - c + 1\big)$-connected. Thus, the map $T_n\alpha : T_nF \to T_nG$ satisfies condition $O_n(c-1, \kappa)$. Inductively on $\ell \geqslant 0$, the map $T_n^{\ell}\alpha(K)$ is $((n+1)k - c + \ell)$-connected for every space K of connectivity $k \geqslant \kappa$. Taking the colimit for increasing ℓ, the map $P_n\alpha(K)$ is infinitely connected and thus an equivalence. ☐

Lemma 4.1.8. *The map $R^{\star(n+1)} \to 1$ in $Fun(\mathrm{Fin}, \mathcal{S})$ is perfect.*

Proof. Let us show that the map $c_{n+1} : R^{\star(n+1)} \to 1$ satisfies condition $O_n(c, 0)$ for some $c \in \mathbb{Z}$. By definition, $c_{n+1}(K)$ is the map $K^{\star(n+1)} \to 1$ for any finite space K. But the space $K^{\star(n+1)}$ is $((n+1)k + 2n)$-connected if K is k-connected by **??**. Hence the map c_{n+1} satisfies condition $O_n(-2n, 0)$. It then follows from Proposition 4.1.7 that the map $P_n(c_{n+1})$ is invertible. □

Theorem 4.1.9. *(Goodwillie) The functor P_n defined above is a left exact reflector onto the subcategory of T_n-closed objects of $Fun(\mathrm{Fin}, \mathcal{S})$.*

Proof. This follows from 4.1.8 and 3.4.32. □

We will next prove in 4.1.12 after Goodwillie [] that a functor $F \in Fun(\mathrm{Fin}, \mathcal{S})$ is T_n-closed if and only if it is n-excisive. Our proof is somewhat different from the original proof.

For every $K \in \mathrm{Fin}$, the diagonal map $\delta(K) : R \to R^K$ is the image of the map $K \to 1$ by the Yoneda functor $R^{(-)} : \mathrm{Fin}^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{Fin}, \mathcal{S})$.

Lemma 4.1.10. *The map $\delta(K_1)\square \cdots \square\delta(K_{n+1})$ is a P_n-equivalence for every $(n+1)$-tuple of finite spaces $(K_1, \ldots, K_{n+1})$.*

Proof. We estimate the connectivity of the map $\alpha := \delta(K_1)\square \cdots \square\delta(K_{n+1})$ evaluated on some fixed ℓ-connected space $L \in \mathrm{Fin}$. Let us suppose that K is of dimension $\leqslant k$ (which means that K can be represented by a CW complex of dimension $\leqslant k$). Then the connectivity of the diagonal map $\delta(K)(L) : L \to \mathrm{map}(K, L)$ is $\geqslant \ell - k - 1$ by a classical result **??**. It then follows from **??** that if K_i is of dimension $\leqslant k$ for every i, then the connectivity of the map $\delta(K_1)(L)\square \cdots \square\delta(K_{n+1})(L)$ is $\geqslant (n+1)(\ell - k + 1) - 2 = (n+1)\ell - c$ with $c := (n+1)(k-1) + 2$. This shows by Definition 4.1.6 that the map $\delta(K_1)\square \cdots \square\delta(K_{n+1})$ satisfies condition $O_n(c, -2)$. It then follows from 4.1.7 that the map $P_n(\alpha)$ is invertible. □

Lemma 4.1.11. *Let c_n be the map $R^{\star n} \to 1$. For every $K \in \mathrm{Fin}$, the map*

$$R^K \otimes c_n : R^K \otimes R^{\star n} \to R^K$$

is the fiberwise n-fold join power of the map $\delta(K) : R \to R^K$.

Proof. The functor $A \star (-) : \mathrm{Fin} \to \mathrm{Fin}$ preserves pushouts for every $A \in \mathrm{Fin}$, since it preserves contractible colimits by **??**. Hence the monoidal category $(\mathrm{Fin}, \star, \varnothing)$ satisfies condition (G) A.2.2. It follows by A.2.3 that the dilation functor

$$R^K \otimes (-) : \mathcal{V}/1 \to \mathcal{V}/R^K$$

is a morphism of logoi. But $R^{\star(n)}$ is the n-fold join power of R. It follows that the object of $\mathcal{V}/R^K$ defined by the map $R^K \otimes c_n$ is the n-fold join power of the object defined by the map $R^K \otimes c$. This proves the result, since $R^K \otimes c$ is the diagonal map $\delta(K) : R \to R^K$. □

Theorem 4.1.12. *(Goodwillie) A functor $F : \mathrm{Fin} \to \mathcal{S}$ is T_n-closed if and only if it is n-excisive.*

Proof. Let $\mathscr{P}_n$ be the class of maps in $Fun(\mathrm{Fin}, \mathcal{S})$ inverted by the reflector P_n and let $\mathscr{J} \subset Fun(\mathrm{Fin}, \mathcal{S})$ be the class of maps inverted by the evaluation functor $F \mapsto F(1)$. Let us show that $\mathscr{P}_n \subset \mathscr{J}^{(n+1)}$. By 3.4.33, it suffices to show that the map $R^K \otimes c_{n+1} : R^K \otimes R^{\star(n+1)} \to R^K$ belongs to $\mathscr{J}^{(n+1)}$ for every $K \in \mathrm{Fin}$. But the map $R^K \otimes c_{n+1}$ is the fiberwise $(n+1)$-fold join power of the map $\delta(K) : R \to R^K$ by 4.1.11. It follows that the map $R^K \otimes c_{n+1}$ belongs to $\mathscr{J}^{(n+1)}$, since the map $\delta(K) : R \to R^K$ belongs to $\mathscr{J}$. Thus, $\mathscr{P}_n \subset \mathscr{J}^{(n+1)}$. Conversely, let us show that $\mathscr{J}^{(n+1)} \subset \mathscr{P}_n$. The set Σ of diagonal maps $\delta(K) : R \to R^K$ for $K \in \mathrm{Fin}$ is a lex generator of the congruence $\mathscr{J}$ by **??**. It follows by **??** that the set $\Sigma^{\square(n+1)}$ is a lex generator of the congruence $\mathscr{J}^{(n+1)}$. But we have $\Sigma^{\square(n+1)} \subset \mathscr{P}_n$ by 4.1.10. It follows that $\mathscr{J}^{(n+1)} \subset \mathcal{P}_n$. □

5 Orthogonal calculus revisited

Orthogonal calculus was devised by Weiss [**?**, **?**]. Weiss states that he was inspired by Goodwillie's work. The motivation is similar: with Goodwillie calculus one attempts to extrapolate the value of a functor on a particular space by its values on highly connected spaces; in orthogonal calculus one tries to extrapolate the value of a functor on a particular finite dimensional vector space by its values on vector spaces with much higher dimension. However, in Goodwillie's calculus of homotopy functors one uses nice categorical properties of the source category: the existence of finite colimits and of a terminal object. The source category in orthogonal calculus is the category $\mathbb{W}$ of finite dimensional Euclidean vector spaces, a category without finite colimits (except in trivial cases) and without a terminal object. The question arose whether there exists a common frame work. We provide one by proving that the orthogonal tower is a completion tower in our sense.

In Section 5.2 we prove that the category $\mathbb{W}$ is filtered and formulate Theorem 5.2.2 stating that Weiss' orthogonal calculus is a special case of our completion tower. In 5.3 we provide generating sets of maps for the stages of the tower. Using the Ganea construction in the category $\mathscr{S}^{\mathbb{W}}$ a sequence of augmented objects is exhibited in Section 5.4. Then, in Section 5.5, we apply the machinery from Section 3 to the category $\mathbb{W}$ and obtain in 5.6 Weiss' formula for n-th stage reflector. In Section 5.7 we follow Weiss in proving that P_n is the reflector onto n-polynomial functors and in Section 5.8 we give the proof of the main theorem 5.2.2 stating that the orthogonal tower is a completion tower. The key facts that link our abstract setup with the concrete combinatorics of $\mathbb{W}$ are Weiss' Propositions 5.4.2 and 5.7.1.

5.1 Summary

Let $\mathbb{W}$ be the category of finite dimensional real euclidian vector spaces and isometric embeddings. Its objects are finite dimensional $\mathbb{R}$-vector spaces equipped with a positive definite non-degenerate inner product. The space of maps $\mathbb{R}^m \to \mathbb{R}^n$ in $\mathbb{W}$ is the Stiefel manifold $St(m, n)$ of unitary orthogonal m-frames in $\mathbb{R}^n$. In particular, $St(n, n)$ is the orthogonal $O(n)$. In general, we shall denote by $St(U, V)$ the space of isometric embedding $U \to V$. The category $\mathbb{W}$ is enriched over the category of topological spaces, and it can be viewed as an ∞-category by the Bergner-Dwyer-Kan equivalence []. For us, $\mathbb{W}$ is said to be a category, since in this paper, ∞-categories are gnerally called categories Orthogonal calculus as devised by Weiss is concerned with the functor category $[\mathbb{W}, \mathcal{S}]$.

The orthogonal sum of two finite dimensional real euclidian vector spaces U and V is a finite dimensional real euclidian vector space $U \oplus V$. The orthogonal sum $(U, V) \mapsto U \oplus V$ can be extended to maps and it defines a symmetric monoidal structure on the category $\mathbb{W}$, with the null vector space 0 as the unit object. The category $\mathcal{V} = \mathscr{S}^{\mathbb{W}}$ is then equipped a symmetric monoidal closed structure defined by Day convolution.

We shall denote by $\mathrm{St}(V)$ the corepresentable functor $\mathrm{St}(V, -) = \mathrm{map}_{\mathbb{W}}(V, -)$ The canonical inclusion $i_k : \mathbb{R}^k \to \mathbb{R}^{k+1}$ induces natural transformations

$$j_k := \mathrm{St}(i_k) : \mathrm{St}(\mathbb{R}^{k+1}) \to \mathrm{St}(\mathbb{R}^k)$$

between representable functors in $\mathscr{S}^{\mathbb{W}}$. The functor $Z := \mathrm{St}(\mathbb{R}) =: \mathbb{W} \to \mathscr{S}$ is the *unit sphere* functor.

Let $Z^{\star n}$ be the n-fold join power of Z in $\mathcal{V}$ and let $z_n : Z^{\star n} \to 1$.

Lemma 5.1.1. *(Weiss, Prop. 5.4)*

$$Z^{\star n} = \operatorname*{colim}_{0 \neq U \subseteq \mathbb{R}^n} \mathrm{St}(U, -) \quad \textit{and} \quad [Z^{\star n}, F] = \lim_{0 \neq U \subseteq \mathbb{R}^n} F(U \oplus -)$$

for every F in $\mathcal{V}$.

Let us put $T_n = [Z^{\star(n+1)}, -]$ and $t_n = [z_{n+1}, -] : \mathrm{Id} \to T_n$.

Definition 5.1.2 (Weiss Orth, Def(5.1))**.** A functor $F : \mathbb{W} \to \mathscr{S}$ is said to be *polynomial* of degree $\leqslant n$ if the map $F \to T_n(F)$ is invertible.

Let P_n be the endofunctor of $\mathscr{S}^{\mathbb{W}}$ defined by letting

$$P_n := \mathrm{colim}\big(\mathrm{Id} \xrightarrow{t_n} T_n \xrightarrow{t_n T_n} T_n^2 \xrightarrow{t_n T_n^2} T_n^3 \xrightarrow{t_n T_n^3} T_n^4 \to \ldots\big)$$

and let $p_n : \mathrm{Id} \to P_n$ be the canonical map.

Theorem 5.1.3. (Weiss) *The functor P_n defined above is a left exact reflector onto the subcategory of T_n-closed objects of $\mathscr{S}^{\mathbb{W}}$.*

By 3.4.32, the theorem is a consequence of the following lemma:

Lemma 5.1.4. *The map $z_{n+1} : Z^{\star(n+1)} \to 1$ in $\mathscr{S}^{\mathbb{W}}$ is perfect.*

By definition 3.4.6, it suffices to verify two conditions: (1) the functor $Z^{\star(n+1)} : \mathbb{W} \to \mathscr{S}$ is compact; (2) $P_n(Z^{\star(n+1)}) = 1$. But condition (1) holds since $Z^{\star(n+1)}$ is a finite colimit of representables $\mathrm{St}(U)$ by Lemma 5.1.1. It remains to show that $P_n(Z^{\star(n+1)}) = 1$. Following Weiss, we will prove this by estimating the connectivity of the space $Z^{\star(n+1)}(W)$ for $W \in \mathbb{W}$ and by applying

Lemma 5.1.5. *Let $\alpha : F \to G$ be a morphism in $\mathscr{S}^{\mathbb{W}}$ the connectivity of the map $\alpha(W) : F(W) \to G(W)$ is $\geqslant (n+1)\dim(W) - c$ for all $W \in \mathbb{W}$ of dimension $\geqslant \kappa$. Then $P_n(\alpha) : P_n F \to P_n$ is invertible.*

Proof. Under the assumptions on α, Weiss shows in [?, e.3 Lemma] that the connectivity of the map $T_n(\alpha)$ is $\geqslant (n+1)\dim(W)-c+1$ for all $W \in \mathbb{W}$ of dimension $\geqslant \kappa-1$. It follows by induction on $\ell \geqslant 0$ that the connectivity of the map $T^\ell\alpha(W)$ is $\geqslant (n+1)\dim(W)-c+\ell$ for all $W \in \mathbb{W}$ of dimension $\geqslant \kappa - l$. Hence the connectivity of the map $T^\ell(\alpha)(W)$ tends to infinity with ℓ for all $W \in \mathbb{W}$. □

Proof. of 5.1.4. If W is of dimension m, then $Z(W)$ is a sphere S^{m-1}. Therefore, $Z(W)^{\star(n+1)}$ is a sphere $S^{(n+1)m-1}$ and its connectivity is $(n+1)m-2 = (n+1)\dim(W)-2$ for all $W \in \mathbb{W}$. Hence the map $z_{n+1} : Z^{\star(n+1)} \to 1$ satisfies the hypothesis of Proposition 5.1.5 with $c = 2$ and $\kappa = 0$. □

Let $i(U)$ be the inclusion $U \to U\oplus\mathbb{R}$ and let us put $j(U) := \mathrm{St}(i(U)) : \mathrm{St}(U\oplus\mathbb{R}) \to \mathrm{St}(U)$. By construction $j(U) = \mathrm{St}(U) \otimes z : \mathrm{St}(U) \otimes Z \to \mathrm{St}(U)$. If $\dim(U) < \dim(V)$, then the map $j(U)(V) : \mathrm{map}(U \oplus \mathbb{R}, V) \to \mathrm{map}(U, V)$ is a bundle of spheres of dimension $\dim(V) - \dim(U) - 1$.

Lemma 5.1.6. *Let z_{n+1} be the map $Z^{\star(n+1)} \to 1$. For every $U \in \mathbb{W}$, the map*

$$R^U \otimes z_{n+1} : \mathrm{St}(U) \otimes Z^{\star(n+1)} \to \mathrm{St}(U)$$

is the fiberwise $(n+1)$-fold join power of the map $j(U) = \mathrm{St}(U)\otimes z : \mathrm{St}(U)\otimes Z \to \mathrm{St}(U)$.

Proof. It is easy to see that the co-dilation functor $U\oplus(-) : \mathbb{W} \to U\backslash\mathbb{W}$ is an equivalence of categories. It follows that the dilation functor

$$\mathrm{St}(U) \otimes (-) : \mathcal{V}/1 \to \mathcal{V}/\mathrm{St}(U)$$

is an equivalence of categories. But the map $z_{n+1} : Z^{\star(n+1)} \to 1$ is the $(n+1)$-fold join power of the map $z : Z \to 1$. Hence the map $\mathrm{St}(U) \otimes z_{n+1}$ is the fibrewise $(n+1)$-fold join power of the map $j(U) := \mathrm{St}(U) \otimes z : \mathrm{St}(U) \otimes Z \to \mathrm{St}(U)$. □

Lemma 5.1.7. *The map $j(U_1)\square\cdots\square j(U_{n+1})$ is a P_n-equivalence for every $(n+1)$-tuple of euclidian vector spaces $(U_1, \ldots, U_{n+1}) \in \mathbb{W}^{n+1}$.*

Proof. If $\dim(U) < \dim(V)$, then the map $j(U)(V) : \mathrm{St}(U\oplus\mathbb{R}, V) \to \mathrm{St}(U, V)$ is a bundle of spheres of dimension $\dim(V)-\dim(U)-1$. Hence the connectivity of the map $j(U)(V)$ is $\geqslant \dim(V) - \dim(U) - 2$. Let $\kappa = \max_i \dim(U_i)$. By Lemma **??** the connectivity of the map $j(U_1)(V)\square\ldots\square j(U_{n+1})(V)$ is at least

$$2n + \sum_{i=1}^{n+1} \dim(V) - \dim(U_i) - 2 \geqslant (n+1)\dim(V) - (n+1)\kappa - 2$$

Hence the map $j(U_1)\square\cdots\square j(U_{n+1})$ satisfies the hypothesis of Lemma 5.1.5 with constants $\kappa = 1 + \max_i k_i$ and $c = (n+1)\kappa + 2$. It is thus a P_n-equivalence. □

Theorem 5.1.8. *(Weiss) A functor $F : \mathbb{W} \to \mathcal{S}$ is T_n-closed (= is a polynomial of degree $\leqslant n$) if and only if it is n-excisive.*

Proof. Let $\mathscr{P}_n$ be the class of maps in $\mathscr{S}^{\mathbb{W}}$ inverted by the reflector P_n and let $\mathscr{J} \subset \mathscr{S}^{\mathbb{W}}$ be the class of maps inverted by the functor $F \mapsto F(\infty)$. Let us show that $\mathscr{P}_n \subset \mathscr{J}^{(n+1)}$. By 3.4.33, it suffices to show that the map $R^U \otimes z_{n+1} : R^U \otimes Z^{\star(n+1)} \to R^U$ belongs to $\mathscr{J}^{(n+1)}$ for every $U \in \mathbb{W}$. But the map $R^U \otimes z_{n+1}$ is the fiberwise $(n+1)$-fold join power of the map $R^{U \oplus \mathbb{R}} \to R^U$ by 5.1.6. It follows that the map $R^U \otimes z_{n+1}$ belongs to $\mathscr{J}^{(n+1)}$, since the map $R^{U \oplus \mathbb{R}} \to R^U$ belongs to $\mathscr{J}$. This shows that $\mathscr{P}_n \subset \mathscr{J}^{(n+1)}$. Conversely, let us show that $\mathscr{J}^{(n+1)} \subset \mathscr{P}_n$. The set Σ of maps $R^{U \oplus \mathbb{R}} \to R^U$ for $U \in \mathbb{W}$ is a lex generator of the congruence $\mathscr{J}$ by **??**. It follows by **??** that the set $\Sigma^{\square(n+1)}$ is a lex generator of the congruence $\mathscr{J}^{(n+1)}$. But we have $\Sigma^{\square(n+1)} \subset \mathscr{P}_n$ by 5.1.7. It follows that $\mathscr{J}^{(n+1)} \subset \mathcal{P}_n$. □

5.2 The category $\mathbb{W}$

Orthogonal calculus as devised by Weiss is concerned with functors from the category $\mathbb{W}$ to spaces. Here $\mathbb{W}$ is the category of finite dimensional Euclidean vector spaces. Its objects are finite dimensional $\mathbb{R}$-vector spaces equipped with a positive definite non-degenerate inner product. The morphisms are given by Stiefel manifolds, i.e. spaces of linear maps preserving the inner product.

The category $\mathbb{W}$ contains the vector spaces $\mathbb{R}^k, k \geqslant 0$. These objects together with the canonical inclusions

$$i_k : \mathbb{R}^k \to \mathbb{R}^{k+1} \ , \ i_k(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 0)$$

form a (non-full) subcategory of $\mathbb{W}$ that is isomorphic to the 1-category $\mathbb{N}$ from Example **??**. We denote the inclusion functor by $r : \mathbb{N} \to \mathbb{W}$.

In the context of orthogonal calculus we denote by

$$\mathrm{St} : \mathbb{W}^{\mathrm{op}} \to \mathscr{S}^{\mathbb{W}}, V \mapsto \mathrm{St}(V, -) = \mathrm{map}_{\mathbb{W}}(V, -)$$

the Yoneda embedding (different from our previous notation R^V). The canonical inclusion $i_k : \mathbb{R}^k \to \mathbb{R}^{k+1}$ induces natural transformations

$$j_k := \mathrm{St}(i_k) : \mathrm{St}(\mathbb{R}^{k+1}, -) \to \mathrm{St}(\mathbb{R}^k, -)$$

between representable functors in $\mathscr{S}^{\mathbb{W}}$.

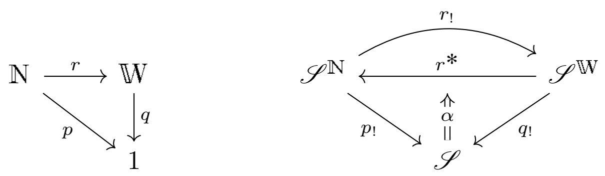
Lemma 5.2.1. *The category $\mathbb{W}$ is filtered.*

Note that the $(\infty, 1)$-category $\mathbb{W}$ is filtered in the $(\infty, 1)$-categorical sense, but the associated 1-category is very far from being filtered in the 1-categorical sense.

Proof. The category $\mathbb{N}$ from Example **??** is filtered. It therefore suffices to prove that the functor $r^* : \mathscr{S}^{\mathbb{W}} \to \mathscr{S}^{\mathbb{N}}$ given by precomposition with the inclusion $r : \mathbb{N} \to \mathbb{W}$ induces isomorphisms on the respective colimits.

Let $p : \mathbb{N} \to 1$ and $q : \mathbb{W} \to 1$ denote the respective functors to the terminal category. If we denote by p^* and q^* the respective precomosition functors, then r^*, p^* and q^* have left adjoints $r_!, p_!$ and $q_!$ given by left Kan extension. Since $q \circ r = p$, we have: $q_! \circ r_! = p_!$.

Adjoint to this isomorphism there is a natural transformation $q_! \xrightarrow{\alpha} p_! \circ r^*$ (written below as a 2-cell).



Since $p_! = \operatorname{colim}_{\mathbb{N}}$ and $q_! = \operatorname{colim}_{\mathbb{W}}$, our aim is to prove that the natural transformation

$$\alpha : q_! = \operatorname*{colim}_{\mathbb{W}} \to \operatorname*{colim}_{\mathcal{N}} r^*(-) = p_! \circ r^*$$

is in fact an isomorphism. All three functors $p_!, q_!$ and r^* are cocontinuous. So it is enough to check on representable functors $\mathrm{St}(V, -)$ for all V in $\mathbb{W}$. On the left the space $q_! \mathrm{St}(V, -) = \operatorname{colim}_{V \in \mathbb{W}} \mathrm{St}(V, -)$ is contractible since it is the nerve of the category of elements of $\mathrm{St}(V, -)$ (Grothendieck construction) and this category has an initial object. On the right side it is well known that the inifinite dimensional Stiefel manifold

$$p_! r^* \mathrm{St}(V, -) = \operatorname*{colim}_{k \in \mathbb{N}} \big(\ldots \to \mathrm{St}(V, \mathbb{R}^k) \xrightarrow{(i_k)_*} \mathrm{St}(V, \mathbb{R}^{k+1}) \to \ldots \big) = \mathrm{St}(V, \mathbb{R}^\infty) = *$$

is contractible as well. Thus, α is an isomorphism on representable functors. Hence it is an isomorphism overall and the lemma is proved. □

As a consequence of Proposition 5.2.1 and the results from Section **??** the logos $\mathscr{S}^{\mathbb{W}}$ has a point at ∞:

$$\operatorname*{colim}_{\mathbb{W}} : \mathscr{S}^{\mathbb{W}} \to \mathscr{S}, \; F \mapsto \operatorname*{colim}_{\mathbb{W}} F.$$

This left exact reflector is the 0-th level of Weiss' orthogonal tower and he denotes it by $T_0 = F(\mathbb{R}^\infty)$. We will denote it by P_0.

We choose this notation because we want to make the point that we can treat the Weiss tower and the Goodwillie tower on equal footing. In Weiss' notation the reflector P_0 and all the other reflectors P_n in the tower are denoted by T_n, and Weiss' functor τ_n is denoted T_n here.

Theorem 5.2.2. *The completion tower associated to the point at ∞ of $\mathbb{W}$, denoted here by $P_0 = \operatorname{colim}_{\mathbb{W}}$, is Weiss' orthogonal tower.*

The proof of this theorem will be given in Subsection 5.8.

5.3 A generating set of maps for the point at ∞

The canonical inclusion $i_k : \mathbb{R}^k \to \mathbb{R}^{k+1}$ induces natural transformations

$$j_k := \mathrm{St}(i_k) : \mathrm{St}(\mathbb{R}^{k+1}, -) \to \mathrm{St}(\mathbb{R}^k, -)$$

between representable functors in $\mathscr{S}^{\mathbb{W}}$ and we set

$$J_0 := \{ j_k \mid k \geqslant 0 \}.$$

Lemma 5.3.1. *The class* $(J_0)^s = (J_0)^c = P_0^{-1}(\mathrm{Iso})$ *is the congruence associated to the point at* ∞ *of the category* $\mathbb{W}$*.*

Proof. Since the category $\mathbb{W}$ is filtered by Proposition 5.2.1, we know from Proposition **??** that the set $\{\mathrm{St}(V,-) \to \mathrm{St}(U,-) \,|\, U \to V \in \mathbb{W}^{\to}\}$ of all maps between representable functors generates the congruence $P_0^{-1}(\mathrm{Iso})$ associated to the point at ∞ as a saturated class. But in $\mathbb{W}$ every object is isomorphic to $\mathbb{R}^n$ for $n = \dim U$ and every map $U \to V$ is isomorphic to a composition the maps i_k. Thus every map $\mathrm{St}(V,-) \to \mathrm{St}(U,-)$ is isomorphic to a composition of maps in J_0. And although this isomorphism is not canonical, it is enough to prove the statement. □

Let us introduce some abbreviations. Let $\mathcal{L}_0 = P_0^{-1}(\mathrm{Iso}) = J_0^s$ be the congruence associated to the point at ∞ of the category $\mathbb{W}$. It is the left class of the left exact modality whose right class we will denote by $\mathcal{R}_0$. The right class consists of the P_0-local maps: $f : F \to G$ in $\mathscr{S}^{\mathbb{W}}$ such that

$$\begin{array}{ccc} F & \xrightarrow{p_0 F} & P_0 F = \mathrm{colim}_n\, F(\mathbb{R}^n) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle P_0 f} \\ G & \xrightarrow{p_0 G} & P_0 G = \mathrm{colim}_n\, G(\mathbb{R}^n) \end{array}$$

is a cartesian square. The fact that $\mathcal{R}_0$ has this description follows because this is true for any left exact modality by [ABFJ22, Propositions 3.1.10 and 4.1.6].

Lemma 5.3.1 states in particular, that J_0 serves as a lex generators for $\mathcal{L}_0$. Therefore, according to Corollary **??**, the set

$$J_n := J_0^{\square n+1} = \{ j_{k_1} \square \ldots \square j_{k_{n+1}} \}_{k_1, \ldots, k_{n+1} \geqslant 0}$$

is a lex generator for the congruence $J_n^a = \mathcal{L}_n$ at the n-th level of the completion tower associated to $\mathcal{L}_0$. We write $J_n^{\perp\!\!\perp} = \mathcal{R}_n$ for the corresponding right class.

5.4 The Ganea construction and an augmented object

For every $k, n \geqslant 0$ let us consider the Ganea construction from Section **??** for the map j_k. It yields a cartesian square

$$\begin{array}{ccc} Z_{n+1}^k & \longrightarrow & W_{n+1}^k \\ {\scriptstyle \gamma_{n+1}^k}\downarrow & & \downarrow{\scriptstyle j_k^{\square n+1}} \\ \mathrm{St}(\mathbb{R}^k,-) & \xrightarrow{\Delta} & \mathrm{St}(\mathbb{R}^k,-)^{n+1} \end{array} \tag{5.4.1}$$

where W_{n+1}^k and Z_{n+1}^k denote the domains of the respective map γ_{n+1}^k and $j_k^{\square n+1}$.

For every $k \geqslant 0$ and every real vector space U in $\mathbb{W}$ there is the canonical inclusion $\mathbb{R}^k \to \mathbb{R}^k \oplus U$. These inclusions are compatible for varying $U \subset \mathbb{R}^{n+1}, n \geqslant 0$. Therefore they induce a map

$$g_{n+1}^k : \underset{0 \neq U \subset \mathbb{R}^{n+1}}{\mathrm{colim}} \mathrm{St}(\mathbb{R}^k \oplus U, -) \to \mathrm{St}(\mathbb{R}^k, -)$$

on covariant representable functors. The suggestive notation for the colimit is copied from Weiss. More precisely, the colimit is taken over a finite space, a flag manifold whose strata are given by Grassmannians $\mathrm{Gr}_\ell^{n+1}, 0 < \ell \leqslant n+1$.

Proposition 5.4.2. *For every $k, n \geqslant 0$ there are isomorphism*

$$Z_{n+1}^k = \operatorname*{colim}_{0 \neq U \subset \mathbb{R}^{n+1}} \mathrm{St}(\mathbb{R}^k \oplus U, -)$$

and

$$\gamma_{n+1}^k = g_{n+1}^k.$$

In particular, Z_{n+1}^k is finitely presented.

Proof. This is proved in [?, Prop. 5.4]. The fact that Z_{n+1}^k is finitely presented follows from the fact that the space, over which the colimit is taken, is finite. □

Let $Z := \mathrm{St}(\mathbb{R}, -) : \mathbb{W} \to \mathrm{Fin} \subset \mathscr{S}$ be the unit sphere functor sending an Euclidean vector space U to its unit sphere $Z(U) = \mathbb{S}^{\dim U - 1}$. Note that for $k = 0$ and all $n \geqslant 0$ we have

$$Z^{\star n+1} = \operatorname*{colim}_{0 \neq U \subset \mathbb{R}^{n+1}} \mathrm{St}(U, -) = Z_{n+1}^0.$$

by Example **??**(i) and hence

$$j_0^{\square n+1} = \gamma_{n+1}^0 : Z^{\star n+1} \to 1.$$

This augmented object can now be fed into the machinery of Section 3.

5.5 The Day convolution of $\mathscr{S}^{\mathbb{W}}$ with respect to $\oplus$

The category $\mathbb{W}$ equipped with the direct sum $\oplus$ becomes a symmetric monoidal category whose unit is the initial object $0 = \mathbb{R}^0$. From Section **??** one gets an associated Day convolution product $\widehat{\oplus}$ on $\mathscr{S}^{\mathbb{W}}$. It becomes a symmetric monoidal closed category with inner hom denoted by

$$[\![-,-]\!]^{\widehat{\oplus}} : \left(\mathscr{S}^{\mathbb{W}}\right)^{\mathrm{op}} \times \mathscr{S}^{\mathbb{W}} \to \mathscr{S}^{\mathbb{W}}.$$

Its unit is given by the terminal funtor $\mathrm{St}(0, -) = 1$. There are canonical isomorphisms

$$F \,\widehat{\oplus}\, 1 = F \quad \text{and} \quad [\![1, F]\!]^{\widehat{\oplus}} = F.$$

Lemma 5.5.1. *There is an isomorphism*

$$\gamma_{n+1}^k = \mathrm{St}(\mathbb{R}^k, -) \,\widehat{\oplus}\, \zeta_{n+1}$$

of maps in $\mathscr{S}^{\mathbb{W}}$.

Proof. The calculation is straightforward because $\widehat{\oplus}$ preserves colimits in both variables:

$$\begin{aligned}
\mathrm{St}(\mathbb{R}^k, -) \,\widehat{\oplus}\, Z_{n+1}^k &= \mathrm{St}(\mathbb{R}^k, -) \,\widehat{\oplus}\, Z^{\star n+1} \\
&= \mathrm{St}(\mathbb{R}^k, -) \,\widehat{\oplus} \big(\operatorname*{colim}_{0 \neq U \subset \mathbb{R}^{n+1}} \mathrm{St}(U, -) \big) \\
&= \operatorname*{colim}_{0 \neq U \subset \mathbb{R}^{n+1}} \big(\mathrm{St}(\mathbb{R}^k, -) \,\widehat{\oplus}\, \mathrm{St}(U, -) \big) \\
&= \operatorname*{colim}_{0 \neq U \subset \mathbb{R}^{n+1}} \mathrm{St}(\mathbb{R}^k \oplus U, -) = Z_{n+1}^k
\end{aligned}$$

□

5.6 The functors T_n and P_n

Definition 5.6.1. For $n \geqslant 0$ let us define an object Z_{n+1} together with a map

$$(\zeta_{n+1} : Z_{n+1} \to 1) := \left(\gamma^0_{n+1} : Z^0_{n+1} = \operatorname*{colim}_{0 \neq U \subset \mathbb{R}^{n+1}} \mathrm{St}(U, -) \to \mathrm{St}(\{0\}, -) = 1\right)$$

in the category $\mathscr{S}^{\mathbb{W}}$.

The map ζ_{n+1} makes Z_{n+1} into an augmented finitely presented object in the sense of Section **??**. From Definitions **??** and 3.4.1 we now obtain two endofunctors.

Definition 5.6.2. For $n \geqslant 0$ let $T_n : \mathscr{S}^{\mathbb{W}} \to \mathscr{S}^{\mathbb{W}}$ be defined as

$$\begin{aligned} T_n F &:= [\![Z_{n+1}, F]\!]^{\widehat{\oplus}} = [\![\operatorname*{colim}_{0 \neq U \subset \mathbb{R}^{n+1}} \mathrm{St}(U, -), F]\!]^{\widehat{\oplus}} = \lim_{0 \neq U \subset \mathbb{R}^{n+1}} [\![\mathrm{St}(U, -), F]\!]^{\widehat{\oplus}} \\ &= \lim_{0 \neq U \subset \mathbb{R}^{n+1}} F(- \oplus U) \end{aligned}$$

together with the coaugmentation

$$(t_n F : F \to T_n F) := [\![\zeta_{n+1} : Z_{n+1} \to 1, F]\!]^{\widehat{\oplus}}.$$

Then one obtains $P_n : \mathscr{S}^{\mathbb{W}} \to \mathscr{S}^{\mathbb{W}}$ by setting

$$P_n F = \operatorname{colim}(F \xrightarrow{t_n F} T_n F \xrightarrow{t_n T_n F} T^2 F \xrightarrow{t_n T_n^2} T_n^3 F \to \dots)$$

and

$$p_n F : F \to P_n F.$$

A functor $F : \mathbb{W} \to \mathscr{S}$ is called *n-polynomial functors* if the map

$$t_n F(V) : F(V) \to \lim_{0 \neq U \subset \mathbb{R}^{n+1}} F(V \oplus U) = T_n F(V)$$

is an isomorphism for all V in $\mathbb{W}$, see [**?**, Def. 5.1]. A map $\alpha : F \to G$ in $\mathscr{S}^{\mathbb{W}}$ is *T_n-local* if the following square

$$\begin{array}{ccc} F & \xrightarrow{t_n F} & T_n F \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle T_n \alpha} \\ G & \xrightarrow{t_n G} & T_n G \end{array} \tag{5.6.3}$$

is cartesian.

Lemma 5.6.4. *For all $n, \ell \geqslant 1$ we have: $T_n^\ell F = [\![Z_{n+1}^{\widehat{\oplus}\ell}, F]\!]^{\widehat{\oplus}}$.*

Proof. This is a general fact, see Equation **??**. Alternatively one can write it out

$$\left(\operatorname*{colim}_{0 \neq U \subset \mathbb{R}^{n+1}} \mathrm{St}(U, -)\right) \widehat{\oplus} \left(\operatorname*{colim}_{0 \neq V \subset \mathbb{R}^{n+1}} \mathrm{St}(V, -)\right) = \operatorname*{colim}_{0 \neq U, V \subset \mathbb{R}^{n+1}} \mathrm{St}(U \oplus V, -).$$

and use induction. ☐

Lemma 5.6.5. *Fix $n \geqslant 0$. For a map α in $\mathscr{S}^{\mathbb{W}}$ the following statements are equivalent:*

(i) *The map α is T_n-local.*

(ii) $\alpha \in \{\gamma_{n+1}^k \,|\, k \geqslant 0\}^{\perp}$

Proof. If $\alpha : F \to G$ is T_n-local then the square (5.6.3) is cartesian. In particular it is cartesian when evaluated at $\mathbb{R}^k$ for all $k \geqslant 0$. This implies $\alpha \in \{\gamma_{n+1}^k \,|\, k \geqslant 0\}^{\perp}$ because the map γ_{n+1}^k corepresents the map $t_n(-)(\mathbb{R}^k)$. The reverse direction is obtained by observing that every object in $\mathbb{W}$ is isomorphic to one of the form $\mathbb{R}^k$ for some $k \geqslant 0$. □

5.7 P_n is a reflector

Following the steps in Section 3 we need to prove that the map $P_n\zeta_{n+1}$ is an isomorphism. Weiss provides a tool based on connectivity estimates. We remind the reader of the definition of connectivity given in Section **??**. This is not the convention used by Weiss. Nevertheless the next two statements remain the same since the difference ± 1 is absorbed by the constant c.

Proposition 5.7.1. *Let $\alpha : F \to G$ be a morphism in $\mathscr{S}^{\mathbb{W}}$. Suppose that there exist integers c and κ such that $\alpha(W) : F(W) \to G(W)$ is $\big((n+1)\dim(W) - c\big)$-connected for all W in $\mathbb{W}$ with $\dim W \geqslant \kappa$. Then $P_n\alpha : P_nF(W) \to P_nG(W)$ is an isomorphism.*

Proof. In [**?**, e.3 Lemma] Weiss shows that under the given condition on the map α, the induced map $T_n\alpha$ is $\big((n+1)\dim(W) - c + 1\big)$-connected for all W in $\mathbb{W}$ with $\dim W \geqslant \kappa - 1$. By induction the map $T^\ell\alpha(W)$ is $\big((n+1)\dim(W) - c + \ell\big)$-connected. As ℓ tends infinity, the connectivity tends to infinity and the colimit $P_n\alpha(W)$ is an equivalence. This is true for any W as κ reaches 0 after finitely many steps. □

Proposition 5.7.2. *The map $P_n\zeta_{n+1}$ is an isomorphism.*

Proof. Let U be ℓ-dimensional and V be m-dimensional with $\ell \leqslant m$. Then the Stiefel manifold $\mathrm{St}(U, V)$ is $(m - \ell - 1)$-connected as a space. So the map $\mathrm{St}(U, V) \to *$ is $(m - \ell - 1)$-connected (recall Section **??** for our convention for connectivity of maps). So the map

$$j_0(V) : Z(V) = \mathrm{St}(\mathbb{R}^1, V) \to \mathrm{St}(\{0\}, V) = 1$$

is $(m-2)$-connected as $\ell = 1$. Recall that $\zeta_{n+1} = \gamma_{n+1}^0 = j_0^{\square n+1}$. Therefore, by Lemma **??**, the space $Z(V)^{\star n+1}$ and hence the map $\zeta_{n+1}(V)$ has connectivity

$$(n+1)(m-2) + 2n = (n+1)m - 2$$

for all V. With $\dim V \geqslant 3$ the map ζ_{n+1} satisfies the hypothesis of Proposition 5.7.1 with constants $c = 2$ and $\kappa = 3$. □

Corollary 5.7.3. *The functor P_n from* Definition 5.6.2 *is a left exact reflector onto the subcategory of P_n-local objects. The associated congruence has as its left class the P_n-equivalences $P_n^{-1}(\mathrm{Iso})$ and as its right class the P_n-local maps.*

Recall that we denote by $Z = \mathrm{St}(\mathbb{R}, -) : \mathbb{W} \to \mathscr{S}$ the unit sphere functor.

Proof. For all $n \geqslant 0$ the object $Z_{n+1} = Z^{\star n+1}$ is finitely presentable by Proposition 5.4.2. Together with Proposition 5.7.2 this tells us that Theorem **??** applies. □

5.8 The Weiss tower as a completion tower

This section is devoted to a proof of Theorem 5.2.2. The situation is as follows:

1. The category $\mathbb{W}$ is filtered and hence admits a point at ∞: $P_0 = \operatorname{colim}_{\mathbb{W}} : \mathscr{S}^{\mathbb{W}} \to \mathscr{S}$. This gives us a congruence $\mathcal{L}_0 = P_0^{-1}(\mathrm{Iso})$ to which we can associated a completion tower. In particular, in the notation of Theorem **??** we set $\Phi_0 = P_0$.

   Thus we consider the nested sequence of congruences $\ldots \subset \mathcal{L}_n \subset \mathcal{L}_{n-1} \subset \ldots$, each one given as acyclic power of the ground stage: $\mathcal{L}_n = \mathcal{L}_0^n$. The congruence $\mathcal{L}_n$ sits at the n-stage of the completion tower. The associated left exact localization in the completion tower is $\Phi_n : \mathscr{S}^{\mathbb{W}} \to \mathscr{S}^{\mathbb{W}} /\!/ \mathcal{L}_n$.

   Since J_0 is a lex generator for $\mathcal{L}_0$, we know from Corollary **??** that $J_0^{\square n+1}$ is a lex generator for $\mathcal{L}_n = (J_0^{\square n+1})^a$. The corresponding right class was denoted by $\mathcal{R}_n$. It is defined as $\mathcal{R}_n = \mathcal{L}_n^{\perp} = (J_0^{\square n+1})^{\perp\!\!\!\perp}$. This summarizes Sections 5.2 and 5.3.

2. On the other hand, in Sections 5.4 through 5.7, there is Weiss' construction of the orthogonal tower accomodated to our language. We have chosen the augmented object $\zeta_{n+1} : Z_{n+1} = Z^{\star n+1} \to 1$ where $Z = \mathrm{St}(\mathbb{R}, -)$ is the unit sphere functor. As it turns out that $Z^{\star n+1} = \operatorname{colim}_{0 \neq U \subset \mathbb{R}^{n+1}} \mathrm{St}(U, -)$. With the machinery of Section 3 one arrives at Weiss' construction P_n at the n-stage of the orthogonal tower. Corollary 5.7.3 proves that P_n is indeed a left exact reflector and yields the congruence of P_n-equivalences $P_n^{-1}(\mathrm{Iso})$. Let us denote this congruence by $\mathcal{L}_n^W$ (W for Weiss). We denote the corresponding right class by $(\mathcal{L}_n^W)^{\perp} = \mathcal{T}_n$. By Lemma **??** this right class is given by T_n-local or equivalently P_n-local maps. We have seen in Lemma 5.6.5 that $\mathcal{T}_n = \{\gamma_{n+1}^k \,|\, k \geqslant 0\}^{\perp}$. Equivalently this can be expressed by $\{\gamma_{n+1}^k \,|\, k \geqslant 0\}^s = \{\gamma_{n+1}^k \,|\, k \geqslant 0\}^a = \mathcal{L}_n^W$.

The goal is to show the equivalent statements $\mathcal{L}_n = \mathcal{L}_n^W, \mathcal{R}_n = \mathcal{T}_n$ and $P_n = \Phi_n$. This proves Theorem 5.2.2 and shows that Weiss' tower is indeed a completion tower in our sense.

We start by showing $\mathcal{R}_n \subset \mathcal{T}_n$. Recall again the class $J_n^{\perp\!\!\!\perp} = \mathcal{R}_n$ with

$$J_n := J_0^{\square n+1} = \{j_{k_1} \,\square\, \ldots \,\square\, j_{k_{n+1}}\}_{k_1, \ldots, k_{n+1} \geqslant 0}.$$

The maps γ_{n+1}^k arise from the Ganea construction in Diagram (5.4.1). Hence, γ_{n+1}^k is a base change of $j_k^{\square n+1} \in J_n$. Thus for all $k \geqslant 0$ we have $\gamma_{n+1}^k \in J_n^a$, because the acyclic class J_n^a is closed under base change. Hence:

$$\{\gamma_{n+1}^k \,|\, k \geqslant 0\}^s \subset \{\gamma_{n+1}^k \,|\, k \geqslant 0\}^a \subset J_n^a$$

Therefore on the right side of the factorization systems:

$$\mathcal{R}_n = J_n^{\perp\!\!\!\perp} \subset \{\gamma_{n+1}^k \,|\, k \geqslant 0\}^{\perp\!\!\!\perp} \subset \{\gamma_{n+1}^k \,|\, k \geqslant 0\}^{\perp} = \mathcal{T}_n$$

Now we show the reverse inclusion $\mathcal{T}_n \subset \mathcal{R}_n$. We will prove that each map in J_n is a P_n-equivalence. Take an arbitrary map $j_{k_1} \,\square\, \ldots \,\square\, j_{k_{n+1}} \in J_0^{\square n+1} = J_n$ and let $\kappa = \max_i k_i$. For fixed V in $\mathbb{W}$ the space $\mathrm{St}(\mathbb{R}^{k_i}, V)$ is $(m - k_i - 1)$-connected. If we start with vector spaces such that $m = \dim V \geqslant \kappa + 1$, then, for all i, the space $\mathrm{St}(\mathbb{R}^{k_i}, V)$ is $(m - k_i - 1)$-connected with $m - k_i - 1 \geqslant m - \kappa \geqslant 0$. Then the map

$$j_{k_i}(V) : \mathrm{St}(\mathbb{R}^{k_i+1}, V) \to \mathrm{St}(\mathbb{R}^{k_i}, V)$$

is at least $(m-\kappa)$-connected, referring to our convention about connectivity of maps in Section ??. By Lemma ?? the pushout product

$$\big(j_{k_1} \,\square\, \dots \,\square\, j_{k_{n+1}}\big)(V) = j_{k_1}(V) \,\square\, \dots \,\square\, j_{k_{n+1}}(V)$$

has connectivity

$$\sum_{i=1}^{n+1}(m-k_i-1)+2n \geqslant \sum_{i=1}^{n+1}(m-\kappa)+2n = (n+1)m-(n+1)\kappa+2n$$

Hence $j_{k_1} \,\square\, \dots \,\square\, j_{k_{n+1}}$ satisfies the hypothesis of Corollary 5.7.1 with constants $\kappa = 1+\max_i k_i$ and $c=(n+1)\kappa-2n$. We conclude that it is a P_n-equivalence: i.e. $J_n \subset \{P_n\text{-equiv}\}$. Hence:

$$\mathcal{T}_n = \{P_n\text{-equiv}\}^{\perp\!\!\!\perp} \subset J_n^{\perp\!\!\!\perp} = \mathcal{R}_n.$$

This concludes the proof.

5.9 The monogenic part of the orthogonal tower

Corollary 5.9.1. *The monogenic part* $\mathscr{S}^{\mathbb{W}} \to (\mathscr{S}^{\mathbb{W}}/\!/(P_0^{-1}(\mathrm{Iso})^{\mathrm{mono}})$ *of the orthogonal tower is forcing all maps between representable functors to become surjective. The monogenic congruence* $P_0^{-1}(\mathrm{Iso})^{\mathrm{mono}} = \mathcal{L}_0^{\mathrm{mono}}$ *is generated as an acyclic class by the monomorphic parts* $\mathrm{im}\, j_k$ *of the maps* $j_k : \mathrm{St}(\mathbb{R}^{k+1},-) \to \mathrm{St}(\mathbb{R}^k,-)$ *for all* $k \geqslant 0$.

Proof. Since $\mathbb{W}$ is filtered and we now know that the orthogonal tower is the completion tower of the point at ∞ of $\mathbb{W}$, Theorem ?? applies. It only remains to show the statement about the generators which follows from Remark ??4. ☐

5.10 Blakers-Massey theorems

Since the orthogonal tower can be constructed as a completion tower, all general theorems from Section ?? apply. In particular, the Blakers-Massey theorem ?? and its "dual" version ?? hold when P_n is interpreted as the reflectors in the orthogonal tower. As far as we know, this is a new result.

5.11 Variants

Taggart [?] has developped unitary calculus by adapting Weiss' approach to functors from finite dimensional complex vector spaces equipped with a positive definite Hermitian form. Tynan [?] and Taggart [?] independently construct a calculus of functors from finite dimensional complex inner product spaces taking into account complex conjugation. These are further examples of completion towers.

Another variant is to take a category $\mathscr{B}$ whose underlying nerve is finite. For example $\mathscr{B}$ could be just a finite space B. Then one can consider the category $\mathscr{V} = \mathscr{B}^{\mathbb{W}}$ of Euclidean vector bundles over $\mathscr{B}$. Then $\mathscr{V}$ is still filtered with point at ∞ given by the colimit

$$P_0F = \operatorname*{colim}_n F(\mathbb{R}^n \oplus -),$$

where $\mathbb{R}^n$ denotes the trivial bundle. A symmetric monoidal structure on $\mathscr{V}$ is given by the Whitney sum. In this case one replaces the augmented object $\mathrm{St}(\mathbb{R}, -) \to 1$ by the trivial unit sphere bundle mapping to the zero bundle. The associated completion tower on the category $\mathscr{S}^{\mathscr{V}}$ is a fiberwise orthogonal tower. Of course, this can also be directly obtained from Weiss' articles.

A Enriched category theory

A.1 Enrichments of limits and colimits

Let $\mathcal{V} = Fun(\mathbb{V}, \mathcal{S})$ be a symmetric monoidal category. Recall that an *enrichement* of a functor $F : \mathcal{A} \to \mathcal{B}$ between $\mathcal{V}$-categories is a natural transformation $\theta^F(X, Y) : \mathcal{A}(X, Y) \to \mathcal{B}(FX, FY)$ respecting composition and units.

Recall that a $\mathcal{V}$-category $\mathcal{M}$ is said to be *tensored* by $\mathcal{V}$ if for every object $A \in \mathcal{V}$ and every object $M \in \mathcal{M}$, the $\mathcal{V}$-functor $N \mapsto [A, [M, N]]$ is representable by an object $A \otimes M \in \mathcal{M}$. Dually, $\mathcal{M}$ is said to be *cotensored* by $\mathcal{V}$ if for every object $A \in \mathcal{V}$ and every object $N \in \mathcal{M}$, the contravariant $\mathcal{V}$-functor $M \mapsto [A, [M, N]]$ is representable by an object $\{A, N\} \in \mathcal{V}$.

If $\mathcal{A}$ and $\mathcal{B}$ are tensored the $\mathcal{V}$-categories, then the enrichement of a $\mathcal{V}$-functor $F : \mathcal{A} \to \mathcal{B}$ can be described by an *assembly map* $\lambda^F(A, X) : A \otimes FX \to F(A \otimes X)$ satisfying standard associativity and unitary conditions. Dually, if $\mathcal{A}$ and $\mathcal{B}$ are cotensored $\mathcal{V}$-categories, then the enrichement of F can be described by a *co-assembly map* $\gamma^F(A, X) : F\{A, X\} \to \{A, FX\}$ satisfying a standard associativity and unitary conditions.

Recall also that a natural transformation $\alpha : F \to G$ between $\mathcal{V}$-enriched functors $F, G : \mathcal{A} \to \mathcal{B}$ is said to be *strong* if the following square commutes for every $X, Y \in \mathcal{A}$.

$$\begin{array}{ccc}
\mathcal{A}(X,Y) & \xrightarrow{\theta^G} & \mathcal{B}(GX,GY) \\
{\scriptstyle \theta^F}\downarrow & & \downarrow{\scriptstyle (-)\circ\alpha(X)} \\
\mathcal{B}(FX,FY) & \xrightarrow{\alpha(Y)\circ(-)} & \mathcal{B}(FX,GY)
\end{array}$$

If $\mathcal{A}$ and $\mathcal{B}$ are tensored over $\mathcal{V}$, then a natural transformation $\alpha : F \to G$ is strong if and only if the following square commutes for every $K \in \mathcal{V}$ and $A \in \mathcal{A}$.

$$\begin{array}{ccc}
K \otimes F(X) & \xrightarrow{K\otimes\alpha(X)} & K \otimes G(X) \\
{\scriptstyle \lambda^F(K,X)}\downarrow & & \downarrow{\scriptstyle \lambda^G(K,X)} \\
F(K \otimes X) & \xrightarrow{\alpha(K\otimes X)} & G(K \otimes X)
\end{array}$$

Dually, $\mathcal{A}$ and $\mathcal{B}$ are cotensored over $\mathcal{V}$, then a natural transformation $\alpha : F \to G$ is strong if and only if the following square commutes for every $K \in \mathcal{V}$ and $X \in \mathcal{A}$.

$$\begin{array}{ccc} F\{K,X\} & \xrightarrow{\alpha\{K,X\}} & G\{K,X\} \\ {\scriptstyle \gamma^F(K,X)}\downarrow & & \downarrow{\scriptstyle \gamma^G(K,X)} \\ \{K,FX\} & \xrightarrow{\{K,\alpha X\}} & \{K,GX\} \end{array}$$

If a $\mathcal{V}$-category $\mathcal{A}$ is cotensored then the endo-functor $S_Z := \{Z,-\} : \mathcal{A} \to \mathcal{A}$ is enriched over $\mathcal{V}$ for any object Z in $\mathcal{V}$. The coassembly map $\gamma : S_Z\{A,X\} \to \{A, S_Z X\}$ the composite of the natural isomorphisms

$$\gamma : \{Z,\{A,X\}\} \simeq \{Z \otimes A, X\} \simeq \{A \otimes Z, X\} \simeq \{A,\{Z,X\}\}$$

Notice that $S_{Z_1} \circ S_{Z_2} = S_{Z_1 \otimes Z_2}$ and $S_I = \mathrm{Id}_{\mathcal{A}}$. In particular, the endofunctor $T_Z := [Z,-] : \mathcal{V} \to \mathcal{V}$ is enriched over $\mathcal{V}$. The coassembly map $\gamma : T_Z[A,X]] \to [A, T_Z X]$ is the composite of the natural isomorphisms

$$[Z,[A,X]] \simeq [Z \otimes A, X] \simeq [A \otimes Z, X] \simeq [A,[Z,X]]$$

We have $T_{Z_1} \circ T_{Z_2} = T_{Z_1 \otimes Z_2}$ and $T_I = \mathrm{Id}_{\mathcal{V}}$.

Proposition A.1.1. *Let $\mathcal{V}$ be a symmetric monoidal closed category and $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{V}$-closed $\mathcal{V}$-categories. If $\mathcal{B}$ is cocomplete (resp. complete), then the category $[\mathcal{A},\mathcal{B}]$ of $\mathcal{V}$-functors $\mathcal{A} \to \mathcal{B}$ is cocomplete (resp. complete) and colimits (resp. limits) are computed pointwise.*

Proof. Let us show that the pointwise colimit $F : \mathcal{A} \to \mathcal{B}$ of a diagram of $\mathcal{V}$-functors $D : I \to [\mathcal{A},\mathcal{B}]$ is a $\mathcal{V}$-functor. For every $i \in I$, the enrichment of the functor $D(i) : \mathcal{A} \to \mathcal{B}$ is described by an assembly map

$$\lambda(i)(K,A) := \lambda(D(i))(K,A) : K \otimes D(i)(A) \to D(i)(K \otimes A)$$

for $K \in \mathcal{V}$ and $A \in \mathcal{A}$. The following square of natural transformations commutes

$$\begin{array}{ccc} K \otimes D(i)(A) & \xrightarrow{K\otimes D(u)} & K \otimes D(j)(A) \\ {\scriptstyle \lambda(i)(K,A)}\downarrow & & \downarrow{\scriptstyle \lambda(j)(K,A)} \\ D(i)(K\otimes A) & \xrightarrow{D(u)(K\otimes A)} & D(j)(K \otimes A) \end{array} \qquad \text{(A.1.2)}$$

for every map $u : i \to j$ in I since the natural transformation $D(u) : D(i) \to D(j)$ is strong. Hence the assembly maps $\lambda(i)(K,A)$ for $i \in I$ are defining a natural transformation $\lambda(i)(K,A) : K \otimes D(A) \to D(K \otimes A)$ between two diagrams $I \to \mathcal{B}$. The functor $K \otimes (-) : \mathcal{B} \to \mathcal{B}$ preserves colimits, since the category $\mathcal{B}$ is cotensored. If $F = \mathrm{colim}_{i \in I} D(i)$ and $\lambda(K,A) := \mathrm{colim}_{i \in I} \lambda(i)(K,A)$ then $\lambda(K,A) : K \otimes F(A) \to F(K \otimes A)$ is the assembly map defining the enrichement of F. It is easy to see the $K \otimes F = \mathrm{colim}_{i \in I} K \otimes D(i)$ The proof that the pointwise limit of a diagram $D : I \to [\mathcal{A},\mathcal{B}]$ has the structure of a $\mathcal{V}$-functor is dual, using co-assembly maps. □

A.2 Dilation and codilation

Definition A.2.1. The *dilation* of a functor $F : \mathscr{C} \to \mathscr{D}$ at an object C in $\mathscr{C}$ is the functor

$$F_C : \mathscr{C}/C \to \mathscr{D}/FC, \quad (p : X \to C) \mapsto (F(p) : FX \to FC).$$

For example, if $\mathcal{V} := (\mathcal{V}, \otimes, I)$ is a symmetric monoidal category and $\mathcal{M}$ a tensored $\mathcal{V}$-category, then for every object A in $\mathcal{V}$ the dilation of the functor $A \otimes (-) : \mathcal{M} \to \mathcal{M}$ at the object B in $\mathcal{M}$ is the functor

$$A \otimes (-) : \mathcal{M}/B \to \mathcal{M}/(A \otimes B)$$

which takes a map $p : X \to B$ to the map $A \otimes p : A \otimes X \to A \otimes B$.

Definition A.2.2. We shall say that a symmetric monoidal category $(\mathbb{V}, \oplus, 0)$ *satisfies condition* (G) if it has pushouts and the functor $A \oplus (-) : \mathbb{V} \to \mathbb{V}$ preserves pushouts for every object $A \in \mathbb{V}$.

Examples: The following symmetric monoidal categories satisfies condition (G):

1. the monoidal category $(\mathrm{Fin}, \star, 0)$;
2. the monoidal category $({}^\bullet\mathrm{Fin}, \wedge, \mathrm{S}^0)$;

Lemma A.2.3. *Let* $\mathbb{V} = (\mathbb{V}, \oplus, 0)$ *be a symmetric monoidal category satisfying condition (G). Suppose that the functor category* $\mathcal{V} := Fun(\mathbb{V}, \mathscr{S})$ *is equipped with the symmetric monoidal structure* $(\mathcal{V}, \otimes, I)$ *defined by Day convolution. Then the dilation functor*

$$R^A \otimes (-) : \mathcal{V}/R^B \to \mathcal{V}/R^A \otimes R^B \tag{A.2.4}$$

is a morphism of logoi for every $A, B \in \mathbb{V}$.

Proof. The functor $A \oplus (-) : \mathbb{V} \to \mathbb{V}$ preserves pushouts since the category $\mathbb{V}$ satisfies condition (G). The category $B\backslash\mathbb{V}$ has finite colimits, since it has pushouts and an initial object. Hence the co-dilation functor $\phi := A \oplus (-) : B\backslash\mathbb{V} \to (A \oplus B)\backslash\mathbb{V}$ preserves finite colimits, since it preserves pushouts and initial objects. Hence the left Kan extension

$$\phi_! : Fun(B\backslash\mathbb{V}, \mathscr{S}) \to Fun((A \oplus B)\backslash\mathbb{V}, \mathscr{S})$$

of the functor ϕ^{op} preserves finite limits, since the functor ϕ preserves finite colimits. The result follows, since the dilation functor A.2.4 is equivalent to the functor $\phi_!$. □

A.3 Slicing adjunctions

Let $F \vdash G$ be an adjuntion

$$F : \mathcal{C} \leftrightarrow \mathcal{D} : G$$

and let $\theta : Hom(FX, G) \mapsto Hom(X, G(Y))$ be the adjunction isomorphism. Let $A \in \mathcal{C}$, $B \in \mathcal{D}$, $u : FA \to B$ and $v := \theta(u) : A \to G(B)$.

Let us denote by $F/u : \mathcal{C}/A \to \mathcal{D}/B$ the functor which takes a map $f : X \to A$ to the composite of the map maps

$$FX \xrightarrow{Ff} FA \xrightarrow{u} B$$

By definition, the functor F/u is the composite of the functors

$$\mathcal{C}/A \xrightarrow{F/A} \mathcal{D}/FA \xrightarrow{u_!} \mathcal{D}/B$$

were F/A is the dilation of the functor F at A.

Let us denote by $G/v : \mathcal{D}/B \to \mathcal{C}/A$ the functor which takes a map $g : Y \to B$ to the map p_1 in the pullback square

$$\begin{array}{ccc} A \times_{GB} GY & \xrightarrow{p_2} & GY \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle Gg} \\ A & \xrightarrow{v} & GB \end{array}$$

By definition, the functor G/v is the composite of the functors

$$\mathcal{D}/B \xrightarrow{G/B} \mathcal{C}/GB \xrightarrow{v^\star} \mathcal{C}/A$$

were G/B is the dilation of the functor G at B.

Proposition A.3.1. *With the notation above, we have an adjunction*

$$F/u : \mathcal{C}/A \longleftrightarrow \mathcal{D}/B : G/v$$

A.4 Slicing monoidal categories

If $\mathcal{V} = (\mathcal{V}, \otimes, I)$ is a symmetric monoidal category, then so is the category $\mathcal{V}/I$ with

$$(X, u) \otimes (Y, v) := (X \otimes Y, u \otimes v)$$

with structure map $X \otimes Y \xrightarrow{u \otimes v} I \otimes I = I$.

Proposition A.4.1. *If $\mathcal{V} = (\mathcal{V}, \otimes, I)$ is a symmetric monoidal closed category with pullbacks, then the symmetric monoidal category $\mathcal{V}/I$ is closed. Moreover, $\mathcal{V}/I$ is confined when $\mathcal{V}$ is confined.*

Proof. The internal hom $[(X, u), (Y, v)]$ between two objects (X, u) and (Y, v) of $\mathcal{V}/I$ is the map $p_1 Z \to I$ defined by base change

$$\begin{array}{ccc} Z & \xrightarrow{p_2} & [X, Y] \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle [X,v]} \\ I & \xrightarrow{[u,I]} & [X, I] \end{array}$$

in $\mathcal{V}$. The evaluation map $[(X,u),(Y,v)]\otimes(X,u)\to(Y,v)$ is the composite of the map $p_2\otimes X : Z\otimes X\to[X,Y]\otimes X$ with the evaluation map $[X,Y]\otimes X\to Y$. The category $\mathcal{V}/I$ is ω-presentable, since $\mathcal{V}$ is ω-presentable. Moreover, $\mathrm{Comp}(\mathcal{V}/I)=\mathrm{Comp}(\mathcal{V})/I$. If X and Y are compact objects of $\mathcal{V}$, then $X\otimes Y$ is compact, since $\mathcal{V}$ is confined. It follows that the object $(X,u)\otimes(Y,v):=(X\otimes Y,u\otimes v)$ is compact in $\mathcal{V}/I$ for any maps $u:X\to I$ and $v:Y\to I$. The object I is compact in $\mathcal{V}$, since $\mathcal{V}$ is confined. Hence the object (I,id_I) is compact in $\mathcal{V}/I$. □

Let us say that a monoidal closed category $\mathcal{V}=(\mathcal{V},\otimes,I)$ is *semi-cartesian* if its unit object I is the terminal object $1\in\mathcal{V}$. If $\mathcal{V}$ is semi-cartesian, for every object $A\in\mathcal{V}$ let us denote the unique map $A\to 1$ by $\tau(A)$. If $A,B\in\mathcal{V}$, we shall say that the map $p_A:=A\otimes\tau(B)$ is the *first projection* and that $p_B:=\tau(A)\otimes B$ is the *second projection*.

$$A \xleftarrow{A\otimes\tau(B)} A\otimes B \xrightarrow{\tau(A)\otimes B} B$$

For every $X\in\mathcal{V}$, we shall say that the map $[\tau(A),X]:X\to[A,X]$ is the *diagonal*.

For any object $C\in\mathcal{V}$ the category $\mathcal{V}/C$ is equipped with a natural action of $\mathcal{V}$: by definition, $A\otimes(X,f):=(A\otimes X,\tau(A)\otimes f)$ for every object $A\in\mathcal{V}$ and every map $f:X\to C$.

It is easy to verify the associativity law $A\otimes(B\otimes(X,f))=(A\otimes B)\otimes(X,f)$ and the unit law $1\otimes(X,f)=(X,f)$. The forgetful functor $U:\mathcal{V}/C\to\mathcal{V}$ preserves the action of $\mathcal{V}$ on these categories (with $\mathcal{V}$ acting in the obvious way on itself).

Proposition A.4.2. *If $\mathcal{V}$ is semi-cartesian and has pullbacks, then for every object $C\in\mathcal{V}$ the category $\mathcal{V}/C$ has the structure of a closed $\mathcal{V}$-module. The cotensor $\{A,(X,f)\}$ of an object $(X,f)\in\mathcal{V}/C$ by an object $A\in\mathcal{V}$ is constructed by the following pullback square*

$$\begin{array}{ccc} \{A,(X,f)\} & \longrightarrow & [A,X] \\ \big\downarrow & & \big\downarrow{\scriptstyle [A,f]} \\ C & \xrightarrow{[\tau(A),C]} & [A,C] \end{array} \tag{A.4.3}$$

The forgetful functor $U:\mathcal{V}/C\to\mathcal{V}$ is a morphism of $\mathcal{V}$-modules and top horizontal map of the square A.4.5 is the coassembly map $\gamma^U(A,(X,f)):U\{A,(X,f)\}\to[A,U(X,f)]$. The $\mathcal{V}$-module $\mathcal{V}/C$ and the functor U are cofined when $\mathcal{V}$ is confined.

Proof. By definition, $A\otimes(X,f):=(A\otimes X,\tau(A)\otimes f)$ for every object $A\in\mathcal{V}$ and every map $f:X\to C$. Observe first that the map $\tau(A)\otimes f$ fits into the following commutative square:

$$\begin{array}{ccc} A\otimes X & \xrightarrow{\tau(A)\otimes X} & X \\ {\scriptstyle A\otimes f}\big\downarrow & \searrow^{\tau(A)\otimes f} & \big\downarrow{\scriptstyle f} \\ A\otimes C & \xrightarrow[\tau(A)\otimes C]{} & C \end{array}$$

Let us now show that the functor $(-)\otimes(X,f):\mathcal{V}\to\mathcal{V}/C$ has a right adjoint $[(X,f),-]$ for every object $(X,f)\in\mathcal{V}/C$. The formula $\tau(A)\otimes f=f\circ(\tau(A)\otimes X)$ shows that the functor $(-)\otimes(X,f):\mathcal{V}\to\mathcal{V}/C$ is the composite

$$\mathcal{V}\xrightarrow{(-\otimes X)/1}\mathcal{V}/X\xrightarrow{\quad f_! \quad}\mathcal{V}/C$$

where $(-\otimes X)/1$ is a dilation of the functor $(-)\otimes X:\mathcal{V}\to\mathcal{V}$ and the functor $f_!$ is composition with $f:X\to C$. By Propositon A.3.1, the functor $(-)\otimes(X,f)$ is left adjoint to the composite

$$\mathcal{V}/C\xrightarrow{[X,-]}\mathcal{V}/[X,C]\xrightarrow{\ulcorner f\urcorner^\star}\mathcal{V}$$

where the map $\ulcorner f\urcorner:1\to[X,C]$ corresponds to the map $f:X\to C$ via the adjunction isomorphism $\mathrm{map}(1,[X,C])\cong\mathrm{map}(X,C)$. This shows that the external hom $[(X,f),(Y,g)]$ between two objects of $\mathcal{V}/C$ is the object of $\mathcal{V}$ constructed by the following pullback square:

$$\begin{array}{ccc}
[(X,f),(Y,g)] & \longrightarrow & [X,Y] \\
\downarrow & & \downarrow{\scriptstyle [X,g]} \\
1 & \xrightarrow{\ulcorner f\urcorner} & [X,C]
\end{array}$$

Let us now show that the enriched category $\mathcal{V}/C$ is cotensored. For every object $A\in\mathcal{V}$ the formula $\tau(A)\otimes f=(\tau(A)\otimes C)(A\otimes f)$ shows that the functor $A\otimes(-):\mathcal{V}/C\to\mathcal{V}/C$ is the composite

$$\mathcal{V}/C\xrightarrow{(A\otimes-)/C}\mathcal{V}/(A\otimes C)\xrightarrow{(\tau(A)\otimes C)_!}\mathcal{V}/C,$$

where $(A\otimes -)/C$ denotes a dilation **ref** of the functor $A\otimes(-):\mathcal{V}\to\mathcal{V}$ and where $(\tau(A)\otimes C)_!$ denotes composition by $\tau(A)\otimes C$. By Proposition A.3.1, the functor $A\otimes(-):\mathcal{V}/C\to\mathcal{V}/C$ is left adjoint to the composite

$$\mathcal{V}/C\xrightarrow{[A,-]/C}\mathcal{V}/[A,C]\xrightarrow{[\tau(A),C]^\star}\mathcal{V}/C$$

where $[A,-]/C$ is a dilation of the functor $[A,-]:\mathcal{V}\to\mathcal{V}$ and $[\tau(A),C]^\star$ is the base change functor along the map $[\tau(A),C]$. This shows that the cotensor $\{A,(X,f)\}$ is constructed by the following pullback square:

$$\begin{array}{ccc}
\{A,(X,f)\} & \longrightarrow & [A,X] \\
\downarrow & & \downarrow{\scriptstyle [A,f]} \\
C & \xrightarrow{[\tau(A),C]} & [A,C]
\end{array}$$

We have proved that the category $\mathcal{V}/C$ has the structure of a closed $\mathcal{V}$-module. The forgetful functor $U:\mathcal{V}/C\to\mathcal{V}$ preserves obviously the tensorial action $\mathcal{V}$. We leave to the reader the verification that the top horizontal map of the square A.4.5 is the coassembly map $\gamma^U(A,(X,f)):U\{A,(X,f)\}\to[A,U(X,f)]$. If the category $\mathcal{V}$ is ω-presentable then $\mathcal{V}/C$ is ω-presentable by **ref**. Moreover, an object $(X,f)\in\mathcal{X}/C$ is compact if and only if the object $X\in\mathcal{V}$ is compact. It follows that the forgetful functor $U:\mathcal{V}/C\to\mathcal{C}$ is confined. If the symmetric monoidal category $\mathcal{V}$ is confined, then the object $A\otimes(X,f):=(A\otimes X,\tau(A)\otimes f)$ of $\mathcal{V}/C$ is compact for every compact objects $A\in\mathcal{V}$ and $X\in\mathcal{V}$. Hence the $\mathcal{V}$-module $\mathcal{V}/C$ is confined. □

We saw in A.4.2 that if a confined symmetric monoidal closed category $\mathcal{V} = (\mathcal{V}, \otimes, I)$ is semi-cartesian then the category $\mathcal{V}/C$ has the structure of confined $\mathcal{V}$-module for any object $C \in \mathcal{V}$. Moreover, the forgetful functor $U : \mathcal{V}/C \to \mathcal{V}$ is a confined morphism of $\mathcal{V}$-module.

Lemma A.4.4. *With the hypothesis above, let $Z \to 1$ be a perfect object in $\mathcal{V}$. Then a map $u : (X, f) \to (Y, g)$ in $\mathcal{V}/C$ is Q-closed (resp. is a Q-equivalence) if and only if the map $u : X \to Y$ in $\mathcal{V}$ is P-close (resp. is a P-equivalence).*

Proof. Let $T : \mathcal{V} \to \mathcal{V}$ be the endo-functor defined by letting $T(X) = [Z, X]$ and let $t := [\tau(Z), -] : Id \to T$. Similarly, let $S : \mathcal{V}/C \to \mathcal{V}/C$ be the endo-functor defined by letting $S(X, f) = \{Z, (X, f)\}$ and let $s := \{\tau(Z), -\} : Id \to S$. The construction of $S(X, f) = \{Z, (X, f)\}$ in A.4.5 shows that the map $s(X, f) : (X, f) \to \{Z, (X, f)\}$ is the cartesian gap map of the following naturality square

$$\begin{array}{ccc} X & \xrightarrow{[\tau(Z),X]} & [Z,X] \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle [Z,f]} \\ C & \xrightarrow{[\tau(Z),C]} & [Z,C] \end{array} \tag{A.4.5}$$

Hence the following diagram commutes

$$\begin{array}{cccc} X & & & \\ & \searrow^{s(X,f)} & & \\ & S(X,f) & \xrightarrow{\gamma(X,f)} & [Z,X] \\ & \downarrow & & \downarrow{\scriptstyle [Z,f]} \\ & C & \xrightarrow{[\tau(Z),C]} & [Z,C] \end{array} \quad X \xrightarrow{[\tau(Z),X]} [Z,X], \quad X \xrightarrow{f} C \tag{A.4.6}$$

where $\gamma(X, f) := \gamma^U(Z, (X, f))$ is a coassembly map of the of the forgetful functor $U : \mathcal{V}/C \to \mathcal{V}$. Let us now show that a map $u : (X, f) \to (Y, g)$ in $\mathcal{V}/C$ is S-closed if and only the map $u : X \to Y$ in $\mathcal{V}$ is T-closed. The composite square of the following diagram is cartesian by **??**, since $[Z, f] = [Z, g][Z, u]$.

$$\begin{array}{ccc} \{Z,(X,f)\} & \xrightarrow{\gamma(X,f)} & [Z,X] \\ {\scriptstyle \{Z,u\}}\downarrow & & \downarrow{\scriptstyle [Z,u]} \\ \{Z,(Y,g)\} & \xrightarrow{\gamma(Y,g)} & [Z,Y] \\ \downarrow & & \downarrow{\scriptstyle [Z,g]} \\ C & \xrightarrow{[\tau(Z),C]} & [Z,C] \end{array} \tag{A.4.7}$$

The bottom square is also cartesian by **??**. Hence the top square of the diagram is cartesian by cancellation. In other words, the coassembly map $\gamma : US \to TU$ is a cartesian natural transformation. By A.4.6, the top and bottom triangles of the following diagram commutes:

$$\begin{array}{ccccc} & & \overset{tX}{\frown} & & \\ X & \xrightarrow{s(X,f)} & \{Z,(X,f)\} & \xrightarrow{\gamma(X,f)} & [Z,X] \\ \Big\downarrow{\scriptstyle u} & & \Big\downarrow{\scriptstyle \{Z,u\}} & & \Big\downarrow{\scriptstyle [Z,u]} \\ Y & \xrightarrow{s(Y,g)} & \{Z,(Y,g)\} & \xrightarrow{\gamma(Y,g)} & [Z,Y] \\ & & \underset{tY}{\smile} & & \end{array} \tag{A.4.8}$$

since $tX := [\tau(Z), X]$ and $tY := [\tau(Z), Y]$. But the right hand square of the diagram is cartesian by A.4.8. It follows that the left hand square is cartesian if and only if the composite square is cartesian. This shows that the map $u : (X, f) \to (Y, g)$ in $\mathcal{V}/C$ is S-closed if and only the map $u : X \to Y$ in $\mathcal{V}$ is T-closed. It follows by **??** that the map $u : (X, f) \to (Y, g)$ is Q-closed if and only the map $u : X \to Y$ in $\mathcal{V}$ is P-closed. It remains to show that the map $u : (X, f) \to (Y, g)$ is a Q-equivalence if and only the map $u : X \to Y$ in $\mathcal{V}$ is P-equivalence. If $\mathcal{L}$ is the class of P-equivalences in $\mathcal{V}$ and $\mathcal{R}$ is the class of P-closed maps, then the pair $(\mathcal{L}, \mathcal{R})$ is a factorisation system in $\mathcal{V}$ by **??**. Similarly, if $\mathcal{L}_C$ is the class of Q-equivalences in $\mathcal{V}/C$ and $\mathcal{R}_C$ is the class of Q-closed maps in $\mathcal{V}/C$, then the pair $(\mathcal{L}_C, \mathcal{R}_C)$ is a factorisation system in $\mathcal{V}/C$ by **??**. The pair $(U^{-1}\mathcal{L}, U^{-1}\mathcal{R})$ is also a factorisation system in $\mathcal{V}/C$ by **??**. But we saw above that $U^{-1}\mathcal{R} = \mathcal{R}_C$. It follows by orthogonality that $U^{-1}\mathcal{L} = \mathcal{L}_C$. □

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