1^{st} order logic

signature Σ
- sorts S
- fct symbols F
- rel. symbols R

generates
- contexts
- terms
- formula
- sequents.

Theory = signature + Axioms

models in categories.

sorts $\to$ objects

fct symb $\to$ arrows.
terms

rel symb $\to$ subobjects.
formulas

Constructors on formulas

$=, \top, \wedge, \bot, \vee, \exists, \forall, \Rightarrow, \neg$

are interpreted as operators on subobjects

C lex category : functor of subobjects

$$\begin{array}{ccc} C^{op} & \longrightarrow & \mathrm{MLat} \\ X & \longmapsto & \mathrm{Sub}(X) \\ f \downarrow & & \uparrow f^* \\ Y & \longmapsto & \mathrm{Sub}(Y) \end{array}$$

$\rightarrow$ interpret T, $\wedge$ on formula

C regular category : $\underline{\underline{f_!}} \dashv f^*$ (direct image of subobjects)

+ Beck-Chevalley condition

$\longrightarrow$ semantics for $\exists$ quantifiers.

$\mathrm{Sub}(X \times Y) \xrightarrow{\exists x. = p_!} \mathrm{Sub}(Y)$ $\quad p : X \times Y \longrightarrow Y$

to interpret $\perp$ and $\vee$ need sub(X) to be a <u>join semi lattice</u>

$(x \mid \perp) \longrightarrow$ interpred as minimal element of Sub(X)

$(x \mid \varphi(x) \vee \psi(x)) \longrightarrow$ interpreted as the join in Sub(X)

$$[\varphi \vee \psi] = [\varphi] \vee [\psi]$$

the compatibility with substitution:

$y \mid t$ of sort x $\quad (\varphi \vee \psi)[t/x] = \varphi[t/x] \vee \psi[t/x]$

$Y \xrightarrow{[t]} X$ $\quad [t]^* : \mathrm{sub}(X) \longrightarrow \mathrm{Sub}(Y)$ morphism of join semilattices.

this condition has the consequence that
the two structures of meet semi lattice and
join ——— on Sub(X)

are going to distribute on each other.

$A, B, C \in \mathrm{Sub}(X)$ $\quad A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$

in other words Sub(X) is a <u>Distributive lattice</u>.

<u>proof</u>: exercise.

Coherent category = regular category such that

$\forall X$, $\quad$ sub(X) is a Dist. lattice.

$\forall X \xrightarrow{f} Y \quad f^*: \text{Sub}(Y) \longrightarrow \text{Sub}(X)$ morph. of Dlat.

in a coherent cat can interpret 1st order logic

with constructor $\top, \wedge, \bot, \vee, \exists$

To interpret $\forall$ need . for any $f : X \to Y$

$$\mathrm{Sub}(X) \underset{\forall_f = f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{Sub}(Y)$$

$$f^* \dashv \underline{f_*}$$

right adjoint to inverse image.

compatibility with substitution.

$$\big(\forall x\, \varphi(x,y)\big)[t/y] = \forall x\, \big(\varphi(x,y)[t/y]\big)$$

demands some Beck-Chevalley conditions on $\forall$

$$\left(p'_* f'^* = f^* p_* \right)$$

lemma: if $f_! \dashv f^* \dashv f_* : \mathrm{Sub}(X) \to \mathrm{Sub}(Y)$

if $f_!$ satisfies BC condition, then so does f_*.

lemma the existence of f_* for any $f : X \to Y$

turns $\mathrm{Sub}(X)$ into a Heyting algebra

($x \wedge -$ has a right adjoint $x \Rightarrow -$
Heyting implication)

Each $\mathrm{Sub}(X)$ is a model for intuitionist logic.

Heyting category : Coherent category with
f_* for all $f : X \to Y$
(BC conditions are automatic).

Heyting categories provide semantics for 1^{st} order logic
with $=, \top, \wedge, \bot, \vee, \exists, \forall, \Rightarrow$ $(\neg := \Rightarrow \bot)$

Rem a Heyting Cat. is called a Boolean category if $\neg X := (X \Rightarrow \bot)$
is such that $\neg\neg X = X$

examples of 1st order theories

- Regular theories $(\top, \wedge, \exists)$
  - divisible groups. $\mathbb{Q} \ni x$, $\mathbb{N} \ni n$ $\quad \frac{x}{n}$ exist in $\mathbb{Q}$

    does not always exist. $x \in \mathbb{Z}$, $n \in \mathbb{N}$ $\quad \frac{x}{n}$ may not exist in $\mathbb{Z}$.

    a group such that the division by n exist is called divisible

    need to add the following axioms to group axioms

    for each n, $\quad x \vdash \exists y, \, ny = x$

- Coherent theories $(\top, \wedge, \perp, \vee, \exists)$

- local rings.

add axioms

$$0 = 1 \vdash \perp$$

$$x \vdash \exists y,\ xy = 1 \vee \exists y\ (1-x)y = 1$$

in a local ring either x or $1-x$ is invertible

- torsion groups : add axiom $x, \vdash \bigvee_n nx = 0$

$$\underbrace{x + x + x \cdots + x}_{n \text{ time}} = 0$$

example : the clock group

$\{0, 1, 2, \ldots, 11\} = \mathbb{Z}/12\mathbb{Z}$

we've seen how to interpret the differents kinds of 1^{st} order logic (Horn, regular, coh, Heyting) into corresponding categories (lex, regular, coh, Heyting).

this defines <u>functors of models</u> for a theory T,

$$\mathrm{Mod}(T) : \begin{array}{l} \mathrm{Cat}^{lex} \\ \text{or } \mathrm{Cat}^{reg} \\ \text{or } \mathrm{Cat}^{coh} \\ \text{or } \mathrm{Cat}^{Heyt.} \end{array} \longrightarrow \mathrm{Cat}$$

$$C \longmapsto \mathrm{Mod}(T, C)$$

We focus on the regular case.

a functor $F : C \to D$ between regular categories is called <u>regular</u> if

- it is lex (preserves finite limits) ($\Rightarrow$ pres mono)
- it preserve the image factorization. ($\Leftarrow$ pres eff-epi)

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\text{cover}"'] & & Y \\ & Z \arrow[ur, "\text{mono}"'] & \end{tikzcd} \quad \text{in } C$$

cover '' (eff epi)

$$\begin{tikzcd} F(X) \arrow[rr, "F(f)"] \arrow[dr, "\text{cover}"'] & & F(Y) \\ & F(Z) \arrow[ur, "\text{mono}"'] & \end{tikzcd} \quad \text{in } D$$

cover '' (eff epi)

T regular theory and C reg cat.

a <u>model of T of C</u> is an interpretation of T in C

a <u>morphism of model</u> (left as exercise)

$\rightarrow$ category $\mathrm{Mod}(T, C)$ of models.

if $C \xrightarrow{F} D$ regular functor. induces a functor

$$\mathrm{Mod}(T, C) \longrightarrow \mathrm{Mod}(T, D)$$ between cat. of models.

thus defines a functor $\mathrm{Mod}(T) : \mathrm{Cat}^{\mathrm{reg}} \longrightarrow \mathrm{Cat}$

$$C \longmapsto \mathrm{Mod}(T, C)$$

the syntactic category of T $Syn(T)$ is defined as the regular category representing the functor $Mod(T)$.

$$Mod(T, C) \underset{\substack{\text{equivalence}\\ \text{of cat}}}{\simeq} Fun^{\text{regular}}(Syn(T), C)$$

Thm: the syntactic category of any regular theory
coh
Heyting

exist.

Construction of the syntactic category of a regular theory T

$T \to$ contexts $\longrightarrow$ object.

$T \to$ terms $\longrightarrow$ arrows. $\longrightarrow$ "cheating": replace them by formulas.

$T \to$ formula $\longrightarrow$ (sub)object any formula defines an object

Syn(T) : <u>objects</u> : formulas in context (x, φ)

((x, T) correspond to the object interpreting the context x)

morphism $(x, \varphi) \xrightarrow{u} (y, \psi)$

any term $(y \mid t)$ of sort x give a morphism $(x, T) \longrightarrow (y, T)$

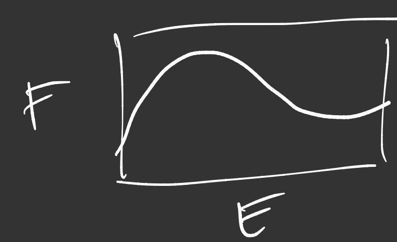
not enough. $t \leftrightarrow (x, y, t(y) = x)$

$$\boxed{u = (x, y \mid \rho(x, y))}$$

= formula in context (x, y). which are "functional"

- $x, y \mid \rho \vdash \varphi \wedge \psi$ (if $\rho(x,y)$ true $\varphi(x)$ and $\psi(y)$ have to be true)
- $x, y, y' \mid \rho(x, y), \rho(x, y') \vdash y = y'$ (x has a single image by ρ)
- $x \mid \varphi \vdash \exists y\ \rho(x, y)$ (any x s.t. $\varphi(x)$ true has an image by ρ)

Recall in Set that $u : E \to F$



graph of u $\Gamma_u \subset E \times F$

$$\Gamma_u = \{(x,y) \mid y = u(x)\} = \{(x, u(x)),\ x \in E\}$$

$\Gamma_u \subset E \times F$ is a functional relation between E and F.

- $\Gamma_u(x,y) \wedge \Gamma_u(x,y') \vdash_{x,y,y'} y = y'$
- $\vdash_x \exists y\ \Gamma_u(x,y)$

$$[(x,\varphi)] \xrightarrow{\;u\;} [(y,\psi)]$$

$$[(x,y,\rho)] = \text{graph of } u.$$

$$\cap$$

$$[(x,T)] \times [(y,T)]$$