Categorical Logic

Spring 2021

Homework 2 – Due March 22

Propositional logic

1. (The theory of monotone functions) Being given two posets $(A, \leq)$ and $(B, \leq)$ define a propositional theory such that the models in $\{0,1\}$ are precisely the monotone functions $A \to B$. (Simply give the relation symbols and the list of axioms, no need to write the proof that the models are monotone functions)

2. (Powerposet) For a poset X, the analogue of the powerset is the poset $P(X) = \mathrm{Hom}(X^{op}, \underline{2})$ ($\underline{2} = \{0 < 1\}$). This is also an analog of presheaf category over a category and we have a "Yoneda embedding" $X \to P(X)$ sending an element x to the function $\widehat{x} : y \mapsto [y \leq x]$ (where $[y \leq x]$ is 1 or 0 depending whether it is true or not that y is lower than x).

   A join lattice is called *complete* if it admits arbitrary join. Let $\mathcal{CL}\mathrm{at}$ be the category of complete lattices and maps preserving arbitrary joins. There is a forgetful functor $U : \mathcal{CL}\mathrm{at} \to \mathcal{P}\mathrm{oset}$.

   (a) Prove that $P(X)$ is a complete lattice (= prove that arbitrary joins exist in $P(X)$).

   (b) Let C be a complete lattice and X a poset. The composition with the embedding $X \to P(X)$ gives a map $\mathrm{Hom}(P(X), C) \to \mathrm{Hom}(X, C)$. Prove that this map restricts to an isomorphism of posets $\mathrm{Hom}_{\mathcal{CL}\mathrm{at}}(P(X), C) \simeq \mathrm{Hom}(X, C)$.

   (c) Deduce that the functor $X \mapsto P(X) = \mathrm{Hom}(X^{op}, \underline{2})$ is left adjoint to the forgetful functor U.

3. (Frames v. Boolean algebras) The purpose of the exercise is to define an adjunction

$$\mathcal{BA}\mathrm{lg} \underset{B}{\overset{Idl}{\rightleftarrows}} \mathcal{F}\mathrm{rame}$$

   (a) An element x of a frame F is called *complemented* if there exists an element y such that $x \wedge y = 0$ and $x \vee y = 1$. Prove that the subposet $B(F)$ of F spanned by complemented elements is a Boolean algebra and that the inclusion $B(F) \to F$ is a morphism of lattices (preserves finite joins and finite meets).

   (b) The *frame envelope* of a Boolean algebra B is defined as the subposet $Idl(B)$ of $P(B) = Hom(B^{op}, \underline{2})$ consisting of functions $B^{op} \to \underline{2}$ sending finite joins in B to finite meets in $\underline{2}$ (elements of $Idl(B)$ are called *ideals* of B).

      i. Prove that the Yoneda map of Exercise 2 defines a monotone map $B \to Idl(B)$ which is a morphism of lattices (Hint: use the strong Yoneda lemma in $P(B)$: $F(x) = Hom(\widehat{x}, F)$).

      ii. (REMOVED) Construct an adjunction $Idl \dashv B$ (Hint: one can use the previous map).

      iii. (REMOVED) Prove that the functor Idl is fully faithful.

Remark on topological interpretation: If $F = O(X)$ is a the frame of open subsets of a topological space X. The Boolean algebra $B(F)$ is the collection of clopens of X (subset that are both closed and open) and the corresponding Stone space is the space $\pi(X)$ of *connected components* of X. The canonical map $X \to \pi(X)$ sends a point to its connected component.