Categorical Logic

Spring 2021

Homework 1 – Due March 3

Algebraic Theories

1. (The theory with one sort) Let T_1 be the signature with one sort, no function symbols and no axioms.

   (a) Describe the set $T(x_1, \dots, x_n)(\Sigma)$ of all terms in the context $\vec{x} = (x_1, \dots, x_n)$.

   (b) Describe the morphisms $(x_1, \dots, x_n) \to (y_1, \dots, y_m)$ in the syntactic category and prove that they are in bijection with functions $\{1, 2, \dots, m\} \to \{1, 2, \dots, n\}$.

   (c) Let $\mathcal{F}\mathrm{in}$ be the category of finite sets. It is easy to see that $\mathcal{F}\mathrm{in}$ has finite sums. In consequence, its opposite category $\mathcal{F}\mathrm{in}^{op}$ has finite products.
   Construct an equivalence of categories $\mathcal{S}\mathrm{yn}(\Sigma) \simeq \mathcal{F}\mathrm{in}^{op}$. (Hint: use the fully faithful + essentially surjective characterization of equivalences)

   (d) Describe the isomorphisms in $\mathcal{S}\mathrm{yn}(\Sigma)$.

   (e) A Σ-structure M is just a set. Find a morphism in $\mathcal{S}\mathrm{yn}(\Sigma)$ whose interpretation is the diagonal $\Delta : M \to M \times M$.

   Remark: More generally, if Σ is the signature with set of sorts S and no function symbols, one can construct an equivalence $\mathcal{S}\mathrm{yn}(\Sigma) \simeq (\mathcal{F}\mathrm{in}^S)^{op}$ (where $\mathcal{F}\mathrm{in}^S$ is the category of functors $S \to \mathcal{F}\mathrm{in}$, i.e. the category of S-families of finite sets). It is a good exercise to try to prove it.

2. (Theories without function symbols) Let Σ be a signature with set of sorts S and no function symbols.

   (a) Describe all the algebraic theories with signature Σ (in other words, list all the possible axioms that can be given).

   (b) How many algebraic theory with signature Σ are there when S is a singleton ?

3. (Universal property of the theory with one sort) Let T_1 be the algebraic theory of **??**. Let $\mathrm{T} = (\Sigma, \mathrm{A})$ another algebraic theory, and X a distinguished sort in Σ.

   (a) Using X, construct a function $T(\mathrm{T}_1) \to T(\mathrm{T})$ between the set of terms.

   (b) Prove that this function extends to a cartesian functor $\mathcal{F}\mathrm{in}^{op} = \mathcal{S}\mathrm{yn}(\mathrm{T}_1) \to \mathcal{S}\mathrm{yn}(\mathrm{T})$ (no need to prove that is actually a functor, just explain what are the maps between the sets of objects and arrows, and justify that it is a cartesian functor).

4. (Equivalence of theories)

   (a) Two theories are called "Lawvere equivalent" if they have equivalent syntactic categories. Consider the theory with two sorts X and Y and two function symbols $f : X \to Y$ and $g : Y \to X$ with axioms $f(g(y)) = y$ and $g(f(x)) = x$. Prove that the syntactic category is equivalent to the one of the theory T_1.

(b) Two theories are called "Morita equivalent" if they have equivalent categories of models in $\mathcal{S}\mathrm{et}$. consider the theory T' with two sorts X and Y and two function symbols $r : X \to Y$ and $s : Y \to X$ with axioms $r(s(y)) = y$ and the theory T', with one sort X and one function symbols $e : X \to X$ with axioms $e(e(x)) = e(x)$. Construct an equivalence between the categories of models in $\mathcal{S}\mathrm{et}$ of both theories (Hint: use that, for a set X with an idempotent function $e : X \to X$, the set Y of the first theory is the set of fixed points of e).

(c) Prove that Lawvere equivalence $\Rightarrow$ Morita equivalence, but not the contrary (use Question **??** for the second part)