Tidy maps and their localizations

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Abstract

Motivated by Weiss' construction of orthogonal calculus, we introduce the notion of a tidy map in a symmetric monoidal cocomplete category and show how it provides a localization which is both symmetric monoidal and left exact. This allows us to build Goodwillie's calculus and Weiss' orthogonal calculus within a unified framework. We further show that the orthogonal tower is an instance of a completion tower in the sense of our previous paper. As an application we obtain Blakers-Massey-type theorems for orthogonal calculus.

Contents

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1 Introduction

Our goals for this paper are the following:

- We give a general setup for Weiss' ideas in [Weiss95] and [Weiss98] on how to construct reflectors. The main Theorem 4.2.16 shows how to construct a symmetric monoidal left exact localization from a tidy map in a symmetric monoidal category. Theorem 8.1.5 is a corresponding version for module categories.
- We show in Theorems 6.2.6, 6.3.4, 6.4.2 and 8.2.1 that Goodwillie's reflector P_n onto n-excisive functors can be constructed in this way.
- As to be expected, Weiss' construction of orthogonal calculus can be rephrased in terms of our framework. This is stated in Theorem 7.2.5.
- We deduce in Theorem 7.3.4 that orthogonal calculus is a special case of a completion tower in the sense of [ABFJ24b]. As a consequence we derive Blakers-Massey-type Theorems 7.4.1 and 7.4.2 for orthogonal calculus.

Now Goodwillie's homotopy functor calculus and Weiss' orthogonal caluclus have the same construction. In fact, there are now two ways to construct them in the same way: the first one is via completion towers [ABFJ24b], an entirely topos-theoretic concept that has the additional benefit of showing how cubical diagrams emerge from acyclic products of congruences (i.e. pushout products of maps). The second way is the setup presented here via tidy maps which applies outside of topos theory and yields that the localization is symmetric monoidal.

Let us now summarize the paper section by section.

In Section 2 we gather the necessary basics about enriched higher category 2.1 and on ω-compact objects, filtered colimits and ω-presentability 2.2 in a symmetric monoidal $(\infty, 1)$-category $\mathcal{V}$.

We will immediately drop the $(\infty, 1)$-decoration as soon as possible, since it becomes our standard assumption that we are always working in the $(\infty, 1)$-categorical world unless explicitely specified otherwise.

The defining property of compact objects is formulated in terms of mapping spaces, i.e. the enrichment over the category $\mathcal{S}$ of spaces (∞-groupoids). But we need compact objects to behave in the same way with respect to inner homs and cotensors. This forces in $\mathcal{V}$ a compatibility between the monoidal structure and the presentability assumption. This is introduced in Section 3 where we coin the term *confined symmetric monoidal category*. Here we also define confined and docile functors, two classes of, in a certain way, well-behaved functors.

In Section 4 we develop the theory of *confined symmetric monoidal categories* and *tidy maps* generalizing Weiss' construction of a reflection in [Weiss95, Weiss98].

Definition 3.2.1: A symmetric monoidal closed category is confined if it is ω-presentable its unit object is compact and the tensor of two compact objects is compact.

Here is an example: for a small symmetric monoidal category $\mathcal{C}$, the category $\mathcal{V} = \mathrm{Fun}(\mathcal{C}, \mathcal{S})$ of functors to spaces equipped with the Day convolution product is confined symmetric monoidal. If $z : Z \to \mathbb{1}$ is a map in $\mathcal{V}$ to the unit, we study the cotensor $T = [Z, -]$ as an endofunctor of $\mathcal{V}$. Via the associated natural transformation $t = [z, -] : \mathrm{Id} \to T$ we can define an endofunctor

$$P = \operatorname*{colim}_{k \in \mathbb{N}} \left(\dots \to T^k \xrightarrow{tT^k} T^{k+1} \to \dots \right)$$

of $\mathcal{V}$ with a natural transformation $p : \mathrm{Id} \to P$.

Definition 4.1.3: We call the map $z : Z \to \mathbb{1}$ tidy, if Z is compact and the induced map $P(z) : P(Z) \to P(\mathbb{1})$ is an isomorphism.

Main Theorem 4.2.16: If $z : Z \to \mathbb{1}$ is a tidy map in a confined symmetric monoidal category $\mathcal{V}$, then the associated endofunctor P is a confined symmetric monoidal left exact localization of $\mathcal{V}$.

In the short Section 5 we remind ourselves what pushout prodcuts are, define the fiberwise join and gather a few facts.

We equip in Section 6 the category Fin of finite spaces with the join product. Then the category $\mathrm{Fun}(\mathrm{Fin}, \mathcal{S})$ with the Day convolution product becomes a confined symmetric monoidal category. If $\mathrm{Id} : \mathrm{Fin} \to \mathcal{S}$ denotes the inclusion, then the map $\mathrm{Id}^{\star n+1} \to 1$ is tidy, as one can see from estimating its connectivity, and yields the reflector P_n in the Goodwillie tower.

In Section 7 we rephrase Weiss' construction of the orthogonal tower in terms of Section 4. The direct sum in the category $\mathcal{J}$ of finite dimensional Euclidian vector space yields a Day convolution product on $\mathrm{Fun}(\mathcal{J}, \mathcal{S})$. If $\mathrm{Sph}(-) : \mathcal{J} \to \mathcal{S}$ denotes the unit sphere functor, then $\mathrm{Sph}(-)^{\star n+1} \to 1$ is a tidy map (Proposition 7.2.3) by Weiss' connectivity estimates and yields the n-the stage in the orthogonal tower.

The second part of Section 7 is concerned with the proof of Theorem 7.3.4 stating that the orthogonal tower is a completion tower in the sense of [ABFJ24b]. We use in an essential way a "Stiefel combinatorics" result by Weiss 7.1.4 that identifies the map $\big(\mathrm{Sph}(-)^{\star n+1} \to 1\big) \otimes \mathrm{St}(\mathbb{R}^k, -)$ as a fiberwise join power. We close the section with Blakers-Massey theorems 7.4.1 and 7.4.2 that hold for any completion tower, thus in particular for the orthogonal tower.

In Section 8 we describe the notion of confined module categories over $\mathcal{V}$. Since tidy maps in $\mathcal{V}$ act via cotensors, they also act on any confined $\mathcal{V}$-category $\mathcal{M}$ and induce confined symmetric monoidal left exact localizations of $\mathcal{M}$. This is Theorem 8.1.5. As an application we observe that Fin acts via the join on any small category that is finitely cocomplete and has a terminal objects. Thus, as stated in Theorem 8.2.1, the tidy maps $\mathrm{Id}^{\star n+1} \to 1$ in $\mathrm{Fun}(\mathrm{Fin}, \mathcal{S})$ induce Goodwillie's P_n on $\mathrm{Fun}(\mathcal{M}, \mathcal{S})$. Finally we prove in Theorem 8.3.1, that tidy maps are transported via confined functors.

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2 Tools from category theory

2.1 Enriched higher category theory

Symmetric monoidal structures in the context of $(\infty, 1)$-category theory are develloped in Lurie [Lur17] Definition 2.0.0.7 and Remarks 2.1.2.18 and 2.1.2.19, and with an alternative approach in Gepner-Haugseng [GH19]. The theory of modules over symmetric monoidal $(\infty, 1)$-categories (and more general notions) are develloped in the work of Heine, see in particular [Hei] and [Hei23]. We are indebted to Hadrian Heine for explaining to us the intricacies. In this article all categories are $(\infty, 1)$-categories unless we explicitly specify otherwise.

Let $\mathcal{V}$ be a symmetric monoidal closed category. If $\mathcal{M}$ is a $\mathcal{V}$-category, we write

$$[-,-]_{\mathcal{V}} \quad \text{or simply} \quad [-,-] : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{V}$$

for the $\mathcal{V}$-enrichment of $\mathcal{M}$. A *closed $\mathcal{V}$-module category* is a $\mathcal{V}$-enriched category that is tensored and cotensored over $\mathcal{V}$. In this case we write (again)

$$- \otimes - : \mathcal{V} \times \mathcal{M} \to \mathcal{M}$$

for the tensor and denote the cotensor by

$$\{-,-\} : \mathcal{V}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{M}.$$

Then we have natural isomorphisms

$$[A \otimes M, N]_{\mathcal{V}} \cong [A, [M, N]_{\mathcal{V}}] \cong [M, \{A, N\}]_{\mathcal{V}}$$

for A in $\mathcal{V}$ and M, N in $\mathcal{M}$.

Lemma 2.1.1. *Let $A \to B$ be a morphism in $\mathcal{V}$. Let $\mathcal{M}$ be a closed $\mathcal{V}$-module. Then the induced natural transformation $\{B, -\} \to \{A, -\}$ of cotensors (of inner homs $[B, -] \to [A, -]$) is a $\mathcal{V}$-enriched natural transformation of $\mathcal{V}$-enriched functors.*

Proof. It is shown in [Hei, Proposition 2.116] that there is an equivalence of categories between enriched left adjoints and enriched right adjoints given by mapping them to each other. Now one observes that tensoring with a fixed object $A \otimes -$ is (lax) monoidal, and hence $\mathcal{V}$-enriched. Therefore the right adjoint cotensor $\{A, -\}$ (or $[A, -]$) is $\mathcal{V}$-enriched and the map $A \to B$, that induces a $\mathcal{V}$-enriched natural transformation of tensors, in turn induces a $\mathcal{V}$-enriched natural transformations of cotensors. □

Let $\mathrm{Fun}_{\mathcal{V}}(\mathcal{C}, \mathcal{D})$ denote the category of $\mathcal{V}$-enriched functors between $\mathcal{V}$-closed modules with morphisms given by $\mathcal{V}$-enriched natural transformations.

Proposition 2.1.2. *Let $\mathcal{V}$ be a symmetric monoidal closed category and $\mathcal{C}$ and $\mathcal{D}$ two closed $\mathcal{V}$-modules. Let $u : \mathrm{Fun}_{\mathcal{V}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ be the functor forgetting the $\mathcal{V}$-enrichment.*

(i) *The functor u is conservative.*

(ii) *If $\mathcal{D}$ is complete, then the category $\mathrm{Fun}_{\mathcal{V}}(\mathcal{C}, \mathcal{D})$ is complete and the functor u is continuous.*

(iii) *If $\mathcal{D}$ is cocomplete, then the category $\mathrm{Fun}_{\mathcal{V}}(\mathcal{C}, \mathcal{D})$ is cocomplete and the functor $\mathrm{Fun}_{\mathcal{V}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ forgetting the enrichment is cocontinuous.*

Under all these assumptions it follows that in $\mathrm{Fun}_{\mathcal{V}}(\mathcal{C}, \mathcal{D})$ limits and colimits exist and are computed objectwise.

Proof. Heine's results in [Hei] are proved in a more general setting of (weakly) left, right and bienriched $(\infty, 1)$-categories over two non-symmetric ∞-operads $\mathcal{V}$ and $\mathcal{W}$. It is shown that $(\infty, 1)$-categories left enriched in any two non-symmetric ∞-operads $\mathcal{V}$ and $\mathcal{W}$ form an $(\infty, 2)$-category, which is denoted by ${}_{\mathcal{V}}\mathrm{LEnr}_{\mathcal{W}}$ by [Hei, Remark 2.136]. We can specialize to our setting: if $\mathcal{V}$ is a symmetric monoidal $(\infty, 1)$-category and $\mathcal{W}$ is the initial ∞-operad, whose space of colors is empty and which is denoted by $\varnothing$, then left $(\mathcal{V}, \varnothing)$-enriched $(\infty, 1)$-categories are precisely $\mathcal{V}$-enriched $(\infty, 1)$-categories by [Hei, Remark 2.140] and the $(\infty, 2)$-category ${}_{\mathcal{V}}\mathrm{LEnr}_{\varnothing}$ is the $(\infty, 2)$-category of $\mathcal{V}$-enriched $(\infty, 1)$-categories by the same remark.

In [Hei, Remark 2.136] the functor $U : {}_{\mathcal{V}}\mathrm{LEnr}_{\mathcal{W}} \to \mathrm{Cat}_{(\infty,1)}$ forgetting the enrichment is constructed as a functor of $(\infty, 2)$-categories. This forgetful functor U is locally conservative, i.e. induces on morphism-$(\infty, 1)$-categories conservative functors by [Hei, Corollary 2.138]. Therefore our functor u is conservative proving (i).

Theorem 4.43.(2) of [Hei] applied to enrichment states that the forgetful functor u is monadic if $\mathcal{D}$ is (left) tensored over $\mathcal{V}$ and has colimits preserved by the left action. The latter is covered since in the closed $\mathcal{V}$-module $\mathcal{D}$ the tensor has a right adjoint. This implies (ii). Since limits in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ are computed objectwise, the same now holds for the category $\mathrm{Fun}_{\mathcal{V}}(\mathcal{C}, \mathcal{D})$.

For (iii) we can specialize [Hei23, Lemma 3.74(1)] and conclude that the category $\mathrm{Fun}_{\mathcal{V}}(\mathcal{C}, \mathcal{D})$ is cocomplete and for every object X in $\mathcal{C}$ the evaluation functor

$$\mathrm{ev}_X : \mathrm{Fun}_{\mathcal{V}}(\mathcal{C}, \mathcal{D}) \to \mathcal{D} \quad , \quad F \mapsto \mathrm{ev}_X(F) = F(X)$$

preserves colimits. We deduce that the functor u is cocontinuous and that colimits in $\mathrm{Fun}_{\mathcal{V}}(\mathcal{C}, \mathcal{D})$ are computed objectwise. □

We would like to stress the importance of the enriched theory for our work since it is essential in the proof of Proposition 4.1.1.

2.2 ω-presentable categories

Let $\mathcal{E}$ be a category with filtered colimits. An object K in $\mathcal{E}$ is called *compact*, or more accurately *ω-compact*, if the functor $\mathrm{map}(K, -) : \mathcal{E} \to \mathcal{S}$ preserves filtered colimits. Here $\mathcal{S}$ is the category of spaces, aka. ∞-groupoids. We denote the full subcategory of compact objects of $\mathcal{E}$ by $\mathsf{c}(\mathcal{E})$. This subcategory is closed under finite colimits and retracts.

Definition 2.2.1 (Definition 20.4.1.7. [Lur18])**.** A small full subcategory $\mathcal{D}$ of a category $\mathcal{E}$ is said to be *dense* if the identity functor of $\mathcal{E}$ can be given as a left Kan extension of the inclusion $\mathcal{D} \subset \mathcal{E}$.

Lemma 2.2.2. *Let $\mathcal{D}$ be a small full subcategory of $\mathcal{E}$. For an object E in $\mathcal{E}$ let $\mathcal{D}/E$ denote the slice category. The following are equivalent:*

(i) *The subcategory $\mathcal{D}$ is dense in $\mathcal{E}$.*

(ii) *For every object E in $\mathcal{E}$ we have $E \cong \operatorname*{colim}_{\mathcal{D}/E} D$.*

(iii) *The restricted Yoneda functor $\mathcal{E} \to [\mathcal{D}^{\mathrm{op}}, \mathcal{S}]$ is fully faithful.*

Proof. This is proved in Remarks 20.4.1.2 and 20.4.1.5. [Lur18]. ☐

Lemma 2.2.3. *Let $\mathcal{D} \subset \mathcal{E}$ be a small dense full subcategory of a category $\mathcal{E}$. Then a morphism $f : X \to Y$ in $\mathcal{E}$ is invertible if and only if $\mathrm{map}(D, f) : \mathrm{map}(D, X) \to \mathrm{map}(D, Y)$ is invertible for every object D in $\mathcal{D}$.*

Proof. The (restricted) Yoneda functor $\mathcal{E} \to \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S})$ is fully faithful, since the subcategory $\mathcal{D}$ is dense. But any fully faithful functor is conservative. ☐

Definition 2.2.4. A category $\mathcal{E}$ is said to be *ω-presentable* if it is cocomplete and its full subcategory of compact objects $\mathsf{c}(\mathcal{E}) \subset \mathcal{E}$ is small and dense in $\mathcal{E}$.

If a category is ω-presentable it is complete and cocomplete.

Example 2.2.5. Let $\mathcal{D}$ be a small category and consider the category $\mathrm{Fun}(\mathcal{D}, \mathcal{S})$. In it every representable functor is compact and any functor is a colimit of representable ones. Thus the category $\mathrm{Fun}(\mathcal{D}, \mathcal{S})$ is ω-presentable. By definition a functor is finitely presentable if it is a finite colimit of representable functors. It is compact if and only if it is a retract of a finitely presentable functor.

Lemma 2.2.6. *If the category $\mathcal{E}$ is ω-presentable, then every object in $\mathcal{E}$ is a filtered colimit of compact objects.*

Proof. Since $\mathcal{E}$ is ω-presentable, for an arbitrary object E in $\mathcal{E}$, we have $E \cong \mathrm{colim}_{\mathsf{c}(\mathcal{E})/E} D$ by Lemma 2.2.2. But compact objects are closed under finite colimits and colimits in a slice category since they agree with colimits of the ambient category. Thus, the slice category $\mathsf{c}(\mathcal{E})/E$ is filtered. ☐

Consider a natural transformation $f : A \to B$ between two increasing sequences

$$\begin{tikzcd}
 & & & & & & \mathrm{colim}\, A \arrow[ddd, "\mathrm{colim}\, f"] \\
A_0 \arrow[r, "a_0"] \arrow[d, "f_0"'] \arrow[urrrrrr, bend left, "\alpha_0"] & A_1 \arrow[r, "a_1"] \arrow[d, "f_1"'] \arrow[urrrrr, bend left, "\alpha_1"] & A_2 \arrow[r, "a_2"] \arrow[d, "f_2"'] \arrow[urrrr, bend left, "\alpha_2"] & A_3 \arrow[r] \arrow[d, "f_3"'] \arrow[urrr, bend left, "\alpha_3"] & \cdots & & \\
B_0 \arrow[r, "b_0"'] \arrow[drrrrrr, bend right, "\beta_0"'] & B_1 \arrow[r, "b_1"'] \arrow[drrrrr, bend right, "\beta_1"'] & B_2 \arrow[r, "b_2"'] \arrow[drrrr, bend right, "\beta_2"'] & B_3 \arrow[r] \arrow[drrr, bend right, "\beta_3"'] & \cdots & & \\
 & & & & & & \mathrm{colim}\, B
\end{tikzcd}$$

in $\mathcal{E}$ and write $\alpha_n : A_n \to \operatorname{colim} A$ and $\beta_n : B_n \to \operatorname{colim} B$ for the conical maps.

Lemma 2.2.7. *Let $\mathcal{E}$ be an ω-presentable category and let $f : A \to B$ be a natural transformation between two sequences in $\mathcal{E}$. If the square*

$$\begin{tikzcd} A_n \arrow[r, "\alpha_n"] \arrow[d, "f_n"'] & \operatorname{colim}(A) \arrow[d, "\operatorname{colim}(f)"] \\ B_n \arrow[r, "\beta_n"] & \operatorname{colim}(B) \end{tikzcd}$$

has a diagonal filler for every $n \geqslant 0$, then the map $\operatorname{colim}(f)$ is invertible.

Proof. Consider first the case $\mathcal{E} = \mathcal{S}$. By the Whitehead Theorem, the map $\operatorname{colim}(f)$ is invertible if and only if every square

$$\begin{tikzcd} S^n \arrow[r, "x"] \arrow[d] & \operatorname{colim}(A) \arrow[d, "\operatorname{colim}(f)"] \\ 1 \arrow[r, "y"] & \operatorname{colim}(B) \end{tikzcd}$$

has a diagonal filler. But this square can be factored as follows:

$$\begin{tikzcd} S^n \arrow[r, "x'"] \arrow[d] & A_k \arrow[r, "\alpha_k"] \arrow[d, "f_k"'] & \operatorname{colim}(A) \arrow[d, "\operatorname{colim}(f)"] \\ 1 \arrow[r, "y'"] & B_k \arrow[r, "\beta_k"] & \operatorname{colim}(B) \end{tikzcd}$$

for some $k \geqslant 0$, since $S^n \to 1$ is a map between compact spaces. The composite square has a diagonal filler since the right hand square has a diagonal filler by the hypothesis. This proves the lemma in the the case where $\mathcal{E} = \mathcal{S}$.

Let us now return to the general case of an ω-presentable category $\mathcal{E}$. By Lemma 2.2.3 it suffices to show that $\operatorname{map}(K, \operatorname{colim}(f))$ is invertible for every compact object K in $\mathcal{E}$. But the functor $\operatorname{map}(K, -) : \mathcal{E} \to \mathcal{S}$ preserves filtered colimits, since K is compact. Hence $\operatorname{map}(K, \operatorname{colim}(f))$ is the colimit of the natural transformation $\operatorname{map}(K, f) : \operatorname{map}(K, A) \to \operatorname{map}(K, B)$,

$$\begin{tikzcd} \operatorname{map}(K, A_n) \arrow[r, "\operatorname{map}(K,\alpha_n)"] \arrow[d, "\operatorname{map}(K,f_n)"'] & \operatorname{map}(K, \operatorname{colim}(A)) \arrow[d, "\operatorname{map}(K,\operatorname{colim}(f))"] \\ \operatorname{map}(K, B_n) \arrow[r, "\operatorname{map}(K,\beta_n)"] & \operatorname{map}(K, \operatorname{colim}(B)) \end{tikzcd}$$

For every $n \geqslant 0$ this square obtains an induced diagonal filler from the square above. Hence the map $\operatorname{map}(K, \operatorname{colim}(f))$ is invertible by the first part of the proof. It follows that $\operatorname{colim}(f)$ is invertible. □

Lemma 2.2.8. *In an ω-presentable category filtered colimits are left exact.*

Proof. Let $\mathcal{F}$ be a filtered category and $\mathcal{D}$ be a finite category. The claim is that for any functor $G : \mathcal{F} \times \mathcal{D} \to \mathcal{V}$ the canonical map

$$A := \operatorname*{colim}_{\mathcal{F}} \lim_{\mathcal{D}} G(f, d) \xrightarrow{\sigma} \lim_{\mathcal{D}} \operatorname*{colim}_{\mathcal{F}} G(f, d) =: B$$

is an isomorphism. [Lur09, Proposition 5.3.3.3] proves this for $\mathcal{V} = \mathcal{S}$. The claim will follow from Lemma 2.2.3 if we show that $\operatorname{map}(K, \sigma) : \operatorname{map}(K, A) \to \operatorname{map}(K, B)$ is an isomorphism for arbitrary compact K in $\mathcal{V}$. But both sides turn out to be canonically isomorphic to the space $\operatorname{colim}_{\mathcal{F}} \lim_{\mathcal{D}} \operatorname{map}(K, G(f, d))$. □

3 Confined categories

3.1 Confined functors

Definition 3.1.1. A functor is *confined* if it is cocontinuous and takes compact objects to compact objects.

Remark 3.1.2. The category of ω-presentable categories and confined functors is equivalent to that of idempotent complete and finitely cocomplete categories and functors preserving finite colimits. The equivalence is given by extracting the compact objects in one direction and by Ind-completion in the other.

Recall from [Lur09, Corollary 5.5.2.9] that every cocontinuous functor between presentable categories $\phi : \mathcal{E} \to \mathcal{F}$ has a right adjoint $\phi_\star : \mathcal{F} \to \mathcal{E}$.

Proposition 3.1.3. *Let $L : \mathcal{E} \to \mathcal{F}$ be a cocontinuous functor between presentable categories and let $R : \mathcal{F} \to \mathcal{E}$ its right adjoint. Let $\mathcal{D} \subset \mathsf{c}(\mathcal{E})$ be a small full subcategory of the compact objects of $\mathcal{E}$ that is dense in $\mathcal{E}$. Then the following are equivalent:*

(i) *The functor L is confined.*

(ii) *For every D in $\mathcal{D}$ the object $L(D)$ is compact.*

(iii) *The functor R preserves filtered colimits.*

Proof. Obviously (i) implies (ii). Now assume (ii) and let us show that R preserves filtered colimits. By Lemma 2.2.3 it suffices to show that $\mathrm{map}(D, R(-))$ does so for all D in $\mathcal{D}$, since $\mathcal{D}$ is dense in $\mathcal{E}$. But this is true since $\mathrm{map}(D, R(-)) \cong \mathrm{map}(L(D), -)$ and $L(D)$ is compact by assumption. So (ii) implies (iii).

Suppose that R preserves filtered colimits and let K in $\mathcal{E}$ be compact. Then the functor $\mathrm{map}(LK, -) \cong \mathrm{map}(K, R(-)) : \mathcal{F} \to \mathcal{S}$ preserves filtered colimits. Hence LK is compact. So (iii) implies (i). ☐

Let $\mathcal{E}$ be a presentable category. If $\mathcal{A}$ is a small category, then every functor $\phi : \mathcal{A}^{\mathrm{op}} \to \mathcal{E}$ has a left Kan extension $\phi_! : \mathrm{Fun}(\mathcal{A}, \mathcal{E}) \to \mathcal{E}$ along the Yoneda functor $y : \mathcal{A}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{A}, \mathcal{E})$.

Corollary 3.1.4. *If $\phi(\mathcal{A}^{\mathrm{op}}) \subset \mathsf{c}(\mathcal{E})$, then the functor $\phi_! : \mathrm{Fun}(\mathcal{A}, \mathcal{E}) \to \mathcal{E}$ is confined.*

Proof. The subcategory of representable functors is dense and every object in it is compact. Moreover the functor $\phi_!$ takes every representable functor to a compact object in $\mathcal{E}$, since $\phi_!(y(a)) \cong \phi(a)$ is compact for every a in $\mathcal{A}$, y the Yoneda functor. Then Proposition 3.1.3 implies that $\phi_!$ is confined. ☐

Example 3.1.5. If $\phi : \mathcal{A} \to \mathcal{B}$ is functor between small categories, then the functor $\phi^* : \mathrm{Fun}(\mathcal{B}, \mathcal{S}) \to \mathrm{Fun}(\mathcal{A}, \mathcal{S})$ has a left adjoint $\phi_! : \mathrm{Fun}(\mathcal{A}, \mathcal{S}) \to \mathrm{Fun}(\mathcal{B}, \mathcal{S})$ and $\phi_!$ is confined.

3.2 Confined symmetric monoidal categories

Definition 3.2.1. We will say that a symmetric monoidal category is *confined* if it is ω-presentable, the monoidal product of two compact objects is compact, and if the unit object is compact.

Note that ω-presentable, and hence confined symmetric monoidal categories are automatically closed. The assumption that the monoidal product of two compact objects is compact enters in the proof of Lemma 3.4.3. Compactness of the unit seemed natural to ask for, but is never used in this article.

Remark 3.2.2. The category of confined symmetric monoidal categories and confined symmetric monoidal functors is equivalent to the category of symmetric monoids in ω-presentable categories and confined functors. Recall from Remark 3.1.2 that the latter category is equivalent to idempotent complete and finitely cocomplete categories and functors preserving finite colimits. Therefore, the category of confined symmetric monoidal categories is equivalent to the category of symmetric monoids in idempotent complete and finitely cocomplete categories. The equivalence is given by extracting the compact objects in one direction and by Ind-completion in the other.

Lemma 3.2.3. *Let $\mathcal{V}$ be a symmetric monoidal category. Suppose that $\mathcal{V}$ is ω-presentable, and that $\mathcal{D} \subset \mathcal{V}$ is a dense subcategory of compact objects of $\mathcal{V}$. Assume also that $\mathbb{1}$ is compact and that for all X, Y in $\mathcal{D}$ the tensor $X \otimes Y$ is compact. Then $\mathcal{V}$ is confined.*

Proof. For any object A the functor $A \otimes -$ is cocontinuous, since it has a right adjoint $[A, -]$. If A is in $\mathcal{D}$, then $A \otimes \mathcal{D} \subset \mathsf{c}(\mathcal{V})$, since $\mathcal{D} \otimes \mathcal{D} \subset \mathsf{c}(\mathcal{V})$ by hypothesis. Now it follows from Proposition 3.1.3 that the functor $A \otimes - : \mathcal{V} \to \mathcal{V}$ is confined for every A in $\mathcal{D}$. We need to extend this to any A in $\mathsf{c}(\mathcal{V})$.

We have just proved that $\mathcal{D} \otimes B \subset \mathsf{c}(\mathcal{V})$ for every object B in $\mathsf{c}(\mathcal{V})$. Hence the functor $- \otimes B : \mathcal{V} \to \mathcal{V}$ is confined for every object B in $\mathsf{c}(\mathcal{V})$ by Proposition 3.1.3. Thus, $\mathsf{c}(\mathcal{V}) \otimes B \subset \mathsf{c}(\mathcal{V})$ for every object $B \in \mathsf{c}(\mathcal{V})$. Thus, $\mathsf{c}(\mathcal{V}) \otimes \mathsf{c}(\mathcal{V}) \subset \mathsf{c}(\mathcal{V})$ and $\mathcal{V}$ is a confined symmetric monoidal category. ☐

Examples 3.2.4. A short list of examples follows.

(i) The category of spaces equipped with the cartesian monoidal structure is confined.

(ii) The category of pointed spaces with the smash product is confined.

(iii) The category of spectra equipped with the smash product is confined.

(iv) The category of small categories is cartesian closed symmetric monoidal and confined.

(v) The category of simplicial spaces is cartesian closed and confined. The subcategory of representable functors satisfies the conditions of Lemma 3.2.3.

(vi) If $\mathbb{V}$ is a small symmetric monoidal category, then the functor category $\mathrm{Fun}(\mathbb{V}, \mathcal{S})$ equipped with the Day convolution product is a $(\infty, 1)$-symmetric monoidal closed category, see [Lur17, Example 2.2.6.17, Corollary 4.8.1.12, Remark 4.8.1.13] or [Gla16]. It is confined by Lemma 3.2.3 using the representable functors as the small dense subcategory $\mathcal{D}$.

3.3 Confined $\mathcal{V}$-modules

Definition 3.3.1. Let $\mathcal{M}$ be a closed $\mathcal{V}$-module over a confined symmetric monoidal category $\mathcal{V}$. We will say that $\mathcal{M}$ is a *confined $\mathcal{V}$-module* if the category $\mathcal{M}$ is ω-presentable and the tensor product $A \otimes M$ of a compact object M in $\mathcal{M}$ with a compact object $A \in \mathcal{V}$ is compact.

Example 3.3.2. Every confined symmetric monoidal category $\mathcal{V}$ is confined as a closed $\mathcal{V}$-module over itself.

Example 3.3.3. If $\phi : \mathcal{V} \to \mathcal{E}$ is a cocontinuous symmetric monoidal functor between presentable symmetric monoidal categories, then the category $\mathcal{E}$ has the structure of a closed $\mathcal{V}$-module by defining

$$[E, F]_{\mathcal{V}} := \phi_*[E, F]_{\mathcal{E}}, \quad A \otimes E := \phi(A) \otimes E \quad \text{and} \quad \{A, F\} := [\phi(A), F]_{\mathcal{E}}$$

for E, F in $\mathcal{E}$ and A in $\mathcal{V}$. Here ϕ_* is the right adjoint to ϕ. If $\phi : \mathcal{V} \to \mathcal{E}$ is a confined symmetric monoidal functor between confined symmetric monoidal categories, then $\mathcal{E}$ becomes a confined $\mathcal{V}$-module.

Example 3.3.4. Let $\sigma : \mathbb{V} \to \mathbb{E}$ be a symmetric monoidal functor between small symmetric monoidal categories. Then the functor $\sigma^* : \mathrm{Fun}(\mathbb{E}, \mathcal{S}) \to \mathrm{Fun}(\mathbb{V}, \mathcal{S})$ has a left adjoint $\sigma_! : \mathrm{Fun}(\mathbb{V}, \mathcal{S}) \to \mathrm{Fun}(\mathbb{E}, \mathcal{S})$ obtained by left Kan extension along σ. With respect to the Day convolution products on both functor categories, $\sigma_!$ is symmetric monoidal and confined by Example 3.1.5. By Example 3.3.3 the category $\mathrm{Fun}(\mathbb{E}, \mathcal{S})$ becomes a confined $\mathrm{Fun}(\mathbb{V}, \mathcal{S})$-module.

Example 3.3.5. Let $\mathbb{V}$ be a small symmetric monoidal category and $\mathbb{M}$ be a category equipped with an action

$$\oplus : \mathbb{V} \times \mathbb{M} \to \mathbb{M}$$

which is coherently associative and unital. Then the functor category $\mathcal{M} = \mathrm{Fun}(\mathbb{M}, \mathcal{S})$ has the structure of a closed module over the symmetric monoidal closed category $\mathcal{V} = \mathrm{Fun}(\mathbb{V}, \mathcal{S})$. For F in $\mathcal{V}$ and M in $\mathcal{M}$ the *tensor product* $F \otimes M : \mathbb{M} \to \mathcal{S}$ is calculated by the formula

$$(F \otimes M)(n) = \int^{a \in \mathbb{V}} \int^{m \in \mathbb{M}} F(a) \times M(m) \times \mathrm{map}(a \oplus m, n)$$

for every $n \in \mathbb{M}$. If write $R^x = \mathrm{map}(x,-)$ for covariant representable functors, then $R^a \otimes R^m = R^{a\oplus m}$ for every a in $\mathbb{V}$ and m in $\mathbb{V}$. The enrichment $[M,N]_{\mathcal{V}}$ between $M, N \in \mathcal{M}$ is calculated by the formula

$$[M,N]_{\mathcal{V}}(a) = \mathrm{nat}(M, N(a \oplus -))$$

for every a in $\mathbb{V}$. In particular, $[R^m, N]_{\mathcal{V}} = N(- \oplus m) : \mathbb{V} \to \mathcal{S}$. The *cotensor product* $\{F, M\} : \mathbb{M} \to \mathcal{S}$ is calculated by the formula

$$\{F, M\}(m) = \mathrm{nat}(F, M(- \oplus m))$$

for every m in $\mathbb{M}$. In particular, $\{R^a, M\} = M(a \oplus -) : \mathcal{M} \to \mathcal{S}$ for every a in $\mathbb{V}$. The formula $R^a \otimes R^m = R^{a\oplus m}$ shows that the $\mathcal{V}$-module $\mathcal{M}$ is confined.

3.4 Docile functors

Definition 3.4.1. A $\mathcal{V}$-functor $F : \mathcal{V} \to \mathcal{V}$ preserves *compact cotensors* if the coassembly map $\gamma(Z, X) : F[Z, X] \to [Z, FX]$ is invertible for all X and every compact Z.

Compact cotensors can be regarded as a $\mathcal{V}$-enriched version of finite limits and should therefore be taken into account in an enriched version of left exactness.

Definition 3.4.2. We say that a $\mathcal{V}$-functor is

(i) $\mathcal{V}$-*left exact* if it preserves finite limits and compact cotensors.

(ii) *docile* if it is $\mathcal{V}$-left exact and preserves filtered colimits.

The reason for introducing confined categories is to prove the following

Lemma 3.4.3. *Let $\mathcal{V}$ be a confined symmetric monoidal category and $\mathcal{M}$ a confined $\mathcal{V}$-module. If C in $\mathcal{V}$ is compact, then the functors $[C,-] : \mathcal{V} \to \mathcal{V}$ and $\{C,-\} : \mathcal{M} \to \mathcal{M}$ are docile.*

Proof. The functors $[C,-]$ and $\{C,-\}$ are $\mathcal{V}$-enriched by Lemma 2.1.1 and preserve all (not just finite) limits. The functor $\{C,-\}$ also preserves all cotensors (not just compact ones): for all B in $\mathcal{V}$ the coassembly map $\gamma(B,X) : \{B, \{C, X\}\} \to \{C, \{B, X\}\}$ is composed of the natural isomorphisms

$$\gamma(B,X) : \{B, \{C, X\}\} \xrightarrow{\cong} \{C \otimes B, X\} \xrightarrow{\cong} \{C, \{B, X\}\}.$$

Hence the map $\gamma(B, X)$ is invertible for all objects B, C in $\mathcal{V}$ and X in $\mathcal{M}$. And similarly for $[C,-]$.

Since $\mathcal{V}$ and $\mathcal{M}$ are assumed to be confined, the tensor $C\otimes-$ is a confined functor, both as endofunctor of $\mathcal{V}$ and of $\mathcal{M}$. So by Proposition 3.1.3 the right adjoints $[C,-]$ and $\{C,-\}$ preserves filtered colimits. ☐

Let $\mathrm{Fun}_{\mathcal{V}}(\mathcal{C}, \mathcal{D})$ denote the category of $\mathcal{V}$-enriched functors between $\mathcal{V}$-closed modules and morphisms given by $\mathcal{V}$-enriched natural transformations.

Proposition 3.4.4. *Let $\mathcal{V}$ be a confined symmetric monoidal category and $\mathcal{M}$ a confined $\mathcal{V}$-module. Then the full subcategory of docile functors in $\mathrm{Fun}_{\mathcal{V}}(\mathcal{M}, \mathcal{M})$ is closed under finite limits, filtered colimits and composition.*

Proof. We need to show that a filtered colimit of docile functors is left exact and preserves compact cotensors. We start by showing left exactness. Let $\mathcal{F}$ be a filtered category and $\mathcal{D}$ be a finite category and G a $\mathcal{F} \times \mathcal{D}$-diagram of docile functors. We need to show that the canonical map

$$\operatorname*{colim}_{\mathcal{F}} \lim_{\mathcal{D}} G(f,d) \xrightarrow{\sigma} \lim_{\mathcal{D}} \operatorname*{colim}_{\mathcal{F}} G(f,d)$$

is an isomorphism in $\mathrm{Fun}_{\mathcal{V}}(\mathcal{M}, \mathcal{M})$. By Proposition 2.1.2 the forgetful functor to $\mathrm{Fun}(\mathcal{M}, \mathcal{M})$ preserves limits and colimits and sends σ to the corresponding canonical map σ' in $\mathrm{Fun}(\mathcal{M}, \mathcal{M})$. Since colimits are computed objectwise in $\mathrm{Fun}(\mathcal{M}, \mathcal{M})$, and filtered colimits are left exact by Lemma 2.2.8, the map σ' is invertible. But the functor forgetting the enrichment is conservative by Proposition 2.1.2, it follows that σ is invertible.

We proceed to show that filtered colimits commute with compact cotensors. The comparison map $\mathrm{colim}_{\mathcal{F}}[C, G(f)] \to [C, \mathrm{colim}_{\mathcal{F}} G(f)]$ for a compact C in $\mathcal{V}$ is an isomorphism if and only if for every compact object K of $\mathcal{V}$ the induced map

$$[K, \underset{\mathcal{F}}{\mathrm{colim}}[C, G(f)]] \to [K, [C, \underset{\mathcal{F}}{\mathrm{colim}}\, G(f)]]$$

is an isomorphism. The cotensor $[K, -]$ commutes with filtered colimits by Lemma 3.4.3. Now the isomorphism can be checked directly. We have shown that the subcategory of docile functors is closed under filtered colimits.

Closure under finite limits is similar, but easier. It is left to the reader. It is clear that the composition of docile functors is again docile. □

Definition 3.4.5. We fix a map $z : Z \to \mathbb{1}$ in $\mathcal{V}$, where $\mathbb{1}$ is the monoidal unit and Z is an object of $\mathcal{V}$. We obtain a functor

$$T := [Z, -] : \mathcal{V} \to \mathcal{V}$$

and a natural transformation $t := [z, -] : \mathrm{Id} \to T$. Now let $P : \mathcal{V} \to \mathcal{V}$ be the colimit of the sequence

$$P := \mathrm{colim}\big(\mathrm{Id} \xrightarrow{t} T \xrightarrow{tT} T^2 \xrightarrow{tT^2} T^3 \xrightarrow{tT^3} T^4 \to \dots\big)$$

in the category $\mathrm{Fun}_{\mathcal{V}}(\mathcal{V}, \mathcal{V})$ and let $p : \mathrm{Id} \to P$ be the canonical natural transformation.

Lemma 3.4.6. *Let $\mathcal{V}$ be a confined symmetric monoidal category. In the situation of* Definition 3.4.5 *the following holds:*

(i) *The functor T is a $\mathcal{V}$-functor and t is a $\mathcal{V}$-enriched natural transformation.*

(ii) *The (underlying) functor P is isomorphic to the objectwise colimit of the defining sequence above. It is a $\mathcal{V}$-functor and the natural transformation p is $\mathcal{V}$-enriched.*

If the object Z is compact, then the endofunctors T and P are docile.

Proof. Lemma 2.1.1 is statement (i). As a composition of $\mathcal{V}$-functors, for all n the functor T^n is a $\mathcal{V}$-functor and the natural transformations in the defining diagram of P are $\mathcal{V}$-enriched. Hence, the defining colimit of F is indeed a diagram in the category $\mathrm{Fun}_{\mathcal{V}}(\mathcal{V}, \mathcal{V})$ of $\mathcal{V}$-enriched functors and $\mathcal{V}$-enriched natural transformations.

Note that $\mathcal{V}$ is cocomplete since it is a presentable category. Thus the claim (ii) follows entirely from Proposition 2.1.2. The colimit P exists as a $\mathcal{V}$-functor, but can also be computed objectwise. The map p is a colimit of $\mathcal{V}$-enriched natural transformations and is therefore itself $\mathcal{V}$-enriched.

Lemma 3.4.3 says that $T = [Z, -]$ is a docile functor if Z is compact. The remaining claims follows from Proposition 3.4.4 and Example 3.3.2. For all n the functor T^n is docile as a composition of docile functors. The functor P is docile since it is a sequential (filtered) colimit of docile functors in the category $\mathrm{Fun}_{\mathcal{V}}(\mathcal{V}, \mathcal{V})$. □

4 Tidy maps and their localizations

4.1 Tidy maps and the Weiss trick

The following idea is due to Weiss [Weiss98, (e.12)]. It leads us to the notion of tidy map in Definition 4.1.3 and will be used in Lemma 4.1.5.

Proposition 4.1.1. *Let $\mathcal{V}$ be a symmetric monoidal closed category, $R : \mathcal{V} \to \mathcal{V}$ a $\mathcal{V}$-functor and $r : \mathrm{Id} \to R$ a $\mathcal{V}$-enriched natural transformation. For a map $f : A \to B$ in $\mathcal{V}$ the following square*

$$\begin{tikzcd} {[B, X]} \arrow[r, "{[B, rX]}"] \arrow[d, "{[f, X]}"'] & {[B, RX]} \arrow[d, "{[f, RX]}"] \\ {[A, X]} \arrow[r, "{[A, rX]}"'] & {[A, RX]} \end{tikzcd} \tag{4.1.2}$$

commutes for every object X in $\mathcal{V}$. If the map $Rf : RA \to RB$ is invertible, then the square has a diagonal filler $\delta(X) : [A, X] \to [B, RX]$ which is $\mathcal{V}$-enriched natural in X.

Proof. Both R and r are $\mathcal{V}$-enriched. Therefore the square (4.1.2) with the internal homs commutes. Let θ denote the $\mathcal{V}$-enrichment of R. Then the following square

$$\begin{tikzcd}
[B,X] \arrow[r, "\theta(B,X)"] \arrow[d, "="'] & [RB,RX] \arrow[d, "{[rB,RX]}"] \\
[B,X] \arrow[r, "{[B,rX]}"'] & [B,RX]
\end{tikzcd}$$

commutes as it expresses the fact that $r : \mathrm{Id} \to R$ is a $\mathcal{V}$-enriched natural transformation. Thus, $[B, rX] = [rB, RX]\theta(B, X)$, and similarly $[A, rX] = [rA, RX]\theta(A, X)$. Hence the square (4.1.2) can be factored in the following way:

$$\begin{tikzcd}
[B,X] \arrow[r, "\theta"] \arrow[d, "{[f,X]}"'] & [RB,RX] \arrow[r, "{[rB,RX]}"] \arrow[d, "{[Rf,RX]}"', shift right] & [B,RX] \arrow[d, "{[f,RX]}"] \\
[A,X] \arrow[r, "\theta"'] & [RA,RX] \arrow[u, "{[g,RX]}"', bend right] \arrow[r, "{[rA,RX]}"'] & [A,RX]
\end{tikzcd}$$

But the middle map $[Rf, RX]$ is invertible, since Rf is invertible by hypothesis. It follows that the composed square has a diagonal filler. More precisely, if $g := (Rf)^{-1}$, then $[g, RX] \cong [Rf, RX]^{-1}$ and the map

$$\delta(X) = [rB, RX][g, RX]\theta(A, X) : [A, X] \to [RA, RX] \to [RB, RX] \to [B, RX]$$

is a diagonal filler of the square (4.1.2). Moreover, note that g, as an inverse of an enriched natural transformation, is itself $\mathcal{V}$-enriched. Hence the map $\delta(X)$ is a $\mathcal{V}$-natural transformation in X, since it is a composition of $\mathcal{V}$-natural transformations. $\square$

Definition 4.1.3. Suppose that the symmetric monoidal category $\mathcal{V}$ is confined. We will say that a map $z : Z \to \mathbb{1}$ in $\mathcal{V}$ is *tidy* if the object Z is compact and the map $Pz : PZ \to P\mathbb{1}$ is invertible, where P is the endofunctor of $\mathcal{V}$ from Definition 3.4.5.

The invertibility condition is of course inspired by the Weiss trick in the previous lemma and it is going to be essential in the proofs of Lemma 4.1.5, Lemma 4.2.6, Lemma 4.2.12 and consequently in the proof of the main Theorem 4.2.16. The compactness of Z is necessary to ensure that the functors T and P commute with filtered colimits. This will be used in the constructions of Diagrams (4.2.2) and (4.2.4), in the proofs of Lemma 4.2.6 and Proposition 4.2.8 and everything onwards.

Example 4.1.4. Not every map of the form $Z \to \mathbb{1}$, Z compact, is tidy. Take the category of spaces as a cartesian closed category. It is confined. The map $S^0 \to 1$ is not tidy: the associated cotensor is $T(X) = [S^0, X] = X \times X$, and for the resulting $P = \mathrm{colim}_n T^n$ we have $P(S^0) \neq 1 = P(1)$. This P is not idempotent and therefore not a localization. In fact, in $(\mathcal{S}, \times, 1)$ the only tidy maps are $0 \to 1$ and $1 = 1$. This follows since $\mathcal{S}$ admits only two left exact localizations: the one that inverts every map and the identity functor.

We expect tidy maps to be rare. Because of the self-referential nature of their definition it is usually difficult to decide whether a given map is tidy or not.

Lemma 4.1.5. *If $\mathcal{V}$ is confined and $z : Z \to \mathbb{1}$ is tidy, the following square*

$$\begin{tikzcd}
\mathrm{Id} \arrow[r, "p"] \arrow[d, "t"'] & P \arrow[d, "tP"] \\
T \arrow[r, "Tp"'] & TP
\end{tikzcd}$$

of $\mathcal{V}$-natural transformations has a $\mathcal{V}$-enriched diagonal filler $T \to P$.

Proof. Apply Lemma 4.1.1 to $(z : Z \to \mathbb{1}) = (A \to B)$ and $R = P$. $\square$

4.2 Constructing the localization

Let $\mathcal{V}$ be a confined symmetric monoidal category. We are heading towards Theorem 4.2.16 stating that P from Definition 3.4.5 is a symmetric monoidal left exact localization of $\mathcal{V}$ under the assumption that the map $Z \to \mathbb{1}$ is tidy. First it is necessary to examine carefully the map tP with the goal of showing in Lemma 4.2.6 that it is invertible.

By construction, we have a colimit cone

$$
\begin{tikzcd}
\mathrm{Id} \arrow[r, "t"] \arrow[drrr, "p:=p_0"'] & T \arrow[r, "tT"] \arrow[drr, "p_1"] & T^2 \arrow[r, "tT^2"] \arrow[dr, "p_2"] & T^3 \arrow[r, "tT^3"] \arrow[d, "p_3"] & T^4 \arrow[r, "tT^4"] \arrow[dl, "p_4"] & \cdots \\
 & & & P & &
\end{tikzcd}
\tag{4.2.1}
$$

with conical maps $p_n : T^n \to P$ that are $\mathcal{V}$-natural transformations for every $n \geqslant 0$. In preparation for the proof of Theorem 4.2.16 we postcompose Diagram (4.2.1) by T.

$$
\begin{tikzcd}
T \arrow[r, "Tt"] \arrow[drrr, "Tp=Tp_0"'] & T^2 \arrow[r, "TtT"] \arrow[drr, "Tp_1"] & T^3 \arrow[r, "TtT^2"] \arrow[dr, "Tp_2"] & T^4 \arrow[r, "TtT^3"] \arrow[d, "Tp_3"] & T^5 \arrow[r, "TtT^4"] \arrow[dl, "Tp_4"] & \cdots \\
 & & & TP & &
\end{tikzcd}
$$

Putting the previous diagram back to back with Diagram (4.2.1) we obtain the following diagram that commutes by naturality of the map $t : \mathrm{Id} \to T$.

$$
\begin{tikzcd}
 & & & & & P \arrow[ddd, "tP"] \\
\mathrm{Id} \arrow[r, "t"] \arrow[d, "t"'] \arrow[urrrrr, bend left, "p_0"] & T \arrow[r, "tT"] \arrow[d, "tT"'] \arrow[urrrr, bend left, "p_1"] & T^2 \arrow[r, "tT^2"] \arrow[d, "tT^2"'] \arrow[urrr, bend left, "p_2"] & T^3 \arrow[r] \arrow[d, "tT^3"'] \arrow[urr, bend left, "p_3"] & \cdots & \\
T \arrow[r, "Tt"'] \arrow[drrrrr, bend right, "Tp=Tp_0"'] & T^2 \arrow[r, "TtT"'] \arrow[drrrr, bend right, "Tp_1"] & T^3 \arrow[r, "TtT^2"'] \arrow[drrr, bend right, "Tp_2"] & T^4 \arrow[r] \arrow[drr, bend right, "Tp_3"] & \cdots & \\
 & & & & & TP
\end{tikzcd}
\tag{4.2.2}
$$

Lemma 4.2.3. *If Z is compact in $\mathcal{V}$, then the natural transformation $tP : P \to TP$ is the filtered colimit of the natural transformations $tT^n : T^n \to T^{n+1}$.*

Proof. The top cone of the diagram is a colimit cone by definition of P. The bottom cone is obtained by applying the functor T to the top cone. But the functor T preserves filtered colimits by Lemma 3.4.6, since Z is compact. It follows that the bottom cone is also a colimit cone. Hence the map tP is the colimit of the sequence of maps $tT^n : T^n \to T^{n+1}$. $\square$

In preparation for the proof of Lemma 4.2.6 we precompose Diagram (4.2.1) with T^n to obtain a new colimit diagram:

$$
\begin{tikzcd}
 & & & PT^n & & \\
T^n \arrow[r, "tT^n"'] \arrow[urrr, "pT^n = p_0T^n"] & T^{n+1} \arrow[r, "tT^{n+1}"'] \arrow[urr, "p_1T^n"] & T^{n+2} \arrow[r, "tT^{n+2}"'] \arrow[ur, "p_2T^n"'] & T^{n+3} \arrow[r, "tT^{n+3}"'] \arrow[u, "p_3T^n"'] & T^{n+4} \arrow[r, "tT^{n+4}"'] \arrow[ul, "p_4T^{n+4}"'] & \cdots
\end{tikzcd}
\tag{4.2.4}
$$

Observe that the bottom line of Diagram (4.2.4) is a cofinal sequence of the top line of Diagram (4.2.1). It follows that there is a unique isomorphism $\sigma_n : PT^n \to P$ such that

$$
\sigma_n p_k T^n = p_{n+k} \tag{4.2.5}
$$

for every $k \geqslant 0$. This is depicted in the following diagram:

$$
\begin{tikzcd}
 & & & & PT^n \arrow[dd, "\sigma_n"] \\
T^n \arrow[r, "tT^n"] \arrow[urrrr, bend left, "pT^n = p_0T^n"] \arrow[drrrr, bend right, "p_n"'] & T^{n+1} \arrow[r, "tT^{n+1}"] \arrow[urrr, bend left, "p_1T^n"] \arrow[drrr, bend right, "p_{n+1}"'] & T^{n+2} \arrow[r, "tT^{n+2}"] \arrow[urr, bend left, "p_2T^n"] \arrow[drr, bend right, "p_{n+2}"'] & T^{n+3} \cdots \arrow[ur, bend left, "p_3T^n"] \arrow[dr, bend right, "p_{n+3}"'] & \\
 & & & & P
\end{tikzcd}
$$

Note that the map σ_n goes "backwards" and needed to be constructed carefully. It is a crucial ingredient in the next proof.

Lemma 4.2.6. *Let $\mathcal{V}$ be confined and $z : Z \to \mathbb{1}$ tidy. Then the $\mathcal{V}$-natural transformation $tP : P \to TP$ is invertible.*

Proof. By Lemma 4.2.3, this map is the filtered colimit of the Diagram (4.2.2) of maps $tT^n : T^n \to T^{n+1}$. By Lemma 2.2.7 we can prove that the map $tP : P \to TP$ is invertible by showing that the square

$$
\begin{tikzcd}
T^n \arrow[r, "p_n"] \arrow[d, "tT^n"'] & P \arrow[d, "tP"] \\
T^{n+1} \arrow[r, "Tp_n"] & TP
\end{tikzcd}
$$

has a diagonal filler for every $n \geqslant 0$. If $\sigma_n : PT^n \to P$ is the isomorphism defined in Equation (4.2.5), then we have $p_n = \sigma_n(pT^n)$. Hence the square above is the composite of the following two commutative squares:

$$
\begin{tikzcd}
T^n \arrow[r, "pT^n"] \arrow[d, "tT^n"'] & PT^n \arrow[r, "\sigma_n"] \arrow[d, "tPT^n"] & P \arrow[d, "tP"] \\
T^{n+1} \arrow[r, "TpT^n"] & TPT^n \arrow[r, "T\sigma_n"] & TP
\end{tikzcd}
$$

But the left hand square of this diagram is obtained by precomposing the square in Lemma 4.1.5 with T^n. Hence the left hand square has a diagonal filler, since the square in Lemma 4.1.5 has a diagonal filler. It follows that the composite square has a diagonal filler for every $n \geqslant 0$ proving that $tP : P \to TP$ is invertible. $\square$

Lemma 4.2.7. *Suppose the $\mathcal{V}$-functor $F : \mathcal{V} \to \mathcal{V}$ preserves compact cotensors. Then we have a commutative diagram*

$$\begin{tikzcd} & F \arrow[dl, "Ft"'] \arrow[dr, "tF"] & \\ FT \arrow[rr, "\gamma", "\cong"'] & & TF \end{tikzcd}$$

of $\mathcal{V}$-natural transformations, where $\gamma = \gamma(Z, X) : F[Z, X] \to [Z, FX]$ is the coassembly map of the functor F.

Proof. From the map $z : Z \to \mathbb{1}$ we obtain a commutative square of $\mathcal{V}$-natural transformations in the variable X in $\mathcal{V}$.

$$\begin{tikzcd} F[\mathbb{1}, X] \arrow[r, "\gamma(\mathbb{1}, X) = \mathrm{id}_{FX}"] \arrow[d, "{F[z,X]}"'] & {[\mathbb{1}, FX]} \arrow[d, "{[z,FX]}"] \\ F[Z, X] \arrow[r, "\gamma(Z,X)"] & {[Z, FX]} \end{tikzcd}$$

The coassembly map $\gamma(Z, X) : F[Z, X] \to [Z, FX]$ is invertible, since the functor F preserves compact cotensors and Z is compact. $\square$

Proposition 4.2.8. *If $\mathcal{V}$ is confined and $z : Z \to \mathbb{1}$ is tidy, then the natural transformations*

$$pP : P \to P^2 \text{ and } Pp : P \to P^2$$

are equal and invertible.

Proof. Part 1: Let us show that the natural transformation $pP : P \to P^2$ is invertible. If we precompose the defning colimit of P with P, we obtain a colimit

$$\operatorname{colim}\left(P \xrightarrow{tP} TP \xrightarrow{tTP} T^2P \xrightarrow{tT^2P} T^3P \xrightarrow{tT^3P} \cdots \right) = P^2$$

and the transfinite composition of the maps in the colimit is $pP : P \to P^2$. The map tT^n is isomorphic to the map $T^n t$, since the symmetric monoidal structure on $\mathcal{V}$ yields (canonical) isomorphisms between the maps $z \otimes Z^{\otimes n}$ and $Z^{\otimes n} \otimes z$. Hence the map tT^nP is isomorphic to $T^n tP$, which is invertible, since the map tP is invertible by Lemma 4.2.6. So pP is invertible, since the transfinite composition of isomorphisms is an isomorphism.

Part 2: Let us show that the map $Pp : P \to P^2$ is invertible. The functor P preserves filtered colimits by Lemma 3.4.6. If we postcompose the defining colimit of P with P, we therefore obtain a colimit

$$\operatorname{colim}\left(P \xrightarrow{Pt} PT \xrightarrow{PtT} PT^2 \xrightarrow{PtT^2} PT^3 \xrightarrow{PtT^3} \cdots \right) = P^2$$

The map $tP : P \to TP$ is invertible by Lemma 4.2.6, since the map $z : Z \to \mathbb{1}$ is tidy. It follows that the map $Pt : P \to PT$ is invertible by Lemma 4.2.7, since P is docile. Hence the map $PtT^n : PT^n \to PT^{n+1}$ is invertible for every $n \geqslant 0$. So P^2 is the colimit of an increasing sequence of isomorphisms and the conical map Pp is invertible.

Part 3: Let us show that $pP = Pp$. The following squares commute by naturality of p:

$$\begin{tikzcd} P \arrow[r, "pP"] \arrow[d, "pP"'] & P^2 \arrow[d, "pP^2"] \\ P^2 \arrow[r, "PpP"] & P^3 \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "Pp"] \arrow[d, "pP"'] & P^2 \arrow[d, "pP^2"] \\ P^2 \arrow[r, "P^2p"] & P^3 \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "Pp"] \arrow[d, "Pp"'] & P^2 \arrow[d, "PpP"] \\ P^2 \arrow[r, "P^2p"] & P^3 \end{tikzcd}$$

Since the map pP is invertible by part (i), by the first square we have $pP^2 = PpP$. Similarly, the map $P^2p = P(Pp)$ is invertible, since the map Pp is invertible by part (ii). It follows that

$$pP = (P^2p)^{-1}(pP^2)(Pp) \quad \text{and} \quad Pp = (P^2p)^{-1}(pPp)(Pp)$$

by the second and third squares respectively. Thus, $pP = Pp$. $\square$

Definition 4.2.9. Let $L : \mathcal{C} \to \mathcal{C}$ be an endofunctor of a category $\mathcal{C}$ with a coaugmentation $\ell : \mathrm{Id} \to L$. We say that an object X in $\mathcal{C}$ is *L-closed* if the map $\ell X : X \to LX$ is invertible. We say that a map $f : X \to Y$ in $\mathcal{C}$ is *L-closed* if the naturality square

$$\begin{tikzcd}
X \arrow[r, "\ell X"] \arrow[d, "f"'] & LX \arrow[d, "Lf"] \\
Y \arrow[r, "\ell Y"] & LY
\end{tikzcd}$$

is cartesian. Let $\mathcal{C}^L$ denote the full subcategory of $\mathcal{C}$ formed by L-closed objects.

Lemma 4.2.10. *Let $\mathcal{V}$ be confined and $z : Z \to \mathbb{1}$ tidy. Then we have:*

(i) *A map $f : X \to Y$ in $\mathcal{V}$ is T-closed if and only if it is P-closed.*

(ii) *An object in $\mathcal{V}$ is T-closed if and only if it is P-closed.*

(iii) *The object PX is P-closed for every X in $\mathcal{V}$.*

Proof. To prove (i), let $f : X \to Y$ be P-closed. Consider the following commutative cube:

$$\begin{tikzcd}
& PX \arrow[rr, "tPX"] \arrow[dd, "Pf"'] & & TPX \arrow[dd, "TPf"] \\
X \arrow[ur, "pX"] \arrow[rr, "tX"] \arrow[dd, "f"'] & & TX \arrow[ur, "TpX"'] \arrow[dd] & \\
& PY \arrow[rr, "tPY"] & & TPY \\
Y \arrow[ur, "pY"] \arrow[rr, "tY"] & & TY \arrow[ur, "TpY"'] &
\end{tikzcd}$$

The left hand face of the cube is cartesian, since f is P-closed. The right hand face is also cartesian, since the functor T preserves limits. But the horizontal maps of the back face, tPX and tPY, are isomorphism by Lemma 4.2.6. It follows that the front face is cartesian. Thus, f is T-closed.

Conversely, let $f : X \to Y$ be T-closed. We need to show that the square

$$\begin{tikzcd}
X \arrow[r, "pX"] \arrow[d, "f"'] & PX \arrow[d, "Pf"] \\
Y \arrow[r, "pY"] & PY
\end{tikzcd}$$

is cartesian. But this square is the infinite composition of the following sequence:

$$\begin{tikzcd}
X \arrow[r, "tX"] \arrow[d, "f"'] & TX \arrow[r, "tTX"] \arrow[d, "Tf"'] & T^2X \arrow[r, "tT^2X"] \arrow[d, "T^2f"'] & T^3X \arrow[r] \arrow[d, "T^3f"'] & \cdots \\
Y \arrow[r, "tY"'] & TY \arrow[r, "tTY"'] & T^2Y \arrow[r, "tT^2Y"'] & T^3Y \arrow[r] & \cdots
\end{tikzcd}$$

It suffices to show that each square in the sequence is cartesian, since filtered colimits preserves finite limits in $\mathcal{V}$. An indicidual square looks like this:

$$\begin{tikzcd}
T^nX \arrow[r, "tT^nX"] \arrow[d, "T^nf"] & T(T^nX) \arrow[d, "T(T^nf)"] \\
T^nT \arrow[r, "tT^nY"] & T(T^nY)
\end{tikzcd}$$

The case $n = 0$ is clear, since the map f is T-closed by assumption. In general the square above is the following composition:

$$\begin{tikzcd}
T^nX \arrow[r, "T^ntX"] \arrow[d, "T^nf"] & T^n(TX) \arrow[r, "\gamma"] \arrow[d, "T^n(Tf)"] & T(T^nX) \arrow[d, "T(T^nf)"] \\
T^nY \arrow[r, "T^ntY"] & T^n(TY) \arrow[r, "\gamma"] & T(T^nY)
\end{tikzcd}$$

where γ is the coassembly isomorphism from Lemma 4.2.7 with $F := T^n$. The left hand square is the image by T^n of the case $n = 0$ considered before. So the left hand square is cartesian, since the functor T^n preserves limits. The right hand square is also cartesian, since its horizontal maps are invertible. It follows that the map f is P-closed.

Statement (ii) is a special case of (i).

For (iii) observe that PX is T-closed by Lemma 4.2.6. Thus, PX is P-closed by (ii). □

Definition 4.2.11. We say that a map $\rho : X \to Y$ is a *$\mathcal{V}$-reflection into the full subcategory of P-closed objects* $\mathcal{V}^P$ if Y belongs to $\mathcal{V}^P$ and the map $[\rho, W] : [Y, W] \to [X, W]$ is invertible for every W in $\mathcal{V}^P$.

Lemma 4.2.12. *Suppose that $\mathcal{V}$ is confined and that $z : Z \to \mathbb{1}$ is tidy. Then the map $pX : X \to PX$ is a $\mathcal{V}$-reflection into $\mathcal{V}^P$ for every object X in $\mathcal{V}$.*

Proof. By Lemma 4.2.10 PX is in $\mathcal{V}^P$. It remains to show that the map $[pX, W] : [PX, W] \to [X, W]$ is invertible for every P-closed W. The following square

$$\begin{tikzcd}
{[PX,W]} \arrow[r, "{[PX,pW]}"] \arrow[d, "{[pX,W]}"'] & {[PX,PW]} \arrow[d, "{[pX,PW]}"] \\
{[X,W]} \arrow[r, "{[X,pW]}"] & {[X,PW]}
\end{tikzcd}$$

commutes by applying the map $[pX, -]$ to the map pW. The Weiss trick Lemma 4.1.1 can be applied to this square. For the benefit of the reader we supply the factorization of the square:

$$\begin{tikzcd}
{[PX,W]} \arrow[r, "\theta"] \arrow[d, "{[pX,W]}"'] & {[P^2X,PW]} \arrow[r, "{[pX,PW]}"] \arrow[d, "{[P(pX),PW]}"', "\cong"] & {[PX,PW]} \arrow[d, "{[pX,PW]}"] \\
{[X,W]} \arrow[r, "\theta"] & {[PX,PW]} \arrow[r, "{[pX,PW]}"] & {[X,PW]}
\end{tikzcd}$$

The map $P(pX)$ is invertible by Proposition 4.2.8. Using the inverse of the middle vertical map, the square above has a diagonal filler d

$$\begin{tikzcd}
{[PX,W]} \arrow[r, "{[PX,pW]}"] \arrow[d, "{[pX,W]}"'] & {[PX,PW]} \arrow[d, "{[pX,PW]}"] \\
{[X,W]} \arrow[r, "{[X,pW]}"'] \arrow[ur, "d"] & {[X,PW]}
\end{tikzcd}$$

by Lemma 4.1.1. But pW is invertible, since W in $\mathcal{V}^P$. It follows that all maps in this diagram are invertible proving that the map $[pX, W]$ is invertible. □

Definition 4.2.13. We will say that a map $f : X \to Y$ in $\mathcal{V}$ is a *P-equivalence* if the map $P(f) : PX \to PY$ is invertible.

Lemma 4.2.14. *A map $f : X \to Y$ is a P-equivalence if and only if the map $[f, W] : [Y, W] \to [X, W]$ is invertible for every W in $\mathcal{V}^P$.*

Proof. By definition f is a P-equivalence if and only if the map $P(f)$ is invertible. The image of the commutative square

$$\begin{tikzcd}
X \arrow[r, "pX"] \arrow[d, "f"'] & PX \arrow[d, "P(f)"] \\
Y \arrow[r, "pY"] & PY
\end{tikzcd}$$

by the contravariant functor $[-, W]$ is the following commutative square.

$$\begin{tikzcd}
{[X,W]} & {[PX,W]} \arrow[l, "{[pX,W]}"'] \\
{[Y,W]} \arrow[u, "{[f,W]}"] & {[PY,W]} \arrow[l, "{[pY,W]}"] \arrow[u, "{[P(f),W]}"']
\end{tikzcd}$$

By Yoneda, the map $P(f)$ in $\mathcal{V}^P$ is invertible if and only if the map $[P(f), W]$ in $\mathcal{V}$ is invertible for every object W in $\mathcal{V}^P$. But the horizontal maps of the lower square are invertible, since the maps $pX : X \to PX$ and $pY : Y \to PY$ are $\mathcal{V}$-reflecting into $\mathcal{V}^P$ by Lemma 4.2.12. Hence the map $P(f)$ is invertible if and only if the map $[f, W]$ is invertible for every object W in $\mathcal{V}^P$. □

Lemma 4.2.15. *For any P-equivalence f and any object V the map $V \otimes f$ is a P-equivalence.*

Proof. Let V be an object in $\mathcal{V}$ and W be in $\mathcal{V}^P$. Then the map $[f, W]$ is invertible by Lemma 4.2.14. Hence the map $[V \otimes f, W] = [V, [f, W]]$ is invertible. So, again by Lemma 4.2.14, the map $V \otimes f$ is a P-equivalence. □

From this lemma it follows that for any two P-equivalences f, f' in $\mathcal{V}$ their tensor product $f \otimes f'$ is a P-equivalence since $f \otimes f' = (f \otimes Y')(X \otimes f')$.

Theorem 4.2.16. *Let $\mathcal{V}$ be a confined symmetric monoidal category and let $z : Z \to \mathbb{1}$ be a tidy map. Define $T := [Z, -]$, $t := [z, -] : \mathrm{Id} \to T$, $P = \mathrm{colim}_n T^n$ and $p : \mathrm{Id} \to P$ as in Definition 3.4.5. Let $\mathcal{V}^P$ be the subcategory of P-closed objects of $\mathcal{V}$ and let $\mathsf{c}(\mathcal{V})$ be the subcategory of compact objects of $\mathcal{V}$. Then:*

(i) *An object of $\mathcal{V}$ is T-closed if and only if it is P-closed.*

(ii) *For every A in $\mathcal{V}$ and X in $\mathcal{V}^P$ the cotensor $[A, X]$ is in $\mathcal{V}^P$.*

(iii) *The subcategory $\mathcal{V}^P$ is $\mathcal{V}$-reflective, and the natural transformation $p : \mathrm{Id} \to P$ is $\mathcal{V}$-reflecting into $\mathcal{V}^P$. The reflector $P : \mathcal{V} \to \mathcal{V}^P$ is $\mathcal{V}$-left exact.*

(iv) *The category $\mathcal{V}^P$ is symmetric monoidal closed with tensor $X \otimes_P Y := P(X \otimes Y)$ for every X, Y in $\mathcal{V}^P$. Its unit object is $\mathbb{1}_P = P(\mathbb{1})$. The localization functor $P : \mathcal{V} \to \mathcal{V}^P$ is symmetric monoidal.*

(v) *The symmetric monoidal category $(\mathcal{V}^P, \otimes_P, P(\mathbb{1}))$ is confined and the reflector $P : \mathcal{V} \to \mathcal{V}^P$ is confined. Every compact object of $\mathcal{V}^P$ is a retract of an object in $P(\mathsf{c}(\mathcal{V}))$. The subcategory $P(\mathsf{c}(\mathcal{V}))$ is dense in $\mathcal{V}^P$.*

Proof. (i) This was proved in Lemma 4.2.10.

(ii) Let X be T-closed. We will show that the object $[A, X]$ is T-closed for every A in $\mathcal{V}$. The claim then follows from (i). Consider the following diagram:

$$\begin{tikzcd}
{[\mathbb{1}, [A, X]]} \arrow[r, "\cong"] \arrow[d, "{t[A,X]=[z,[A,X]]}"'] & {[\mathbb{1} \otimes A, X]} \arrow[r, "\cong"] \arrow[d, "{[z \otimes A, X]}"'] & {[A \otimes \mathbb{1}, X]} \arrow[r, "\cong"] \arrow[d, "{[A \otimes z, X]}"'] & {[A, [\mathbb{1}, X]]} \arrow[d, "{[A,[z,X]]=[A,tX]}"] \\
{[Z, [A, X]]} \arrow[r, "\cong"] & {[Z \otimes A, X]} \arrow[r, "\cong"] & {[A \otimes Z, X]} \arrow[r, "\cong"] & {[A, [Z, X]]}
\end{tikzcd}$$

The horizontal maps are coassembly maps $\gamma(\mathbb{1}, X)$ and $\gamma(Z, X)$. They are invertible, since the functor $[A, -]$ preserves all cotensors as explained in the proof of Lemma 3.4.3. Now the map $[A, tX]$ is invertible, since the map tX is invertible by assumption. So $t[A, X]$ is invertible and $[A, X]$ is T-closed.

(iii) The first and second statement are proved in Lemma 4.2.12. As a right adjoint the inclusion $\mathcal{V}^P \subset \mathcal{V}$ preserves finite limits and it preserves cotensors by (ii). The endofunctor $P : \mathcal{V} \to \mathcal{V}$ is docile by Lemma 3.4.6. So the localization functor $P : \mathcal{V} \to \mathcal{V}^P$ preserves finite limits and compact cotensors and is therefore $\mathcal{V}$-left exact.

(iv) Lemma 4.2.15 supplies the condition to apply [Lur17, Proposition 4.1.7.4]: the functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ induces a functor $\otimes^P : \mathcal{V}^P \times \mathcal{V}^P \to \mathcal{V}^P$ yielding a symmetric monoidal structure on $\mathcal{V}^P$ such that

$$X \otimes_P Y = P(X) \otimes_P P(Y) = P(X \otimes Y)$$

for all X, Y in $\mathcal{V}^P$. Its unit is $\mathbb{1}_P = P(\mathbb{1})$ and the symmetry is $\sigma_P(X, Y) = P(\sigma(X, Y)) : P(X \otimes Y) \cong P(Y \otimes X)$. Clearly, the functor $P : \mathcal{V} \to \mathcal{V}^P$ is symmetric monoidal.

It remains to prove that this structure is closed. For every X in $\mathcal{V}$ and every W in $\mathcal{V}^P$, the cotensor $[X, W]$ is in $\mathcal{V}^P$ by (ii). Moreover, $[P(X \otimes Y), W] = [X \otimes Y, W]$ for every X, Y in $\mathcal{V}$ by Lemma 4.2.12. Thus, for all X, Y, W in $\mathcal{V}^P$:

$$[X \otimes_P Y, W] = [X \otimes Y, W] = [Y, [X, W]].$$

(v) As a reflective subcategory of $\mathcal{V}$, the category $\mathcal{V}^P$ is cocomplete and the localization functor $P : \mathcal{V} \to \mathcal{V}^P$ is cocontinous, since it is left adjoint to the inclusion functor $i : \mathcal{V}^P \to \mathcal{V}$. As an endofunctor, $P : \mathcal{V} \to \mathcal{V}$ is docile by Lemma 3.4.6. It follows that filtered colimits are preserved by the inclusion $i : \mathcal{V}^P \subset \mathcal{V}$. Therefore, by Proposition 3.1.3, the localization functor $P : \mathcal{V} \to \mathcal{V}^P$ is confined. (Beware that a compact object in $\mathcal{V}^P$ may not be compact in $\mathcal{V}$.) In particular, $P(\mathsf{c}(\mathcal{V})) \subset \mathsf{c}(\mathcal{V}^P)$.

Since $\mathcal{V}$ is ω-presentable, for every object X in $\mathcal{V}$ we have $X = \operatorname{colim}_{\mathsf{c}(\mathcal{V})/X} F(D)$ by Lemma 2.2.6 and the category $\mathsf{c}(\mathcal{V})/X$ of compact object over X is filtered. Suppose now that X is P-closed. Then $X \cong PX \cong \operatorname{colim}_{\mathsf{c}(\mathcal{V})/X} PF(D)$ is the corresponding colimit reflected into $P(\mathsf{c}(\mathcal{V})) \subset \mathsf{c}(\mathcal{V}^P)$. This implies that $P(\mathsf{c}(\mathcal{V}))$ is dense in $\mathcal{V}^P$, that $\mathcal{V}^P$ is ω-presentable and that every object in $\mathsf{c}(\mathcal{V}^P)$ is a retract of an object in $P(\mathsf{c}(\mathcal{V}))$

It remains to show that the symmetric monoidal category $\mathcal{V}^P$ is confined. Since $\mathbb{1}$ is compact and P is confined, the unit $\mathbb{1}_P = P(\mathbb{1})$ is compact. Further:

$$P(\mathsf{c}(\mathcal{V})) \otimes_P P(\mathsf{c}(\mathcal{V})) = P\big(\mathsf{c}(\mathcal{V}) \otimes \mathsf{c}(\mathcal{V})\big) \subset P(\mathsf{c}(\mathcal{V})) \subset \mathsf{c}(\mathcal{V}^P)$$

by (iv). Since $P(\mathsf{c}(\mathcal{V}))$ is dense in $\mathcal{V}^P$, it follows from Lemma 3.2.3 that the symmetric monoidal closed category $\mathcal{V}^P$ is confined. □

4.3 The associated factorization system

The notion of a saturated class was defined in [Lur09, Def 5.5.5.1]. See also [ABFJ22, Def 3.1.12] and [ABFJ22, Prop 3.1.14]. The left class of a factorization system is always saturated. If $\mathcal{V}$ is presentable and there exists a set S of morphisms such that $\mathcal{R} = S^\perp$ are the maps right orthogonal to S, then $\mathcal{L} = {}^\perp(S^\perp) = S^s$ is the saturated closure of S and $(\mathcal{L}, \mathcal{R})$ form a factorization system in $\mathcal{E}$. See e.g. [Lur09, Prop 5.5.5.7] or [ABFJ22, Prop 3.1.18]. If $\mathcal{V}$ has finite limits, then we define a *left exact modality* [ABFJ22, Def. 4.1.1] as a factorization system whose left class $\mathcal{L}$ is the class of morphisms that is inverted by a left exact localization of $\mathcal{V}$. Equivalently, $\mathcal{L}$ is the left class of a factorization system closed under finite limits [ABFJ22, Lemma 4.1.2]. We call such a class of morphisms a *congruence* [ABFJ22, Def. 4.2.1].

Proposition 4.3.1. *Suppose that $\mathcal{V}$ is confined and that $\mathcal{D}$ is a small dense subcategory of $\mathcal{V}$. Let $z : Z \to \mathbb{1}$ be a tidy map and $S := \{z \otimes D : Z \otimes D \to D \,|\, D \in \mathcal{D}\}$. Further, let $\mathcal{L}$ be the class of P-equivalences and $\mathcal{R}$ the class of P-closed maps. Then:*

(i) *The pair $(\mathcal{L}, \mathcal{R})$ is left exact modality in $\mathcal{V}$, i.e. $\mathcal{L}$ is a congruence.*

(ii) *$\mathcal{R} = S^\perp$ and $\mathcal{L} = {}^\perp(S^\perp) = S^s$.*

Proof. For (i) note that the functor P is a left exact reflector by Theorem 4.2.16. It then follows by [ABFJ22, Proposition 4.1.6] that $(\mathcal{L}, \mathcal{R})$ is a factorization system and therefore a left exact modality.

Let us prove (ii). A map $f : X \to Y$ in $\mathcal{V}$ is P-closed if an only if it is T-closed by Lemma 4.2.10. By definition, $f : X \to Y$ is T-closed if and only if the square

$$\begin{tikzcd} X \arrow[r, "{[z,X]}"] \arrow[d, "f"'] & {[Z,X]} \arrow[d, "{[Z,f]}"] \\ Y \arrow[r, "{[z,Y]}"] & {[Z,Y]} \end{tikzcd}$$

is cartesian. But by Lemma 2.2.3 this square is cartesian if and only if the following square

$$\begin{tikzcd} \operatorname{map}(D,X) \arrow[r, "{\operatorname{map}(D,[z,X])}"] \arrow[d, "{\operatorname{map}(D,f)}"'] & \operatorname{map}(D,[Z,X]) \arrow[d, "{\operatorname{map}(D,[Z,f])}"] \\ \operatorname{map}(D,Y) \arrow[r, "{\operatorname{map}(D,[z,Y])}"] & \operatorname{map}(D,[Z,Y]) \end{tikzcd}$$

is cartesian for every object D in $\mathcal{D}$. This square is isomorphic to the square

$$\begin{tikzcd} \operatorname{map}(D,X) \arrow[r, "{\operatorname{map}(z\otimes D,X)}"] \arrow[d, "{\operatorname{map}(D,f)}"'] & \operatorname{map}(Z\otimes D,X) \arrow[d, "{\operatorname{map}(Z\otimes D,f)}"] \\ \operatorname{map}(D,Y) \arrow[r, "{\operatorname{map}(z\otimes D,Y)}"] & \operatorname{map}(Z\otimes D,Y) \end{tikzcd}$$

which is cartesian if and only if the maps $z \otimes D : Z \otimes D \to D$ are left orthogonal to the map $f : X \to Y$ for all D. In summary, a map f is P-closed if and only it is right orthogonal to every map in S.

This shows that $\mathcal{R} = S^{\perp}$. But $\mathcal{L} = {}^{\perp}\mathcal{R}$, since the pair $(\mathcal{L}, \mathcal{R})$ forms a factorization system by the first part of the proof. Thus, $\mathcal{L} = {}^{\perp}\mathcal{R} = {}^{\perp}(S^{\perp}) = S^{s}$. □

5 Intermezzo

5.1 Cocartesian gap maps and pushout products

Consider the sets $\underline{n} = \{1, \ldots, n\}$ and the poset $[1] = (0 < 1)$. An *n-cube* in the category $\mathcal{C}$ is a functor $\chi : [1]^n \to \mathcal{C}$ or equivalently a functor $\chi : \mathcal{P}(\underline{n}) \to \mathcal{C}$, where $\mathcal{P}(\underline{n})$ is the poset of subsets of $\underline{n}$. For a cube $\chi : P(\underline{n}) \to \mathcal{C}$ we call

$$\mathrm{cogap}(\chi) : \operatorname*{colim}_{U \subsetneq \underline{n}} \chi(U) \to \chi(\underline{n})$$

its *cocartesian gap map*. If $\mathcal{C}$ admits finite products, then we define for two maps $f : A \to B$ and $g : C \to D$ in $\mathcal{C}$ their *pushout product* $f \square g$ to be the cocartesian gap map of the following square:

$$\begin{tikzcd} A \times C \arrow[r, "(\mathrm{id},g)"] \arrow[d, "(f,\mathrm{id})"'] & A \times D \arrow[d, "(f,\mathrm{id})"] \\ B \times C \arrow[r, "(\mathrm{id},g)"] & B \times D \end{tikzcd}$$

Warning: In any symmetric monoidal category it makes sense to replace the cartesian product from above with the monoidal product and one obtains an associated pushout product. In a category, that carries a symmetric monoidal structure additionally to the cartesian one, this leads to an ambiguity. In this paper this happens in the Sections 6 and 7 where we consider categories like $\mathrm{Fun}(\mathrm{Fin}, \mathcal{S})$ and $\mathrm{Fun}(\mathcal{J}, \mathcal{S})$ which have as logoi a (locally) cartesian closed symmetric monoidal structure. On top of that they are equipped with Day convolution monoidal structures that we employ to obtain Goodwillie calculus and orthogonal calculus out of the setup from Section 4. However, *we always use the pushout product defined above using the cartesian product*!

The *external cartesian product* $\chi \boxtimes \psi$ of two cubes $\chi : [1]^m \to \mathcal{C}$ and $\psi : [1]^n \to \mathcal{C}$ is the cube $\chi \boxtimes \psi : [1]^{m+n} \to \mathcal{C}$ defined by putting $(\chi \boxtimes \psi)(a, b) = \chi(a) \times \psi(b)$ for every $(a, b) \in [1]^m \times [1]^n$.

It is easy to check that $\mathrm{cogap}(\chi \boxtimes \psi) = \mathrm{cogap}(\chi) \square \mathrm{cogap}(\psi)$. Then for any sequence $(f_1 : X_1 \to Y_1, \ldots, f_n : X_n \to Y_n)$ of maps, aka. 1-cubes, in $\mathcal{C}$ it follows that

$$f_1 \square \cdots \square f_n = \mathrm{cogap}(f_1 \boxtimes \cdots \boxtimes f_n) : \mathrm{FW} \to Y_1 \times \ldots \times Y_n, \tag{5.1.1}$$

where FW is a relative fat wedge, because in the case of $X_1 = \ldots = X_n = 1$ it is really the fat wedge of $Y_1, \ldots, Y_n$.

5.2 The join product

For two space A and B their join product $A \star B$ is defined as the codomain of the map $(A \to 1) \square (B \to 1) = (A \star B \to 1)$. If A and B are finite space then $A \star B$ is finite as well.

Lemma 5.2.1. *Finite spaces together with the join are a symmetric monoidal $(\infty, 1)$-category.*

Proof. Example 3.2.4 explains that the Day convolution product yields a symmetric monoidal $(\infty, 1)$-category. The pushout product on the category of morphisms $\mathcal{S}^{[1]}$ coming from the cartesian product on $\mathcal{S}$ can be seen as a Day convolution where one employs the minimum on the poset $[1]$ as the monoidal product. Therefore the pushout product is a symmetric monoidal $(\infty, 1)$-structure on $\mathcal{S}^{[1]}$.

The inclusion of the full subcategory of morphisms of the form $A \to 1$ into the category of all morphisms is symmetric monoidal. The join product is by definition the restriction of the pushout product. Hence it is itself a symmetric monoidal $(\infty, 1)$-structure. Now one can restrict further from the category $\mathcal{S}$ to Fin since the join of two finite spaces is again finite. □

For a space K and a set U let K^U be the ordinary cotensor, in other words the $|U|$-fold cartesian power of K.

Lemma 5.2.2. *For every object K in $\mathcal{S}$ and every $n \geqslant 0$ we have*

$$K^{\star n} = \operatorname*{colim}_{\varnothing \neq U \subset \underline{n}} K^U.$$

Proof. The map $K^{\star n} \to 1 \cong (K \to 1)^{\square n}$ is the cocartesian gap map of the n-cube $\chi := (K \to 1) \boxtimes \cdots \boxtimes (K \to 1)$. By construction, $\chi(U) = K^{\complement U}$ for every subset $U \subsetneq \underline{n}$. Hence

$$K^{\star n} = \operatorname*{colim}_{U \subsetneq n} K^{\complement U} = \operatorname*{colim}_{\varnothing \neq U \subset \underline{n}} K^U$$

for every $n \geqslant 0$. $\square$

5.3 Fiberwise joins

For two maps $f : F \to B$ and $g : G \to B$ in $\mathcal{S}$ the following square

$$\begin{tikzcd} F \times_B G \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & G \arrow[d, "g"] \\ F \arrow[r, "f"] & B \end{tikzcd}$$

is cartesian and we write $f \boxtimes_B g$ of f and g for it. We call its cocartesian gap map

$$f \star_B g = \operatorname{cogap}(f \boxtimes_B g) : F \star_B G \to B$$

the *fiberwise join* of f and g. More details are given in [ABFJ20]**reference!**. Since colimits are universal, the following square

$$\begin{tikzcd} F \times_B G \arrow[r] \arrow[d, "f \star_B g"'] & (F \times B) \sqcup_{F \times G} (B \times G) \arrow[d, "f \square g"] \\ B \arrow[r, "\Delta(B)"] & B \times B \end{tikzcd}$$

is cartesian where $\Delta(B)$ is the diagonal map of B.

An n-cube is *strongly cartesian* if any 2-dimensional face is cartesian.

Lemma 5.3.1. *Let $(f_1, \ldots, f_n)$ be a sequence of maps $f_i : F_i \to B$ in $\mathcal{S}$. Then:*

(i) *The n-cube $f_1 \boxtimes \cdots \boxtimes f_n$ is strongly cartesian and its base change along the diagonal map $B \to B^n$ is the strongly cartesian n-cube $f_1 \boxtimes_B \cdots \boxtimes_B f_n$.*

(ii) *The pushout product $f_1 \square \cdots \square f_n$ is the cocartesian gap map of the n-cube $f_1 \boxtimes \cdots \boxtimes f_n$.*

(iii) *The fiberwise join product $f_1 \star_B \cdots \star_B f_n$ is the cocartesian gap map of the n-cube $f_1 \boxtimes_B \cdots \boxtimes_B f_n$.*

(iv) *The map $f_1 \star_B \cdots \star_B f_n$ is the base change of $f_1 \square \cdots \square f_n$ along the diagonal $B \to B^n$.*

Proof. Statement (i), (ii) and (iii) follow by direct inspection. Statement (iv) follows the universality of colimits in $\mathcal{S}$. $\square$

6 Goodwillie calculus revisited

We denote by $\mathrm{Fin} \subset \mathcal{S}$ the category of finite spaces. It is the free finitely cocomplete category on one generator. The category Fin together with the join product $-\star-$ becomes a symmetric monoidal structure as formulated in Lemma 5.2.1. Consider the confined symmetric monoidal category $\mathrm{Fun}(\mathrm{Fin}, \mathcal{S})$ equipped with the associated convolution product. By slight abuse of notation we write Id for the inlcusion functor $\mathrm{Fin} \subset \mathcal{S}$. For each $n \geqslant 0$ we choose $z_{n+1} : \mathrm{Id}^{\star n+1} \to 1 = (\mathrm{Id} \to 1)^{\square n+1}$ as our map $z : Z \to \mathbb{1}$ to start the machinery of Section 4. The fact that z_{n+1} is tidy (Lemma 6.2.5) yields all the good properties of the reflector P_n.

6.1 The reflector P_n

Definition 6.1.1. Define T_n as an endofunctor of $\mathrm{Fun}(\mathrm{Fin}, \mathcal{S})$ by cotensoring $T_n(F) := [\mathrm{Id}^{\star n+1}, F]$.

Lemma 6.1.2. *For every $n \geqslant 0$ we have*

$$\mathrm{Id}^{\star n+1} = \operatorname*{colim}_{\varnothing \neq U \subset \underline{n+1}} \mathrm{map}(U, -)$$

In particular, the $(n+1)$-fold join power $\mathrm{Id}^{\star n+1}$ is finitely presentable and hence compact.

Proof. Follows from Lemma 5.2.2. □

Since the internal hom here is taken with respect to the Day convolution product coming from the join on Fin, and is not the usual internal hom on $\mathrm{Fun}(\mathrm{Fin}, \mathcal{S})$. For every $F : \mathrm{Fin} \to \mathcal{S}$ we have

$$T_n F = [\mathrm{Id}^{\star n+1}, F] = \lim_{\varnothing \neq U \subset \underline{n+1}} F(U \star -)$$

and the canonical map $z_{n+1} : \mathrm{Id}^{\star n+1} \to 1$ yields a natural transformation

$$t_n(F) = [z_{n+1}, F] : F \to T_n(F).$$

As in Definition 3.4.5 let P_n be the endofunctor of $\mathrm{Fun}(\mathrm{Fin}, \mathcal{S})$ defined by

$$P_n := \mathrm{colim}\big(\mathrm{Id} \xrightarrow{t_n} T_n \xrightarrow{t_n T_n} T_n^2 \xrightarrow{t_n T_n^2} T_n^3 \xrightarrow{t_n T_n^3} T_n^4 \to \dots\big) \tag{6.1.3}$$

and let $p_n : \mathrm{Id} \to P_n$ be the canonical map.

6.2 The map z_{n+1} is tidy

This will be proved in Proposition 6.2.5. This follows easily from connectivity estimates. We quote:

Definition 6.2.1. [Good03, Def 1.2] A map $\alpha : F \to G$ in $\mathrm{Fun}(\mathrm{Fin}, \mathcal{S})$ is said to satisfy condition $O_n(c, \kappa)$ for $c \in \mathbb{Z}$ and $\kappa \geqslant -2$ if the connectivity of the map $\alpha(K) : F(K) \to G(K)$ is $\geqslant (n+1)k - c$ for every K in Fin of connectivity $k \geqslant \kappa$.

Proposition 6.2.2. [Good03, Prop 1.6] *If a map $\alpha : F \to G$ in $\mathrm{Fun}(\mathrm{Fin}, \mathcal{S})$ satisfies condition $O_n(c, \kappa)$ for some c, then $P_n(\alpha) : P_n F \to P_n G$ is invertible.*

Although this was stated in [Good03], the proof relies only on connectivity estimates that Goodwillie had developped earlier. Note that if $\alpha : \mathcal{X} \to \mathcal{Y}$ is a map of n-cubes, then it can be viewed as an $(n+1)$-cube which we denote $[\alpha]$.

Lemma 6.2.3. [Good92, Prop 1.6] *Let $\alpha : \mathcal{X} \to \mathcal{Y}$ be a map of n-cubes in $\mathcal{S}$.*

(i) *If the $(n+1)$-cube $[\alpha]$ is k-cartesian and $\mathcal{Y}$ is k-cartesian, then $\mathcal{X}$ is k-cartesian.*

(ii) *If $\mathcal{X}$ is k-cartesian and $\mathcal{Y}$ is $(k+1)$-cartesian, then $[\alpha]$ is k-cartesian.*

The next statement is Goodwillie's Theorem 1.20 [Good92] simplified to the case $T = \underline{1}$.

Lemma 6.2.4. *Let $\alpha : \mathcal{X} \to \mathcal{Y}$ be a morphism of n-cubes in $\mathcal{S}$. Suppose that the $(n+1)$-cube $[\alpha]$ is k-cartesian and that the map $\alpha(U)$ is $(k + |U| - 1)$-connected for every non-empty subset $U \subset \underline{n}$. Then the map $\alpha(\varnothing)$ is k-connected.*

Lemma 6.2.5. *The map $z_{n+1} : \mathrm{Id}^{\star n+1} \to 1$ is tidy.*

Proof. When we evaluate the map $z_{n+1} : \mathrm{Id}^{\star n+1} \to 1$ at a k-connected finite space K, the resulting map $K^{\star n+1} \to 1$ is $((n+1)k + 2n)$-connected. Hence the map z_{n+1} satisfies condition $O_n(-2n, -1)$. By Proposition 6.2.2 the map $P_n(z_{n+1})$ is invertible. □

Theorem 6.2.6. (Goodwillie) *The functor P_n defined in (6.1.3) is a left exact reflector onto the subcategory of T_n-closed objects of $\mathrm{Fun}(\mathrm{Fin}, \mathcal{S})$. The T_n-closed objects are exactly the n-excisive functors.*

Proof. Lemma 6.2.5 states that the map z_{n+1} is tidy, so and Theorem 4.2.16 applies. The fact that T_n-closed objects coincide with n-excisive functors can be proved either in the original way [Good03, Lem. 1.9], [Rezk13] or by an independent method in [ABFJ24b, Theorem 4.4.5.] based on the observation in [ABFJ24b, Lemma 4.4.2.] that all strongly cocartesian cubes are obtained from free cocartesian cubes by cobase change. □

Using Lemma 6.1.2 the maps

$$\begin{aligned}\operatorname{map}(K,-)\otimes z_{n+1} &= \operatorname{map}(K,-)\otimes(\mathrm{Id}^{\star n+1}\to 1)\\ &= \left(\operatorname{map}(K,-)\otimes \operatorname*{colim}_{\varnothing\neq U\subset \underline{n+1}}\operatorname{map}(U,-)\to\operatorname{map}(K,-)\right)\\ &= \left(\operatorname*{colim}_{\varnothing\neq U\subset \underline{n+1}}\operatorname{map}(K\star U,-)\to\operatorname{map}(K,-)\right)\end{aligned}$$

appearing from Proposition 4.3.1 have the T_n-closed maps as their right orthogonal class. The local objects are the n-excisive functors. This calculation has a nice parallel in orthogonal calculus in (7.3.1).

6.3 $\mathrm{Fun}(\mathrm{Fin}_*,\mathcal{S})$

Let Fin_* denote the category of pointed finite spaces together with the smash product and equip the category of functors $\mathrm{Fun}(\mathrm{Fin}_*,\mathcal{S})$ with the induced Day convolution. By Example 3.2.4(vi) $\mathrm{Fun}(\mathrm{Fin}_*,\mathcal{S})$ becomes a confined symmetric monoidal category.

We write $\operatorname{map}_*(A,-)$ for the functor represented by the pointed space A. The functor $X_\circ:\mathrm{Fin}_*\to\mathcal{S}$ forgetting the base point is represented by the pointed space S^0. It is the unit of the monoidal structure. The functor $\operatorname{map}_*(1,-)=1$ is the terminal functor. The map $S^0\to 1$ induces a natural transformation

$$z_0:1=\operatorname{map}_*(1,-)\to\operatorname{map}_*(S^0,-)=X_\circ.$$

For each $n\geqslant 0$ the square

$$\begin{tikzcd} \Gamma_{n+1}(X_\circ) \arrow[r] \arrow[d, "z_{n+1}:=z_0^{\star n+1}"'] & \mathrm{FW}_{n+1}(X_\circ) \arrow[d, "z_0^{\square n+1}"] \\ X_\circ \arrow[r, "\Delta"] & X_\circ^{n+1} \end{tikzcd} \tag{6.3.1}$$

is cartesian square by Lemma 5.3.1(iv). We will use this map z_{n+1} to start the machinery of Section 4. The tidyness of z_{n+1} is proved in Lemma 6.3.3.

Lemma 6.3.2. *For every $n\geqslant 0$, the domain of the map $z_{n+1}:(1\to X_\circ)^{\star n+1}$ is the functor*

$$\Gamma_{n+1}(X_\circ)=\operatorname*{colim}_{\varnothing\neq U\subset\underline{n+1}}\operatorname{map}_*(\Sigma U,-)\ .$$

In particular, it is a finitely presentable object in $\mathrm{Fun}(\mathrm{Fin}_*,\mathcal{S})$.

Proof. By (5.1.1) the codomain of the iterated pushout product $(1\to X_\circ)^{\square\, n+1}$ is the fat wedge:

$$\mathrm{FW}_{n+1}(X_\circ)=\operatorname*{colim}_{\varnothing\neq U\subset\underline{n+1}}X_\circ^{\complement U}\ =\ \operatorname*{colim}_{\varnothing\neq U\subset\underline{n+1}}\operatorname{map}_*((\complement U)_+,X_\circ)\ .$$

The codomain of the iterated join $z_{n+1}=(1\to X_\circ)^{\star n+1}$ is by Diagram (6.3.1) the base change of this colimit along the diagonal $X_\circ\to(X_\circ)^{n+1}$. To compute the pullback observe: for $U\subset\underline{n+1}$ the pullback of $(X_\circ)^{\underline{n+1}-U}\to(X_\circ)^{n+1}\leftarrow X_\circ$ is the Yoneda image of the span of pointed spaces $(\underline{n+1}\backslash U)_+\leftarrow\underline{n+1}_+\to 1_+$, whose pushout is ΣU. This is the unreduced suspension of the unpointed set U, pointed at the north pole say. So the codomain of z_{n+1} is $X_\circ^{\Sigma U}$. Then, by universality of colimits, the codomain of z_{n+1} is $\operatorname{colim}_{\varnothing\neq U\subset\underline{n+1}}X_\circ^{\Sigma U}$. This is a finite colimit of representable functors and proves the last statement. □

As in Definition 3.4.5 one gets T_n as an endofunctor of $\mathrm{Fun}(\mathrm{Fin}, \mathcal{S})$ by cotensoring:

$$\begin{aligned} T_n(F) &:= \left[\operatorname*{colim}_{\varnothing \neq U \subset \underline{n+1}} \mathrm{map}_*(\Sigma U, -), F\right] \\ &= \lim_{\varnothing \neq U \subset \underline{n+1}} F(\Sigma U \wedge -) \\ &= \lim_{\varnothing \neq U \subset \underline{n+1}} F(U \star -). \end{aligned}$$

using the natural isomorphism $A \wedge \Sigma U = A \star U$ for a pointed space A and an unpointed space U. Again Definition 3.4.5 yields P_n which takes the same form as in the previous section.

Lemma 6.3.3. *The map* $z_{n+1} : (1 \to X_\circ)^{\star n+1}$ *is tidy.*

Proof. Proposition 6.2.2 applies here as well since the P_n obtained from z_{n+1} here is the same as in the previous section. Let K be a k-connected space for $k \geqslant -1$. The map $1 \to K$ is $(k-1)$-connected since its fiber is ΩK. So the map $(1 \to K)^{\star n+1}$ is ℓ-connected for

$$\ell \;=\; (n+1)(k-1) + 2n \;=\; (n+1)k + n - 1$$

Hence the map z_{n+1} satisfies condition $O_n(-n+1, -1)$ and $P_n(z_{n+1})$ is invertible. $\square$

The result is the analogous of Theorem 6.2.6:

Theorem 6.3.4. (Goodwillie) *Goodwillie's functor* P_n *is a left exact reflector onto the subcategory of* n*-excisive functors in* $\mathrm{Fun}(\mathrm{Fin}_*, \mathcal{S})$.

6.4 $\mathrm{Fun}_*(\mathrm{Fin}_*, \mathcal{S}_*)$

We denote by $\mathcal{S}_*$ the category of pointed spaces. We write $\mathrm{map}_*(-,-)$ for the pointed mapping space. Let $\mathrm{Fun}_*(\mathrm{Fin}_*, \mathcal{S}_*)$ be the category of $\mathcal{S}_*$-enriched functors from finite pointed spaces to pointed spaces. This category is itself $\mathcal{S}_*$-enriched and we write $\mathrm{nat}_*(-,-)$ for this pointed space of $\mathcal{S}_*$-enriched natural transformations. If we equip both categories Fin_* and $\mathcal{S}_*$ with the smash product, $\mathrm{Fun}_*(\mathrm{Fin}_*, \mathcal{S}_*)$ with the associated Day convolution is a symmetric monoidal closed category. Its unit is the functor $\mathrm{Id}_* := \mathrm{map}_*(S^0, -) = \mathbb{1}$.

In Section 3.2 the concept of confined symmetric monoidal category was defined with respect to unpointed mapping spaces. To prove that $\mathrm{Fun}_*(\mathrm{Fin}_*, \mathcal{S}_*)$ we consider it as being enriched over $\mathcal{S}$. This is possible since forgetting the base point $u : \mathcal{S}_* \to \mathcal{S}$ is a lax symmetric monoidal and therefore yields an enrichment of $\mathrm{Fun}_*(\mathrm{Fin}_*, \mathcal{S}_*)$ over $\mathcal{S}$. It is given by $u(\mathrm{nat}_*(-,-))$.

Lemma 6.4.1. *The category* $\mathrm{Fun}_*(\mathrm{Fin}_*, \mathcal{S}_*)$ *is a confined symmetric monoidal category.*

Proof. The enriched Yoneda functor $\mathrm{Fin}_*^{\mathrm{op}} \to \mathrm{Fun}_*(\mathrm{Fin}_*, \mathcal{S}_*)$ induces a functor

$$N : \mathrm{Fun}_*(\mathrm{Fin}_*, \mathcal{S}_*) \to \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{S}).$$

A functor F is reduced if $F(1) = 1$ and we denote the full subcategory of reduced functors by a superscript $(-)^{\mathrm{red}}$. Now N can be factored in the following way

$$\mathrm{Fun}_*(\mathrm{Fin}_*, \mathcal{S}_*) = \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{S}_*)^{\mathrm{red}} = \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{S})^{\mathrm{red}} \subset \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{S}),$$

where we have first two equivalences of categories and then the inclusion of a full subcategory. This inclusion preserves filtered colimits. Hence N preserves filtered colimits. Therefore every representable functor $\mathrm{map}_*(A, -)$, A pointed, is compact in $\mathrm{Fun}_*(\mathrm{Fin}_*, \mathcal{S}_*)$. Now the result follows from Lemma 3.2.3. $\square$

Let the map $Z \to \mathbb{1}$ be the fiberwise join power $z_{n+1} : (1 \to \mathrm{Id}_*)^{\star n+1}$. The analogue of the cartesian square (6.3.1) coming from Lemma 5.3.1(iv) implies that there is an isomorphism

$$z_{n+1} : (1 \to \mathrm{Id}_*)^{\star n+1} = \left(\operatorname*{colim}_{\varnothing \neq U \subset \underline{n+1}} \mathrm{map}_*(\Sigma U, -) \to \mathrm{Id}_*\right)$$

of maps where Σ is the unreduced suspension. Further for any finite pointed space K:

$$\operatorname{map}_*(K,-)\otimes z_{n+1} = \Big(\operatorname*{colim}_{\varnothing\neq U\subset \underline{n+1}} \operatorname{map}_*(K\star U,-)\to \operatorname{map}_*(K,-)\Big).$$

It is now clear that the endofunctors T and P of $\mathrm{Fun}(\mathrm{Fin}_*,\mathcal{S}_*)$ associated to z_{n+1} described in Definition 3.4.5 are the functors T_n and P_n constructed by Goodwillie in [Good03]. Now Goodwillie's connectivity estimate 6.2.2 can be used to prove the following:

Theorem 6.4.2. *The map $z_{n+1}:(1\to \mathrm{Id}_*)^{\star n+1}$ is tidy. Theorem 4.2.16 applies and $P=P_n$ is a confined symmetric monoidal left exact localization. The reflector P_n is Goodwillie's P_n.*

7 Orthogonal calculus revisited

The construction of the orthogonal tower given in Section 7.2 is really the one from [Weiss98], but we are recasting it in the framework of confined symmetric monoidal categories from Section 4. Our main motivation though is to prove that orthogonal calculus is a special case of a completion tower developped in [ABFJ24b]. This is done in Section 7.3. As a bonus we obtain Blakers-Massey theorems for orthogonal calculus in Section 7.4.

Let us point out that the symmetric monoidal structures appearing in orthogonal calculus were already studied by Hendrian [Hen].

We are using a uniform notation to highlight how parallel the constructions of orthogonal calculus and Goodwillie calculus here are. Weiss is denoting the functor T_n by τ_n, and the functor P_n by T_n. We apologize for possible confusion.

Let $\mathcal{J}$ be the category of finite dimensional Euclidian vector spaces. The space of morphisms in $\mathcal{J}$ is the Stiefel manifold of linear isometries from U to V. We are going to denote this by $\mathcal{J}(U,V)=\mathrm{St}(U,V)$ following Weiss. The orthogonal sum equips $\mathcal{J}$ with a symmetric monoidal structure. This yields an $(\infty,1)$-symmetric monoidal category (the interested reader can consult [Hen, Proposition/Definition 4.1.1.4]). The category $\mathrm{Fun}(\mathcal{J},\mathcal{S})$ can be given the corresponding Day convolution product with the terminal functor $\mathrm{St}(0,-)=1$ as unit. This symmetric monoidal closed category is confined by Example 3.2.4(vi).

In $\mathcal{J}$ there is the inclusion $i_k:\mathbb{R}^k\to\mathbb{R}^{k+1}$ onto the first k coordinates. Via Yoneda it induces the map $j_k:\mathrm{St}(\mathbb{R}^{k+1},-)\to\mathrm{St}(\mathbb{R}^k,-)$. Let $\mathrm{Sph}(-)=\mathrm{St}(\mathbb{R},-):\mathcal{J}\to\mathcal{S}$ denote the unit sphere functor. We feed the map

$$j_0^{\square n+1}:\mathrm{Sph}(-)^{\star n+1}\to 1$$

into the setup of Section 4. This map is tidy by Proposition 7.2.3. This relies on connectivity estimates proved by Weiss [Weiss98]. As presecribed in Definition 3.4.5, the natural transformation $t_n(F)=[z_{n+1},F]:F\to T_n(F)$ gives rise to the endofunctor

$$P_n := \operatorname{colim}\big(\mathrm{Id}\xrightarrow{t_n} T_n\xrightarrow{t_nT_n} T_n^2\xrightarrow{t_nT_n^2} T_n^3\xrightarrow{t_nT_n^3} T_n^4\to\dots\big)$$

of $\mathrm{Fun}(\mathcal{J},\mathcal{S})$ with $p_n:\mathrm{Id}\to P_n$ the canonical map to the colimit.

7.1 Stiefel combinatorics

Lemma 7.1.1. *For every $k\geqslant 0$ there is a pushout*

$$\begin{tikzcd}
\mathrm{St}(\mathbb{R}^{k+1},V)\times\mathbb{S}^{k-1} \arrow[r, "j_k\times\mathrm{id}"] \arrow[d, "\mathrm{pr}_1"'] & \mathrm{St}(\mathbb{R}^{k},V)\times\mathbb{S}^{k-1} \arrow[d] \\
\mathrm{St}(\mathbb{R}^{k+1},V) \arrow[r] & \mathrm{St}(\mathbb{R}^{k},-)\times\mathrm{St}(\mathbb{R},-)
\end{tikzcd}$$

in $\mathrm{Fun}(\mathcal{J},\mathcal{S})$. In particular, the functor $\mathrm{St}(\mathbb{R}^k,-)\times\mathrm{St}(\mathbb{R},-)$ is finitely presentable.

Proof. In this proof we are working in the 1-category of topological spaces and continuous maps. Let us evaluate on a vector space V of dimension $\geqslant k$, since otherwise the values are empty. Consider the pushout:

$$\begin{tikzcd}
\mathrm{St}(\mathbb{R}^{k+1}, V) \times \mathbb{S}^{k-1} \arrow[r, "j_k \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \mathrm{incl}"'] & \mathrm{St}(\mathbb{R}^{k}, V) \times \mathbb{S}^{k-1} \arrow[d] \\
\mathrm{St}(\mathbb{R}^{k+1}, V) \times \mathbb{D}^{k} \arrow[r] & Q
\end{tikzcd}$$

A linear isometry f in $\mathrm{St}(\mathbb{R}^m, V)$ can be represented as an orthonormal frame $(f_1, \dots, f_m)$. Therefore, set-theoretically, Q is defined as the quotient of

$$\big(\mathrm{St}(\mathbb{R}^{k+1}, V) \times \mathbb{D}^k\big) \sqcup \big(\mathrm{St}(\mathbb{R}^{k}, V) \times \mathbb{S}^{k-1}\big)$$

by the relation

$$\big((f_1, \dots, f_k, f_{k+1}), (t_1, \dots, t_k)\big) \sim \big((g_1, \dots, g_k), (t_1, \dots, t_k)\big)$$

if and only if $f_i = g_i$ for all $1 \leqslant i \leqslant k$ and $\|(t_1, \dots, t_k)\| = 1$. Since the left vertical map is a cofibration, this square is a homotopy pushout. We want to show that Q is homeomorphic to $\mathrm{St}(\mathbb{R}^k, -) \times \mathrm{St}(\mathbb{R}, -)$.

Let us describe a map $q : Q \to \mathrm{St}(\mathbb{R}^k, V) \times \mathrm{St}(\mathbb{R}, V)$. To construct the map we use the fact that Q is a pushout. First let

$$q_1 : \mathrm{St}(\mathbb{R}^k, V) \times \mathbb{S}^{k-1} \to \mathrm{St}(\mathbb{R}^k, V) \times \mathrm{St}(\mathbb{R}, V)$$

be given by

$$(f_1, \dots, f_k; t) \mapsto (f_1, \dots, f_k \ ; \ \sum_{i=1}^k t_i f_i).$$

Note that $\|\sum_{i=1}^k t_i f_i\| = \|t\| = 1$, so $\sum_{i=1}^k t_i f_i$ is a vector on the boundary of the unit disc of $U = \mathrm{span}(f_1, \dots, f_k)$. Therefore the map q_1 is well-defined. Note also that U varies continuously with $(f_1, \dots, f_k)$, so q_1 is continuous.

On the part of Q involving $\mathrm{St}(\mathbb{R}^{k+1}, -) \times \mathbb{D}^k$ the idea is to use $t \in \mathbb{D}^k$ in the fiber over $(f_1, \dots, f_k, f_{k+1})$, to tilt f_{k+1} in such a way that its orthogonal projection into the span U of $(f_1, \dots, f_k)$ becomes t. The map

$$q_2 : \mathrm{St}(\mathbb{R}^{k+1}, V) \times \mathbb{D}^k \to \mathrm{St}(\mathbb{R}^k, V) \times \mathrm{St}(\mathbb{R}, V)$$

is defined as follows:

$$(f_1, \dots, f_k, f_{k+1} \ ; \ t) \mapsto (f_1, \dots, f_k \ ; \ t + (1 - \|t\|^2)^{1/2} \, f_{k+1})$$

with $t := \sum_{i=1}^k t_i f_i \in U$. Observe that

$$\|t + (1 - \|t\|^2)^{1/2} \, f_{k+1}\| = 1$$

since $t \cdot f_{k+1} = 0$ and $\|f_{k+1}\| = 1$. So q_2 is well-defined and continuous.

To obtain a map on the pushout Q we need to check that $q_1(j_k \times \mathrm{id}) = (\mathrm{id} \times \mathrm{incl}) q_2$. This is true because of the fact that, if $\|t\| = 1$, then $t + (1 - \|t\|^2)^{1/2} f_{k+1} = t = \sum_{i=1}^k t_i f_i$.

Now let us define a map in the other direction. An element in $\mathrm{St}(\mathbb{R}^k, V) \times \mathrm{St}(\mathbb{R}, V)$ is represented by an orthonormal frame $(f_1, \dots, f_k)$ and a single unit vector $e \in V$. Let U be the span of the frame. If p denotes the orthogonal projection of e into U, then $\|p\| \leqslant \|e\| = 1$, so p is in the unit disc of U using the basis $(f_1, \dots, f_k)$. Explicitly, $p = \sum_i (e \cdot f_i) f_i$ where the dot denotes the scalar product. Note $p \in \mathbb{S}^{k-1}$ if and only if $e = p \in U$. Now the map

$$s : \mathrm{St}(\mathbb{R}^k, V) \times \mathrm{St}(\mathbb{R}, V) \to Q$$

is defined by applying the Gram-Schmidt process whenever possible. So let s be given by

$$\big((f_1, \dots, f_k), e\big) \mapsto \begin{cases} (f_1, \dots, f_k, (e - p)/\|e - p\| \ ; \ p) & \text{if } e \notin U, \\[1ex] (f_1, \dots, f_k \ ; \ p) & \text{if } e \in U. \end{cases}$$

Note that in the case $e \notin U$ we have $e \neq p$ and $(f_1, \dots, f_k, (e-p)/\|e-p\|)$ is an orthonormal $(k+1)$-frame. So the image of s lies in $\mathrm{St}(\mathbb{R}^{k+1}, V) \times (\mathbb{D}^k - \mathbb{S}^{k-1}) \subset Q$. In the case $e \in U$ the image of s lies in $\mathrm{St}(\mathbb{R}^k, V) \times \mathbb{S}^{k-1} \subset Q$. The map s is continuous by the glueing that occurs in Q.

Now it is elementary to check that q and s are mutually inverse. $\square$

Lemma 7.1.2. *For every compact F in $\mathrm{Fun}(\mathcal{J}, \mathcal{S})$ the functor $\mathrm{Sph}(-) \star F$ is compact.*

Proof. Because F is compact, it is the retract of a finitely presentable functor G. Thus it suffices to show that the join with G is compact. Since G is of the form $\mathrm{colim}_{c \in C} \mathrm{St}(\mathbb{R}^{k(c)}, -)$ with C a finite category, the join $\mathrm{Sph}(-) \star G$ is the pushout of the following diagram:

$$\mathrm{Sph}(-) \leftarrow \operatorname*{colim}_C \left(\mathrm{Sph}(-) \times \mathrm{St}(\mathbb{R}^{k(c)}, -)\right) \to \operatorname*{colim}_C \mathrm{St}(\mathbb{R}^{k(c)}, -).$$

The two ends of the pushout are finitely presentable, and the center is a finite colimit of finitely presentable functors by Lemma 7.1.1. The claim follows. $\square$

Corollary 7.1.3. *The functor $\mathrm{Sph}(-)^{\star n+1}$ is compact.*

Proof. This follows inductively from Lemma 7.1.2. $\square$

Lemma 7.1.4 (Weiss)**.** *In $\mathrm{Fun}(\mathcal{J}, \mathcal{S})$ there is for all $n, k \geqslant 0$ a natural isomorphism*

$$\underbrace{j_k \star_B \dots \star_B j_k}_{n+1} = \left(\operatorname*{colim}_{0 \neq U \subset \mathbb{R}^{n+1}} \mathrm{St}(\mathbb{R}^k \oplus U, -) \to B \right)$$

of maps, where $B := \mathrm{St}(\mathbb{R}^k, -)$.

Proof. Using Weiss' notation in [Weiss95, Proposition 4.2] an isomorphism

$$\operatorname*{colim}_{0 \neq U \subset \mathbb{R}^{n+1}} \mathrm{St}(\mathbb{R}^k \oplus U, -) \xrightarrow{\cong} S\gamma_{n+1}(\mathbb{R}^k, -)$$

over B is constructed, where the target is the total space of the unit sphere bundle obtained from the vector bundle $\gamma_{n+1}(\mathbb{R}^k, -)$ over B, whose fiber at $f \in B := \mathrm{St}(\mathbb{R}^k, -)$ is $\mathbb{R}^{n+1} \otimes \mathrm{coker} f$. Then in the proof of [Weiss95, Proposition 5.4] the author notes that $\gamma_{n+1}(\mathbb{R}^k, -)$ is in fact isomorphic as a vector bundle over B to the Whitney sum $\bigoplus_{n+1} \gamma_1(\mathbb{R}^k, -)$. For the corresponding unit sphere bundle one obtains an identification of the structure map $S\gamma_1(\mathbb{R}^k, -) \to B$ with the map $j_k : \mathrm{St}(\mathbb{R}^{k+1}, -) \to \mathrm{St}(\mathbb{R}^k, -)$. For the unit sphere bundle the Whitney sum translates into a fiberwise join

$$\left(S\gamma_{n+1}(\mathbb{R}^k, -) \to B \right) \cong \underbrace{j_k \star_B \dots \star_B j_k}_{n+1}$$

proving the claim. $\square$

For $k = 0$ the target space is $B = \mathrm{St}(0, -) = 1$, so the map in Lemma 7.1.4 reduces to

$$\left(\operatorname*{colim}_{0 \neq U \subset \mathbb{R}^{n+1}} \mathrm{St}(U, -) \to 1 \right) = \underbrace{j_0 \star_B \dots \star_B j_0}_{n+1} = j_0^{\square n+1} = z_{n+1}.$$

Since the internal hom here is taken with respect to the Day convolution product coming from the orthogonal sum on $\mathcal{J}$, for every $F : \mathcal{J} \to \mathcal{S}$ we have

$$T_n F = \left[\mathrm{Sph}(-)^{\star n+1}, F\right] = \left[\operatorname*{colim}_{0 \neq U \subset \mathbb{R}^{n+1}} \mathrm{St}(\mathbb{R}^k \oplus U, -), F \right] = \lim_{0 \neq U \subset \mathbb{R}^{n+1}} F(U \oplus -).$$

7.2 Weiss' connectivity estimates

Lemma 7.2.1 (e.3 Lemma [Weiss98]). *Let $\alpha : F \to G$ be a morphism in $\mathrm{Fun}(\mathcal{J}, \mathcal{S})$. Suppose that there exists an integer b such that $\alpha(W) : F(W) \to G(W)$ is $((n+1)\dim(W) - b)$connected for all W in $\mathcal{J}$. Then $T_n(\alpha) : T_nF(W) \to T_nG(W)$ is $((n+1)\dim(W) - b + 1)$connected for all W.*

It is important and not difficult to note that the conclusion of the lemma remains true even if the assumption is only made for all W of $\dim W \geqslant \kappa$ for some $\kappa \geqslant 0$.

Lemma 7.2.2. *Let $\alpha : F \to G$ be a natural transformation in $\mathrm{Fun}(\mathcal{J}, \mathcal{S})$ such that the connectivity of the map $\alpha(W) : F(W) \to G(W)$ is $\geqslant (n+1)\dim(W) - b$ for all W in $\mathcal{J}$ of dimension $\geqslant \kappa$. Then $P_n(\alpha) : P_nF \to P_n$ is invertible.*

Proof. Under the assumptions on α, Weiss shows in Lemma 7.2.1 that the connectivity of the map $T_n(\alpha)$ is $\geqslant (n+1)\dim(W) - b + 1$ for all W of dimension $\geqslant \kappa - 1$. It follows by induction on ℓ that the connectivity of the map $T^\ell\alpha(W)$ is $\geqslant (n+1)\dim(W) - b + \ell$ for all W of dimension $\geqslant \kappa - l$. Hence the connectivity of the map $T^\ell(\alpha)(W)$ tends to infinity with ℓ for all objects of $\mathcal{J}$. $\square$

Proposition 7.2.3. *The map $z_{n+1} : \mathrm{Sph}(-)^{\star n+1} \to 1$ in $\mathrm{Fun}(\mathcal{J}, \mathcal{S})$ is tidy.*

Proof. The source of z_{n+1} is compact by Corollary 7.1.3. If W in $\mathcal{J}$ is of dimension m, then $\big(\mathrm{Sph}(-)(W)\big)^{\star n+1} = (S^{m-1})^{\star n+1}$ and its connectivity is $(n+1)(m-2) + 2n = (n+1)\dim(W) - 2$. Hence z_{n+1} satisfies the hypothesis of Proposition 7.2.2 with $b = 2$ and $\kappa = 1$. $\square$

Definition 7.2.4. [Weiss95, Def 5.1] A functor $F : \mathcal{J} \to \mathcal{S}$ is *polynomial of degree* $\leqslant n$ if the map $F \to T_n(F)$ is invertible. In other words, a functor $F : \mathcal{J} \to \mathcal{S}$ is polynomial of degree $\leqslant n$ if and only if it is T_n-closed.

Theorem 7.2.5 (Weiss). *Theorem 4.2.16 applies to the orthogonal tower. The functor P_n defined above is a left exact reflector onto the subcategory of polynomial functors of degree $\leqslant n$.*

Proof. Proposition 7.2.3 serves as input into Theorem 4.2.16. By Weiss' definition a T_n-closed object is exactly a polynomial functor of degree $\leqslant n$. So the theorem is proved. $\square$

7.3 The orthogonal tower is a completion tower

The orthogonal tower is a tower $\ldots \to P_2 \to P_1 \to P_0$ of left exact localizations of the functor category $\mathrm{Fun}(\mathcal{J}, \mathcal{S})$. If $\mathscr{A}_n$ denotes the class of P_n-equivalences in $\mathrm{Fun}(\mathcal{J}, \mathcal{S})$, then Proposition 4.3.1 tells us that $\mathscr{A}_n$ is the left class of a factorization system which is in fact a left exact modality. So $\mathscr{A}_n$ is congruence. In [ABFJ24b, Section 4.2] we introduce for any topos and any congruence in it a completion tower as a tower of left exact localizations whose n-th congruence $\mathscr{C}_n$ is obtained as the $(n+1)$-fold acyclic power $\mathscr{C}_n = \mathscr{C}_0 \,\square^a \ldots \square^a\, \mathscr{C}_0$ of its 0-th level. Here $-\square^a-$ denotes the acyclic product of congruences constructed in [ABFJ24b, Section 3.2]. Its main property for us here is that the acyclic product of two congruences is again a congruence [ABFJ24b, Theorem 3.3.3.] and the resulting formula [ABFJ24b, Corollary 3.3.8.] that allows us to keep track of how to generate the powers. In order to show that the orthogonal tower is a completion tower we need to show that $\mathscr{A}_n = \mathscr{C}_n$ for all $n \geqslant 0$.

In any functor category the representable functors form a dense subcategory. In $\mathcal{J}$ any object is isomorphic to $\mathbb{R}^k$ for some k. The representable functors at $\mathbb{R}^k, k \geqslant 0$ form a dense subcategory of compact objects of $\mathrm{Fun}(\mathcal{J}, \mathcal{S})$. Thus, according to Proposition 4.3.1, the congruence $\mathscr{A}_n$ is generated by the set

$$S_n = \left\{\sigma^k_{n+1} = z_{n+1} \otimes \mathrm{St}(\mathbb{R}^k, -) : \mathrm{Sph}(-)^{\star n+1} \otimes \mathrm{St}(\mathbb{R}^k, -) \to \mathrm{St}(\mathbb{R}^k, -) \,|\, k \geqslant 0\right\}$$

as a saturated class. Hence we have $\mathscr{A}_n = S_n^c = S_n^a = S_n^s$ using the congruence closure, the acyclic closure and the saturated closure studied in [ABFJ22]. The tensor product is the Day convolution with respect to the orthogonal sum in $\mathcal{J}$ which preserves colimits in both variables. With Lemma 7.1.4 we calculate:

$$\begin{aligned}\mathrm{Sph}(-)^{\star n+1} \otimes \mathrm{St}(\mathbb{R}^k,-) &= \Big(\operatorname*{colim}_{0\neq U\subset \mathbb{R}^{n+1}} \mathrm{St}(U,-)\Big) \otimes \mathrm{St}(\mathbb{R}^k,-) \\ &= \operatorname*{colim}_{0\neq U\subset \mathbb{R}^{n+1}} \big(\mathrm{St}(\mathbb{R}^k,-)\otimes \mathrm{St}(U,-)\big) \\ &= \operatorname*{colim}_{0\neq U\subset \mathbb{R}^{n+1}} \mathrm{St}(\mathbb{R}^k\oplus U,-)\end{aligned} \tag{7.3.1}$$

In fact, Lemma 7.1.4 identifies the map $\sigma^k_{n+1}\in S_n$ with the $(n+1)$-st fiberwise join power $j_k \star_B \dots \star_B j_k$. This map fits into a cartesian square

$$\begin{tikzcd} \operatorname*{colim}_{0\neq U\subset \mathbb{R}^{n+1}} \mathrm{St}(\mathbb{R}^k\oplus U,-) \arrow[r] \arrow[d, "\sigma^k_{n+1}=j_k\star_B\dots\star_B j_k"'] & F^k_{n+1} \arrow[d, "j_k^{\square n+1}"] \\ B=\mathrm{St}(\mathbb{R}^k,-) \arrow[r, "\Delta"] & \mathrm{St}(\mathbb{R}^k,-)^{n+1} \end{tikzcd} \tag{7.3.2}$$

by Lemma 5.3.1(iv) applied objectwise. (The source F^k_{n+1} of the join power is a relative fat wedge described in Lemma 5.3.1(iii).) For $n=0$ and all k we have

$$\sigma^k_1 = j_k : \mathrm{St}(\mathbb{R}^{k+1},-)\to \mathrm{St}(\mathbb{R}^k,-)$$

and $S_0=\{j_k \,|\, k\geqslant 0\}$. Note that an $F:\mathcal{J}\to\mathcal{S}$ is S_0-local if and only if it is constant and the reflector P_0 can be identified with Weiss' description

$$P_0F = \operatorname*{colim}_{k\in\mathbb{N}} F(\mathbb{R}^k) =: F(\mathbb{R}^\infty). \tag{7.3.3}$$

If we set $\mathscr{C}_0 := \mathscr{A}_0$, then in the completion tower we have

$$\mathscr{C}_n = \underbrace{\mathscr{A}_0 \,\square^a \dots \square^a\, \mathscr{A}_0}_{n+1} = \{j_{k_1}\,\square\,\dots\,\square\, j_{k_{n+1}} \,|\, k_1,\dots,k_{n+1}\geqslant 0\}^a,$$

by [ABFJ24b, Corollary 3.3.8], since S_0 is a lex generator: $S_0^s = S_0^a = S_0^c = \mathscr{A}_0$.

Theorem 7.3.4. *The orthogonal tower is a special case of a completion tower: $\mathscr{A}_n=\mathscr{C}_n$ for all $n\geqslant 0$.*

Proof. Let us define an auxiliary congruence $\mathscr{B}_n=\{j_k^{\square n+1}\,|\,k\geqslant 0\}^c$. Then:

$$\mathscr{A}_n\subset\mathscr{B}_n\subset\mathscr{C}_n.$$

The first inclusion holds since the generating maps of $\mathscr{A}_n=\{\sigma^k_{n+1}\,|\,k\geqslant 0\}^s$ are obtained from the generators of $\mathscr{B}_n$ by base change as exhibited by the cartesian square (7.3.2) and congruences are closed under base change. The second inclusion is clear since the generators of $\mathscr{B}_n$ are among those of $\mathscr{C}_n$.

To show the reverse inclusion we prove that the generators $j_{k_1}\,\square\,\dots\,\square\, j_{k_{n+1}}$ of $\mathscr{C}_n$ belong to $\mathscr{A}_n$. In other words, we need to explain that they are P_n-equivalences. This happens via a connectivity estimate. Let W be a Euclidian vector space of dimension m. Then the fiber of the map $j_\ell:\mathrm{St}(\mathbb{R}^{\ell+1},W)\to \mathrm{St}(\mathbb{R}^\ell,W)$ is a sphere of dimension $m-\ell-1$ as long as $\ell<m$. So j_ℓ is $(m-\ell-2)$-connected. Therefore, if $m>\max\{k_i\}$, the connectivity of the map $j_{k_1}\,\square\,\dots\,\square\, j_{k_{n+1}}$ is

$$\sum_{i=1}^{n+1}(m-k_i-2)+2n = (n+1)m-2-\sum_{i=1}^{n+1}k_i.$$

Hence, $j_{k_1}\,\square\,\dots\,\square\, j_{k_{n+1}}$ satisfies the assumptions of Proposition 7.2.2 which implies that it is a P_n-equivalence. Thus, $\mathscr{C}_n\subset\mathscr{A}_n$ and equality follows. ∎

7.4 Blakers-Massey theorems

For any completion tower there is an associated Blakers-Massey theorem and a "dual" version. As a consequence we obtain new results for the orthogonal tower. The respective versions for the Goodwillie tower were proved as Theorems 3.4.1 and 3.4.2 in [ABFJ18].

Theorem 7.4.1. *Consider in the category* $\mathrm{Fun}(\mathcal{J}, \mathcal{S})$ *a pushout square*

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \downarrow \\ B & \longrightarrow & D \end{array}$$

where f *is a* P_m*-equivalence and* g *is a* P_n*-equivalence. Then the gap map*

$$A \xrightarrow{(f,g)} B \times_D C$$

is a P_{m+n+1}*-equivalence.*

In the proof we are using the notation from the previous section: $\mathscr{A}_n$ are the P_n-equivalences. We also denote the acyclic powers simply as powers (since the cartesian product is never considered).

Proof. By assumption $f \in \mathscr{A}_m$ and $g \in \mathscr{A}_n$. Congruences are closed under finite limits, hence $(\Delta f : A \to A \times_B A) \in \mathscr{A}_m = \mathscr{A}_0^{m+1}$ and $\Delta g \in \mathscr{A}_n = \mathscr{A}_0^{n+1}$ since the orthogonal tower is a completion tower by Theorem 7.3.4. By the generalized Blakers-Massey Theorem 4.1.1. [ABFJ20] we deduce

$$(f,g) \in \left(\Delta f \,\square\, \Delta g\right)^a \subset (\mathscr{A}_0^{m+1} \,\square\, \mathscr{A}_0^{n+1})^a \subset \mathscr{A}_0^{m+n+2}$$

proving the claim. (The inclusion to the right is actually an equality by [ABFJ24b, Thm 3.3.3.], but this is not really needed here.) $\square$

Theorem 7.4.2. *Consider in the category* $\mathrm{Fun}(\mathcal{J}, \mathcal{S})$ *a pullback square*

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow{f} & D \end{array}$$

where f *is a* P_m*-equivalence and* g *is a* P_n*-equivalence. Then the cogap map*

$$B \sqcup_A C \to D$$

is a P_{m+n+1}*-equivalence.*

Proof. Similarly to the proof above, by [ABFJ20, Theorem 3.5.1] the cogap map is in $(\mathscr{A}_0^{m+1} \,\square\, \mathscr{A}_0^{n+1})^a = \mathscr{A}_0^{m+n+2}$. $\square$

In fact, all the appropriate analogues of the statements in [ABFJ18, Section 2.5] about stability and delooping results hold for orthogonal calculus as they hold for any completion tower in the sense of [ABFJ24b]. For example, it follows from the theorems 7.4.1 and 7.4.2 that the category of pointed $(2n-1)$-polynomial functors ($P_{2n-1}F = F$) and n-reduced functors ($P_{n-1}F = 1$) is stable.

8 Localizing module categories

8.1 Tidy Localizations of confined module categories

Definition 8.1.1. From a map $z : Z \to \mathbb{1}$ in a confined symmetric monoidal category $\mathcal{V}$ we obtain by cotensoring a $\mathcal{V}$-functor

$$S := \{Z, -\} : \mathcal{M} \to \mathcal{M}$$

and $s = \{z, -\} : \mathrm{Id} \to S$. We repeat Definition 3.4.5 with S (instead of T) in the context of a confined $\mathcal{V}$-module $\mathcal{M}$. We will denote by $Q : \mathcal{M} \to \mathcal{M}$ the colimit

$$Q := \operatorname{colim}\big(\mathrm{Id} \xrightarrow{s} S \xrightarrow{sS} S^2 \xrightarrow{sS^2} S^3 \xrightarrow{sS^3} S^4 \to \dots\big).$$

Even though z is a map in $\mathcal{V}$, the resulting Q is an endofunctor of $\mathcal{M}$.

The analogue of Lemma 3.4.6 for S, s, Q and q holds and the proof given there goes through because all the statements referenced there, e.g. Lemma 2.1.1, Proposition 2.1.2 and Proposition 3.4.4, were proved in sufficient generality. We point out again how helpful the papers [Hei] and [Hei23] by Heine here are.

The main Theorem 8.1.5 about localizing module categories is analogous to and a consequence of Theorem 4.2.16 using the trick of putting a symmetric monoidal category and a closed module over it into a single symmetric monoidal category. This construction, explained just below, is analogous to a square zero extension from ordinary commutative algebra.

Let us consider the category $\mathcal{V} \times \mathcal{M}$. Since limits and colimits are computed factorwise, for every objects (V, M) in $\mathcal{V} \times \mathcal{M}$ we have

$$(V, M) = (V, 0) \sqcup (0, M) \quad \text{and} \quad (V, M) = (V, 1) \times (1, M)\ .$$

There are fully faithful inclusion functors

$$i_{\mathcal{V}} : \mathcal{V} \cong \mathcal{V} \times \{0\} \to \mathcal{V} \times \mathcal{M}$$

and

$$i_{\mathcal{M}} : \mathcal{M} \cong \{0\} \times \mathcal{M} \to \mathcal{V} \times \mathcal{M}$$

and we will identify the categories $\mathcal{V}$ and $\mathcal{M}$ with the respective full subcategories of $\mathcal{V} \times \mathcal{M}$.

Proposition 8.1.2. *The category $\mathcal{V} \times \mathcal{M}$ has the structure of a symmetric monoidal closed category with the tensor product defined by*

$$(V_1, M_1) \otimes (V_2, M_2) = (V_1 \otimes V_2, V_1 \otimes M_2 \sqcup V_2 \otimes M_1)\ .$$

The unit is $\mathbb{1} := (\mathbb{1}_{\mathcal{V}}, 0)$. Moreover:

$$[(V_1, M_1), (V_2, M_2)]_{\mathcal{V}\times\mathcal{M}} = ([V_1, V_2] \times [M_1, M_2], \{V_1, M_2\})$$

The symmetric monoidal category $\mathcal{V} \times \mathcal{M}$ is confined, if $\mathcal{V}$ and $\mathcal{M}$ are confined.

The proof is left to the reader.

Now let us consider the image of the map $z : Z \to \mathbb{1}$ in $\mathcal{V} \times \mathcal{M}$:

$$i_{\mathcal{V}}(z) = \big((Z, 0) \to (\mathbb{1}, 0)\big) = (z, \mathrm{id}_0).$$

Following the recipe in Definition 3.4.5 we write down the associated cotensor

$$\mathscr{T}(F) := [Z, F]_{\mathcal{V}\times\mathcal{M}}$$

and

$$\tau(F) := [i_{\mathcal{V}}(z), F]_{\mathcal{V}\times\mathcal{M}} : F = (V, M) \to [(Z, 0), (V, M)]_{\mathcal{V}\times\mathcal{M}}$$

for every $F = (V, M)$ in $\mathcal{V} \times \mathcal{M}$, and obtain an endofunctor $\mathscr{T} : \mathcal{V} \times \mathcal{M} \to \mathcal{V} \times \mathcal{M}$ together with a natural transformation $\tau : \mathrm{Id} \to \mathscr{T}$. By construction we have

$$\begin{aligned}\mathscr{T}(F) &= [Z, F]_{\mathcal{V}\times\mathcal{M}} = [(Z, 0), (V, M)]_{\mathcal{V}\times\mathcal{M}} = ([Z, V]_{\mathcal{V}}, \{Z, M\}) \\ &= (TV, SM)\end{aligned}$$

since $[0, M] = 1$. Moreover

$$\tau(F) = [i_{\mathcal{V}}(z), F]_{\mathcal{V}\times\mathcal{M}} = ([z, V], \{z, M\}) = (tV, sM) \tag{8.1.3}$$

with $t : \mathrm{Id} \to T$ and $s : \mathrm{Id} \to S$ from Definitions 3.4.5 and 8.1.1. We also define an endofunctor $\mathscr{P} : \mathcal{V} \times \mathcal{M} \to \mathcal{V} \times \mathcal{M}$ together with a natural transformation $\pi : \mathrm{Id} \to \mathscr{P}$ by the colimit

$$\mathscr{P} := \operatorname{colim}\big(\ \mathrm{Id} \xrightarrow{\tau} \mathscr{T} \xrightarrow{\tau\mathscr{T}} \mathscr{T}^2 \xrightarrow{\tau\mathscr{T}^2} \mathscr{T}^3 \xrightarrow{\tau\mathscr{T}^3} \mathscr{T}^4 \to \dots\ \big).$$

Then for every $F = (V, M)$ in $\mathcal{V}$ we have

$$\mathscr{P}(F) = (PV, QM) \quad \text{and} \quad \pi(F) = (pV, qM). \tag{8.1.4}$$

With this construction in place we are ready to prove the following

Theorem 8.1.5. *Let $\mathcal{V}$ be a confined symmetric monoidal category and let $\mathcal{M}$ be a confined $\mathcal{V}$-module. Suppose that the map $z : Z \to \mathbb{1}$ in $\mathcal{V}$ is tidy. Let $\mathcal{M}^Q$ denote the full subcategory of Q-closed objects of $\mathcal{M}$.*

(i) *An object in $\mathcal{M}$ is S-closed if and only if it is Q-closed.*

(ii) *The subcategory $\mathcal{M}^Q$ is $\mathcal{V}$-reflective, and the natural transformation $q : \mathrm{Id} \to Q$ is $\mathcal{V}$-reflecting into $\mathcal{M}^Q$. The reflector $Q : \mathcal{M} \to \mathcal{M}^Q$ is $\mathcal{V}$-left exact.*

(iii) *The category $\mathcal{M}^Q$ is a closed $\mathcal{V}^P$-module with the action $\otimes_Q$ defined by letting $F \otimes_Q M := Q(F \otimes M)$ for F in $\mathcal{V}^P$ and M in $\mathcal{M}^Q$.*

(iv) *The closed $\mathcal{V}^P$-module $\mathcal{M}^Q$ is confined and $Q : \mathcal{M} \to \mathcal{M}^Q$ is confined. Every compact object of $\mathcal{M}^Q$ is a retract of an object in $Q(\mathsf{c}(\mathcal{M}))$. The subcategory $Q(\mathsf{c}(\mathcal{M}))$ is dense in $\mathcal{M}^Q$.*

Proof. We want to apply Theorem 4.2.16 to the confined symmetric monoidal category $\mathcal{V} \times \mathcal{M}$ but we need to first show that the map $i_{\mathcal{V}}(z) = (z, \mathrm{id}_0)$ is tidy. Observe that $i_{\mathcal{V}}(Z) = (Z, 0)$ and $i_{\mathcal{V}}(\mathbb{1}) = (\mathbb{1}, 0)$ are clearly compact in $\mathcal{V} \times \mathcal{M}$ and

$$\mathscr{P}(i_{\mathcal{V}}(z)) = (P(z), Q(\mathrm{id}_0))$$

is invertible if and only if $P(z)$ is invertible. But $P(z)$ is invertible by the assumption that z in $\mathcal{V}$ is tidy. So $i_{\mathcal{V}}(z) = (z, \mathrm{id}_0)$ is tidy. Now we just need to read off, what Theorem 4.2.16 applied to $\mathcal{V} \times \mathcal{M}$ says for $\mathcal{M}$ viewed as a subcategory via $i_{\mathcal{M}} : \mathcal{M} \to \mathcal{V} \times \mathcal{M}$.

(i): Theorem 4.2.16(i) states for $\mathcal{V} \times \mathcal{M}$ that $\mathscr{T}$-closed is equivalent to being $\mathscr{P}$-closed, and for $\mathcal{V}$ that T-closed is equivalent to being P-closed. It follows from (8.1.3) that an object (V, M) in $\mathcal{V} \times \mathcal{M}$ is $\mathscr{T}$-closed if and only if V is T-closed and M is S-closed. Similarly, from (8.1.4), (V, M) is $\mathscr{P}$-closed if and only if V is P-closed and M is Q-closed. Hence, M in $\mathcal{M}$ is S-closed if and only if it is Q-closed.

Let $(\mathcal{V} \times \mathcal{M})^{\mathscr{P}}$ denote the full subcategory of $\mathscr{P}$-closed objects of $\mathcal{V} \times \mathcal{M}$. Part (i) implies that the standard inclusion factors as $(\mathcal{V} \times \mathcal{M})^{\mathscr{P}} \cong \mathcal{V}^P \times \mathcal{M}^Q \subset \mathcal{V} \times \mathcal{M} = \mathcal{V} \times \mathcal{M}$.

(ii): Theorem 4.2.16(iii) applied to $\mathcal{V} \times \mathcal{M}$ states that $\pi : \mathrm{Id} \to \mathscr{P}$ is $\mathcal{V}$-reflecting onto $(\mathcal{V} \times \mathcal{M})^{\mathscr{P}}$:

$$\begin{aligned}\big([PV_1, V_2] \times [QM_1, M_2], \{PV_1, M_2\}\big) = [\mathscr{P}(V_1, M_1), (V_2, M_2)]_{\mathcal{V}\times\mathcal{M}} \xrightarrow{\cong} \\ [(V_1, M_1), (V_2, M_2)]_{\mathcal{V}\times\mathcal{M}} = \big([V_1, V_2] \times [M_1, M_2], \{V_1, M_2\}\big)\end{aligned}$$

for all (V_1, M_1) in $\mathcal{V} \times \mathcal{M}$ and (V_2, M_2) in $(\mathcal{V} \times \mathcal{M})^{\mathscr{P}} = \mathcal{V}^P \times \mathcal{M}^Q$. Set $V_1 = 0$. Then, for all M_1 in $\mathcal{M}$ and M_2 in $\mathcal{M}^Q$ we have

$$\begin{aligned}\big([QM_1, M_2], 1\big) &= \big([P(0), 0] \times [QM_1, M_2], \{P(0), M_2\}\big) \\ &= \big([0, V_2] \times [M_1, M_2], \{0, M_2\}\big) \\ &= \big([QM_1, M_2], 1\big)\end{aligned}$$

So clearly $q : \mathrm{Id} \to Q$ is $\mathcal{V}$-reflecting $\mathcal{M}$ onto $\mathcal{M}^Q$. The fact, that for all M_1 the object QM_1 is in $\mathcal{M}^Q$, was already clear since $\mathscr{P}(0, M_1) = (0, QM_1)$ is in $(\mathcal{V} \times \mathcal{M})^{\mathscr{P}} = \mathcal{V}^P \times \mathcal{M}^Q$.

The endofunctor $\mathscr{P} = (P, Q)$ is left exact and preserves compact cotensors by Theorem 4.2.16(iii). Since (co-)limits in $\mathcal{V} \times \mathcal{M}$ are computed separately in $\mathcal{V}$ and $\mathcal{M}$, it follows that Q preserves finite limits. Hence:

$$\begin{aligned}[0, Q\{V_1, M_2\}]_{\mathcal{V}\times\mathcal{M}} = \mathscr{P}[(V_1, 0), (0, M_2)]_{\mathcal{V}\times\mathcal{M}} = [(V_1, 0), \mathscr{P}(0, M_2)]_{\mathcal{V}\times\mathcal{M}} \\ = [(V_1, 0), (0, QM_2)]_{\mathcal{V}\times\mathcal{M}} = [0, \{V_1, QM_2\}]_{\mathcal{V}\times\mathcal{M}}\end{aligned}$$

for all compact V_1 in $\mathcal{V}$ and all M_2 in $\mathcal{M}$. So Q also preserves compact cotensors and is therefore $\mathcal{V}$-left exact.

(iii) According to Theorem 4.2.16(iv) is the category $(\mathcal{V} \times \mathcal{M})^{\mathscr{P}}$ a symmetric monoidal $\mathcal{V}^P$-module with tensor

$$\begin{aligned}(V_1, M_1) \otimes_{\mathscr{P}} (V_2, M_2) &= \mathscr{P}\big((V_1, M_1) \otimes (V_2, M_2)\big)\\ &= \mathscr{P}(V_1 \otimes V_2, V_1 \otimes M_2 \sqcup V_2 \otimes M_1)\\ &= \big((P(V_1 \otimes V_2), Q(V_1 \otimes M_2 \sqcup V_2 \otimes M_1)\big)\end{aligned}$$

for all V_1, V_2 in $\mathcal{V}^P$ and M_1, M_2 in $\mathcal{M}^Q$. Setting V_2 and M_1 equal to 0 one obtains an action of $\mathcal{V}^P$ on $\mathcal{M}^Q$ given by

$$V_1 \otimes_Q M_2 = Q(V_1 \otimes M_2)$$

as claimed and Q preserves the tensor action.

(iv) We know from (ii) that the category $\mathcal{M}^Q$ is a reflective subcategory of $\mathcal{M}$. So it is cocomplete and the localization functor $Q : \mathcal{M} \to \mathcal{M}^Q$ is cocontinous. As an endofunctor, $\mathscr{P} = (P, Q)$ is docile by Lemma 3.4.6 applied to $\mathcal{V} \times \mathcal{M}$. But colimits are computed separately in $\mathcal{V}$ and $\mathcal{M}$. Thus, Q commutes with filtered colimits. In (iii) we showed that Q is $\mathcal{V}$-left exact, hence it is a docile endofunctor of $\mathcal{M}$. Thus the inclusion $\mathcal{M}^Q \subset \mathcal{M}$ preserves filtered colimits. By Proposition 3.1.3, the left adjoint localization functor $Q : \mathcal{M} \to \mathcal{M}^Q$ is confined. In particular, $Q(\mathsf{c}(\mathcal{M})) \subset \mathsf{c}(\mathcal{M}^Q)$.

From this it follows that $Q(\mathsf{c}(\mathcal{M}))$ is dense in $\mathcal{M}^Q$, that $\mathcal{M}^Q$ is ω-presentable and that every object in $\mathsf{c}(\mathcal{M}^Q)$ is a retract of an object in $Q(\mathsf{c}(\mathcal{M}))$ exactly as in the proof of Theorem 4.2.16(v).

It remains to show that the closed $\mathcal{V}$-category $\mathcal{M}^Q$ is confined. We have

$$P(\mathsf{c}(\mathcal{V})) \otimes_Q Q(\mathsf{c}(\mathcal{M}) = Q\big(\mathsf{c}(\mathcal{V}) \otimes \mathsf{c}(\mathcal{M})\big) \subset Q(\mathsf{c}(\mathcal{M})) \subset \mathsf{c}(\mathcal{M}^Q).$$

by the definition of $- \otimes_Q -$, by $\mathcal{M}$ being confined as a closed $\mathcal{V}$-module and by (iii). Also by (iii) the category $Q(\mathsf{c}(\mathcal{V}))$ is dense in $\mathcal{M}^Q$. Lemma 3.2.3 allows for an adaption to the module case with the same proof. It follows from this adaption that the closed $\mathcal{V}$-category $\mathcal{M}^Q$ is confined. $\square$

Lemma 8.1.6. *Let $z : Z \to \mathbb{1}$ be tidy. In $\mathcal{M}$ let $\mathcal{L}$ be the class of Q-equivalences and let $\mathcal{R}$ is the class of Q-closed maps. Write $\Sigma = \{Z \otimes M \xrightarrow{z \otimes M} M \mid M \text{ compact}\}$.*

1. *The pair $(\mathcal{L}, \mathcal{R})$ is a left exact modality in $\mathcal{M}$.*
2. *$\Sigma^\perp = \mathcal{R}$ and $\mathcal{L} = \Sigma^s = {}^\perp(\Sigma^\perp)$*

The proof of this lemma is the same as the proof of Proposition 4.3.1.

Proposition 8.1.7. *Let $\mathcal{V}$ be a confined symmetric monoidal category and let $Z \to \mathbb{1}$ be tidy. Let $\phi : \mathcal{M} \to \mathcal{N}$ be a confined functor of confined $\mathcal{V}$-modules. Then the functor ϕ takes Q-equivalences in $\mathcal{M}$ to Q-equivalences in $\mathcal{N}$. Its right adjoint ϕ_* takes Q-closed objects in $\mathcal{N}$ to Q-closed objects in $\mathcal{M}$, and the following squares*

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\phi"] \arrow[d, "Q"'] & \mathcal{N} \arrow[d, "Q"] \\ \mathcal{M}^Q \arrow[r, "\phi^Q"] & \mathcal{N}^Q \end{tikzcd} \qquad \begin{tikzcd} \mathcal{M} & \mathcal{N} \arrow[l, "\phi_*"'] \\ \mathcal{M}^Q \arrow[u, "\text{inc}"] & \mathcal{N}^Q \arrow[l, "\phi_*|"'] \arrow[u, "\text{inc}"'] \end{tikzcd}$$

commute, where $\phi^Q(M) := Q(\phi(M))$ for every M in $\mathcal{M}^Q$ and $\phi^*|$ is the restriction of ϕ^*.

Proof. We write $\mathcal{L}_1, \mathcal{L}_2$ for the classes of Q-equivalences in $\mathcal{M}$ and $\mathcal{N}$ with $\mathcal{R}_2$ the corresponding right class of Q-closed maps. Similarly, we let $\Sigma_1 = \{z \otimes M \mid M \text{ in } \mathsf{c}(\mathcal{M})\}$ and $\Sigma_2 = \{z \otimes N \mid N \text{ in } \mathsf{c}(\mathcal{N})\}$. From Lemma 8.1.6 we know $\mathcal{L}_1 = \Sigma_1^s$ and $\mathcal{L}_2 = \Sigma_2^s$.

Since ϕ is a morphism of $\mathcal{V}$-modules, we have $\phi(z \otimes M) = z \otimes \phi(M)$. Since ϕ is confined, $\phi(M)$ is a compact object. It follows $\phi(\Sigma_1) \subset \Sigma_2 \subset \mathcal{L}_2$ or equivalently $\Sigma_1 \subset \phi^{-1}(\mathcal{L}_2)$. But ϕ is cocontinuous. Therefore $\phi^{-1}(\mathcal{L}_2)$ is saturated and we have $\mathcal{L}_1 = \Sigma^s \subset \phi^{-1}(\mathcal{L}_2)$ or equivalently $\phi(\mathcal{L}_1) \subset \mathcal{L}_2$. This yields the first claim.

Now let g be Q-closed in $\mathcal{N}$. For $\phi_*(g)$ to be Q-closed, it suffices by Lemma 8.1.6 that we have $f \perp \phi_*(g)$ for every $f \in \mathcal{L}_2$. Since the functor ϕ_* is right adjoint to ϕ, this is equivalent to $\phi(f) \perp g$ which is true, since $\phi(\mathcal{L}_1) \subset \mathcal{L}_2 = {}^{\perp}\mathcal{R}_2$ by Lemma 8.1.6.

Consequently, there is an induced functor $\phi_*| : \mathcal{N}^Q \to \mathcal{M}^Q$. The right hand square above then clearly commutes. It is also easy to verify that the functor $\phi^Q : \mathcal{M}^Q \to \mathcal{N}^Q$ is left adjoint to $\phi_*|$ and so the left hand square above commutes. □

8.2 Applications to module categories

The first application of the module version of our localization technique in Theorem 4.2.16 is to construct the Goodwillie tower in the category $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$ for any small category $\mathcal{C}$ possessing finite colimits and a terminal object. The idea is that $\mathcal{C}$ is aciii) the category $Q(\mathsf{c}(\mathcal{V}))$ is dense in $\mathcal{M}^Q$. Lemma 3.2.3 allows for a adaption to the module case with the same proof. It follows from this adaptionted on by Fin via a join product. Below we are going to give a second application for the orthogonal tower.

To construct the action $- \star - : \mathrm{Fin} \times \mathcal{C} \to \mathcal{C}$ note first that Fin acts on any finitely cocomplete category:

$$K \times C := \bigsqcup_K C.$$

This is a tensor action in the $(\infty, 1)$-categorical sense and this is proved in [Lur17, Section 2.4.3]. Note that $1 \times C = C$. Using the terminal object of $\mathcal{C}$ we can promote this action to a join. We define $K \star C$ as the following pushout:

$$\begin{tikzcd} K \times C \arrow[r, "\mathrm{pr}_C"] \arrow[d, "\mathrm{pr}_K"'] & C \arrow[d] \\ K \arrow[r] & K \star C \end{tikzcd}$$

Here the maps pr_C and pr_K are induced by the maps $C \to 1$ and $K \to 1$.

In this way we obtain the action $- \star - : \mathrm{Fin} \times \mathcal{C} \to \mathcal{C}$. Now we deduce from Example 3.3.5, that the functor category $\mathcal{M} = \mathrm{Fun}(\mathcal{C}, \mathcal{S})$ with its Day convolution product derived from this join action is a confined $\mathcal{V}$-module where $\mathcal{V} = \mathrm{Fun}(\mathrm{Fin}, \mathcal{S})$ with its Day convolution product derived from the join on Fin.

In Section 6.2 we showed that the map $z : \mathrm{Id}^{\star n+1} \to 1$ in $\mathrm{Fun}(\mathrm{Fin}, \mathcal{S})$ is tidy and for varying $n \geqslant 0$ produces the n-th stage of the Goodwillie tower. It is a consequence of Theorem 8.1.5 that this is now also the case in $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$. All we need to check is that Goodwillie's endofunctor T_n coincides with the one here. But this is clearly the case since

$$\{\mathrm{Id}^{n+1}, F\} = \left\{ \operatorname*{colim}_{\varnothing \neq U \subset \underline{n+1}} \mathrm{map}(U, -), F \right\} = \lim_{\varnothing \neq U \subset \underline{n+1}} F(U \star -)$$

by Lemma 6.1.2, where $U \star -$ is the action of Fin on $\mathcal{C}$.

Theorem 8.2.1. (Goodwillie) *Let $\mathcal{C}$ be a small finitely cocomplete category with a terminal object. Then the tidy map* $\mathrm{Id}^{\star n+1} \to 1$ *in* $\mathrm{Fun}(\mathrm{Fin}, \mathcal{S})$ *yields Goodwillie's reflector P_n in the category* $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$.

8.3 The image of a tidy map via a confined functor

Consider a confined symmetric monoidal functor $\phi : \mathcal{V} \to \mathcal{E}$ between confined symmetric monoidal categories. In this situation Example 3.3.3 states that $\mathcal{E}$ becomes a confined $\mathcal{V}$-module. Also $\phi(Z)$ is compact in $\mathcal{E}$. Now we define $S := [\phi(Z), -] : \mathcal{E} \to \mathcal{E}$, $s := [\phi(z), -] = \{z, -\} : \mathrm{Id} \to S$, $Q = \mathrm{colim}_n S^n$ and $q : \mathrm{Id} \to Q$ as in Definition 3.4.5. Note that S can be viewed as an $\mathcal{E}$-functor $S = [\phi(Z), -]$ or as a $\mathcal{V}$-functor $S = \{Z, -\}$. Accordingly, $s := [\phi(z), -] = \{z, -\} : \mathrm{Id} \to S$ is a natural transformation that is both $\mathcal{V}$-enriched and $\mathcal{E}$-enriched, and the same is true for the resulting Q. The functor Q is docile both as a $\mathcal{V}$-functor and as an $\mathcal{E}$-functor by Lemma 3.4.6 as it preserves cotensors by compact objects in $\mathcal{V}$ and $\mathcal{E}$.

Theorem 8.3.1. *A confined symmetric monoidal functor $\phi : \mathcal{V} \to \mathcal{E}$ between confined symmetric monoidal categories takes a tidy map $z : Z \to \mathbb{1}$ in $\mathcal{V}$ to a tidy map $\phi(z) : \phi(Z) \to \phi(\mathbb{1}) = \mathbb{1}$ in $\mathcal{E}$.*

Proof. Taking Q as an endofunctor of the closed $\mathcal{V}$-module $\mathcal{E}$, Theorem 8.1.5(ii) applies and the natural transformation $q : \mathrm{Id} \to Q$ reflects $\mathcal{E}$ onto the subcategory $\mathcal{E}^Q$ of Q-closed objects. So now we know already that the localization in $\mathcal{E}$ exists, is confined and $\mathcal{V}$-left exact. Since $\{z, -\} = [\phi(z), -]$, the localization Q is constructed according to Definition 3.4.5, but we do not yet know that $\phi(z)$ is tidy. (In particular, we do not yet know that Q is symmetric monoidal. But this will follow, once tidyness is proved.) It remains to work somewhat in reverse. We claim that $Q(\phi(z))$ is invertible.

Step 1: Take a compact object K and a Q-closed object N in $\mathcal{E}$. Then the triangle

$$\begin{tikzcd} & {[K,N]} \arrow[dl, "{q[K,N]}"'] \arrow[dr, "{[K,qN]}", "\cong"'] & \\ Q[K,N] \arrow[rr, "\cong", "\gamma(K,N)"'] & & {[K,QN]} \end{tikzcd}$$

commutes. Since Q is a docile $\mathcal{E}$-functor, the coassembly map is an isomorphism. The right hand map is an isomorphism since N is Q-closed. It implies that the left hand map $q[K, N]$ is invertible. Hence $[K, N]$ is Q-closed for every compact K in $\mathcal{E}$.

Step 2: Now let M be another object in $\mathcal{E}$ and consider the map $[qM, N]$. We have a commuting diagram

$$\begin{tikzcd} \mathrm{map}(K,[QM,N]) \arrow[r, "{\mathrm{map}(K,[qM,N])}"] \arrow[d, "\cong"'] & \mathrm{map}(K,[M,N]) \arrow[d, "\cong"'] \\ \mathrm{map}(QM,[K,N]) \arrow[r, "{\mathrm{map}(qM,[K,N])}"] & \mathrm{map}(M,[K,N]) \end{tikzcd}$$

where the vertical maps are invertible. By Step 1 the object $[K, N]$ is Q-closed and so the lower map is an isomorphism. Thus the top map is also an isomorphism. Since $\mathcal{E}$ is ω-presentable, by Lemma 2.2.7 the map $[qM, N]$ is invertible.

Step 3: Now we copy the proof of Lemma 4.2.12(ii). For any map $u : M \to M'$ in $\mathcal{E}$ the square

$$\begin{tikzcd} {[M,N]} & {[QM,N]} \arrow[l, "\cong"', "{[qM,N]}"] \\ {[M',N]} \arrow[u, "{[u,N]}"] & {[QM',N]} \arrow[l, "\cong"', "{[qM',N]}"] \arrow[u, "{[Q(u),N]}"'] \end{tikzcd}$$

commutes. The horizontal maps are invertible by Step 2. So the map $[u, N]$ is invertible if and only if the map $[Q(u), N]$ is invertible and, letting N vary in all $\mathcal{E}^Q$, if and only if $Q(u)$ is invertible.

Step 4: $Q(\phi(z))$ is invertible if and only if the map $[\phi(z), N] = \{z, N\} = sN$ is invertible for all Q-closed N. Now Lemma 4.2.10(iii) applies and states that N is also S-closed. And the definition of being S-closed is that the map sN is an isomorphism. So the theorem follows from Step 3. □

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