Today: introduction to topos theory

a topos, two | topoi (geometry, topology)
| toposes (cat. theory, logic)

Topoi have a **dual nature**:

- **topological** (Grothendieck Topoi)
  was looking for generalization of topological spaces.

- **logical** (Lawvere, Elementary topoi)
  was looking for axiomatization of the category of sets

what is needed to do Set theory ?
do logic?

1st order logic :

- sorts → objects
- terms → arrows
- formulas → subobject / monomorphisms
- substitution → composition arrows / fiber products
- quantifiers → left / right adjoint to substitution.

Higher order logic :

- powerset → subobject classifier
- function types (λ-calc) → exponential (cart. closed category)
- depend product → locally cart. closed category.

Definition an Elementary topos is a category

- with finite limits (terminal object + fibre products)
- locally cartesian closed
- with a subobject classifier.

Morally: any kind of logic can be done in an elementary topos

Prop: all other features of 1st order logic can be deduced from these axioms:

- coeq. exist, eff. epi are stable by base change, image factorization exist, dependent Σ
- $\exists$, $\forall$ exist (with Beck-Chevalley condition)

Def: C is <u>cartesian closed</u> if it is cartesian

$$- \times - : C \times C \xrightarrow{\times} C$$

and if for any $A \in C$ $\quad A \times - : C \longrightarrow C$

has a right adjoint, the exponential by A:

$$(-)^A \qquad C^{op} \longrightarrow C$$

$$X \longmapsto X^A$$

Def a cat C is **locally cart. closed** if it is lex (has finite limits) and for any object $A \in C$ the slice category

C/A is cartesian closed.

the slice category over A is sometimes called the "local category of C at A"

C

Rem $C = Set$ $\quad Set/A$ = cat of families of objects of Set parametrized by A.

C/A =

When C is lex C/A can be thought as the cat. of families of objects in C parametrized by the object A in C.

aka dependent types

the cartesian structure of C/A is $- \times_A -$ the fiber product over A.

the exponential in C/A encode depend products for A-dependent types.

The subobject classifier

$\mathcal{C}$ is lex cat. we have the **subobject functor**

$$\mathrm{Sub} : \mathcal{C}^{op} \longrightarrow (\mathrm{MLat} \longrightarrow \mathrm{Poset} \longrightarrow) \ \mathrm{Set}$$

$$X \longmapsto \mathrm{sub}(X)$$

is Sub a representable functor?

Is there an object Ω in $\mathcal{C}$ together with a natural iso

$$\mathrm{Sub}(X) \simeq \mathrm{Hom}_{\mathcal{C}}(X, \Omega)$$

if $C = \mathrm{Set}$

$$\mathrm{Sub} : \mathrm{Set}^{op} \longrightarrow \mathrm{Set}$$

$$X \longmapsto \mathrm{Sub}(X) = P(X) \simeq \mathrm{Hom}_{\mathrm{Set}}(X, \{0,1\})$$

Sub is representable! by $\Omega = \{0, 1\}$

= first example of subobject classifier

$$\begin{array}{ccc} \{x \mid \chi_U(x) = 1\} = U & \longrightarrow & \{1\} \\ \text{subset} \ \cap & \lrcorner & \searrow \\ X & \xrightarrow[\chi_U]{} & \Omega = \{0,1\} \end{array}$$

χ_U = characteristic fct

$$\chi_U(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{else} \end{cases}$$

from a logical perspective

subobject = formula

formula $U = [\varphi]$

$\cap$

context $= X = [\vec{x}] \xrightarrow{\chi_\varphi} \Omega$

$$\frac{\vec{x} \vdash \varphi \quad \text{formula}}{\vec{x} \vdash \chi_\varphi : \Omega}$$

any formula φ defines a term of type Ω

Ω = type/sort | of Boolean

" | of propositions.

also called "prop"

examples of categories with a subobject classifier:

- Set $\quad \Omega = \{0, 1\}$

- $Set/_I$ $\quad$ sub : $(Set/_I)^{op} \longrightarrow Set$

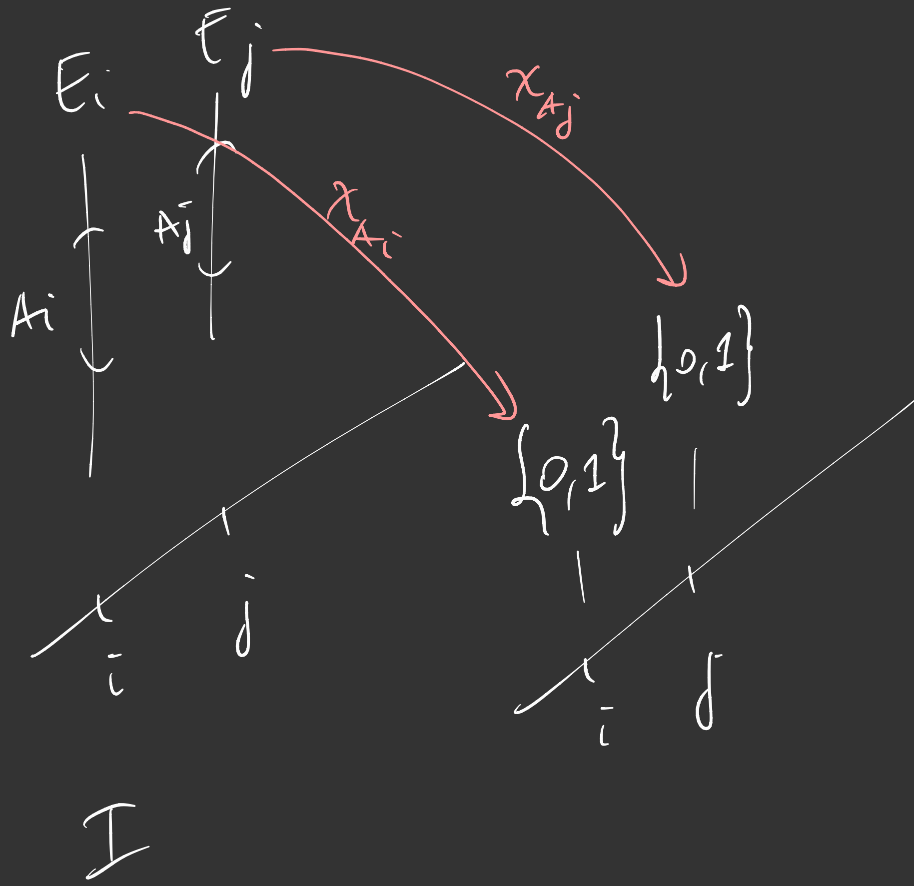
$$(E_i)_{i \in I} \longmapsto \{\text{subfamilies of } (E_i)\} = \prod_i sub(E_i) = \prod_i Hom(E_i, \{0,1\})$$

$$= Hom_{Set/_I}((E_i)_{i \in I}, (\{0,1\})_{i \in I})$$

$(\{0,1\})_{i \in I}$ = constant family with value $\{0, 1\}$.

$$\Omega = (\{0,1\})_{i \in I}$$

(constant family)



E_i
E_j
A_i
A_j
χ_{A_i}
χ_{A_j}
$\{0,1\}$
$\{0,1\}$
i
j
i
j
I

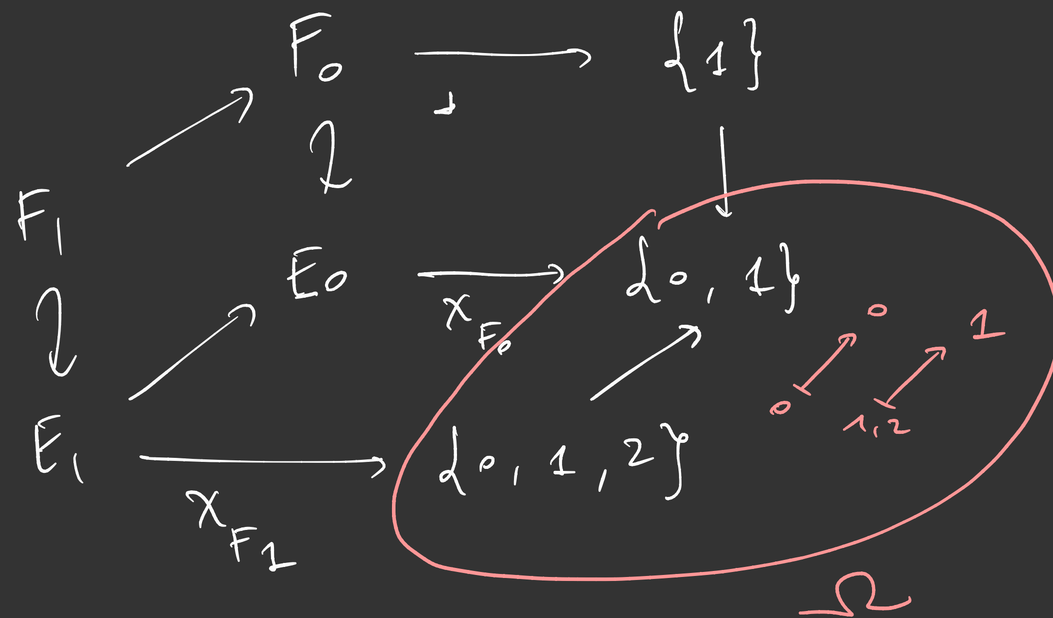
— presheaves categories $Set^{C^{op}}$

$C = \{0 < 1\}$ the arrow category.

$$Set^{\{0<1\}^{op}} = \{ E_0 \leftarrow E_2 \}$$

sub diagram of

$$\begin{array}{ccc} E_0 & \leftarrow & E_1 \\ \text{injection } \cup & & \cup \text{ injection} \\ F_0 & \leftarrow & F_1 \end{array} \quad = \; ?$$



F_0
$\{1\}$
E_0
$\{0, 1\}$
χ_{F_0}
F_1
E_1
$\{0, 1, 2\}$
χ_{F_1}
Ω

$$\mathrm{Set}^{\{0<1<\cdots<n\}^{op}}$$

objets ou

$$\begin{array}{ccccccc}
F_n & \to & F_{n-1} & \to & \cdots & F_1 & \to & F_0 \\
\cap & & \cap & & & \cap & & \cap \\
E_n & \to & E_{n-1} & \to & \cdots & E_1 & \to & E_0 \\
\downarrow & & \downarrow & & & & & \downarrow
\end{array}$$

|| injections.

$$\Omega \quad \left(\{0,1,\ldots,n+1\} \to \{0,1,\ldots,n\} \to \quad \{0,1,2\} \to \{0,1\} \right)$$

$$n+1 \neq i \longmapsto i$$

$$n+1 \longmapsto n$$

general formula for Ω in $Set^{C^{op}} = \hat{C}$

$$\Omega : C^{op} \longrightarrow Set$$
$$X \longmapsto \Omega(X) = Hom_{\hat{C}}(\hat{X}, \Omega)$$

Yoneda

|| by def. of Ω

$$Sub(\hat{X})$$

$$\Omega : C^{op} \longrightarrow Set$$
$$X \longmapsto Sub(\hat{X})$$

$$Sub(\hat{X}) \underset{\text{bij}}{\simeq} \{\text{sieves of } C/X\}$$

if C is a <u>poset</u> like $\{0 < 1 < \cdots\ n\}$

$x \in C$ $\quad \mathrm{sub}(\hat{x}) \underset{\text{bij}}{\simeq} \{$ lower sets of C/x $\}$

<u>ex</u> $\quad C = \{0 < 1 < \cdots\ n\}$

$x = 3$

$C/_3 = \{0 < 1 < 2 < 3\}$

$\mathrm{sub}(\hat{3}) = \{\phi, \{0\}, \{0,1\}, \{0,1,2\}, \{0,1,2,3\}\}$

$\simeq \{0, 1, 2, 3, 4\}$

in general

$\mathrm{sub}(\hat{k}) \simeq \{0, 1, \ldots, k+1\}$

→ recover formula for Ω in $\mathrm{Set}^{\{0,1,\ldots,n\}^{op}}$

What is Ω good for?

- "terms of sort Ω" = arrows with codomain Ω are the formulas of the theory.

  → the existence of Ω makes it possible to quantify over formulas.

$$\forall \varphi, \ \varphi(x) = \bot \quad \ldots$$

- when the category C is Cart-closed can use Ω to define "Powerset" of X: Ω^X

→ can quantify over subsets of X. using Ω^X

Ω^X = sort/type of subsets of X.

possible to write formulas like

$$\forall U : \Omega^X , \quad U \ni x$$

$$\exists U : \Omega^X , \quad \# U = 2 \quad , \ldots$$

→ can iterate this and define

$$\Omega^{\left(\Omega^X\right)} = \text{double power set}.$$

= subset of subsets

can quantify over this also

→ this leads to <u>higher order logic</u>

- if Ω exist and if C has colimits.

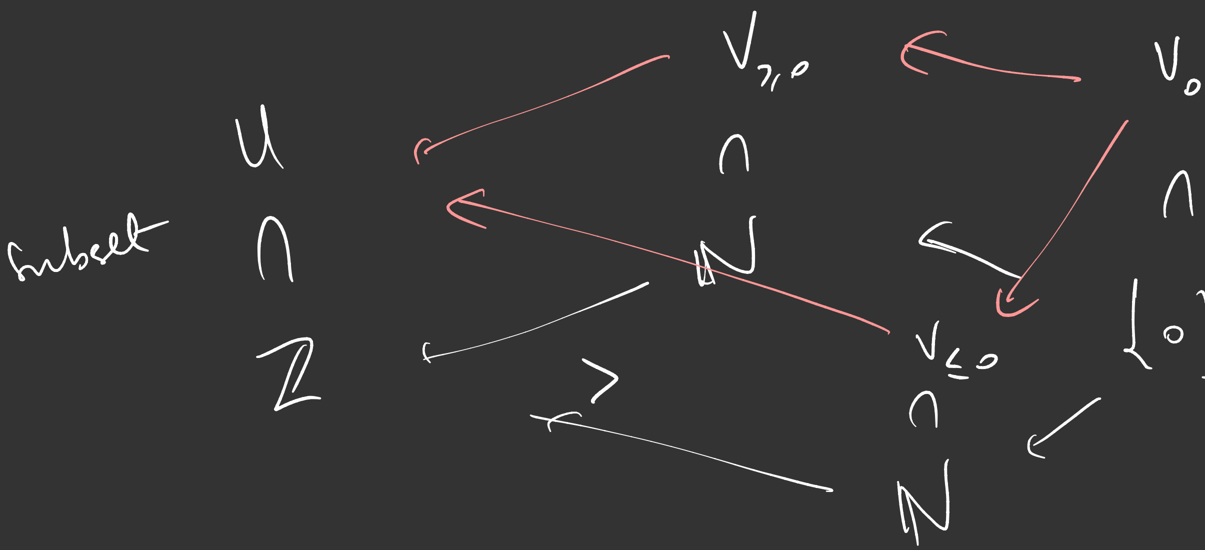
$$X = \operatorname{colim} X_i$$

$$\begin{aligned}
\operatorname{sub}(X) &= \operatorname{Hom}_C(X, \Omega) \\
&= \operatorname{Hom}(\operatorname{colim} X_i, \Omega) \\
&= \lim \operatorname{Hom}(X_i, \Omega) \\
&= \lim \operatorname{sub}(X_i)
\end{aligned}$$

example

$$\begin{array}{ccc}\{0\} & \longrightarrow & \mathbb{N}\\ \downarrow & & \downarrow\\ \mathbb{N} & \longrightarrow & \mathbb{Z}\end{array}$$

pushout in Set.



in terms of logic this means

one can produce a property on $\mathbb{Z}$
formula

from a property of ≥ 0 integer

an ____________ of ≤ 0 integer.

that coincides for 0.