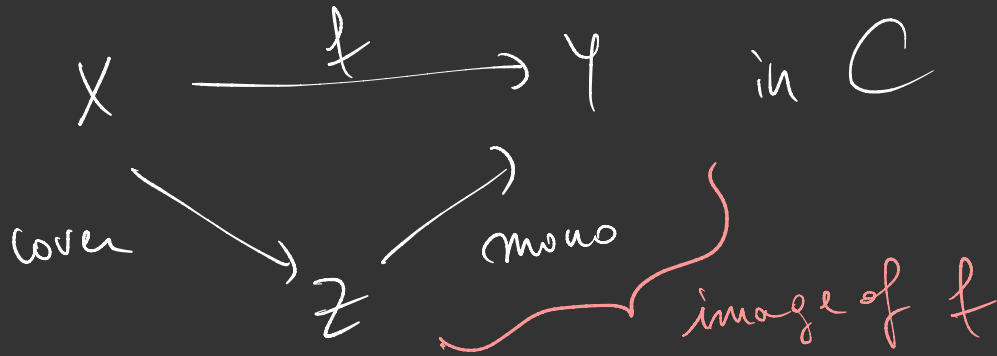


Regular category \mathcal{C} = category with a good notion of image factorization

"surjections" and "injections"
" "
" "
Monomorphism

cover { effective epi
" regular epi
" "
" strong epi



$$\begin{array}{ccc}
 \text{Sub} : C^{\text{op}} & \longrightarrow & \text{MLat} \left(\longrightarrow \text{Poset} \right) \\
 X & \longmapsto & \text{Sub}(X) = \left\{ X' \xrightarrow{\text{mono}} X \right\} \\
 f \downarrow & & \uparrow f^* \\
 Y & \longmapsto & \text{Sub}(Y)
 \end{array}$$

exist if C is lex and $\text{Sub}(X)$ is a meet semilattice

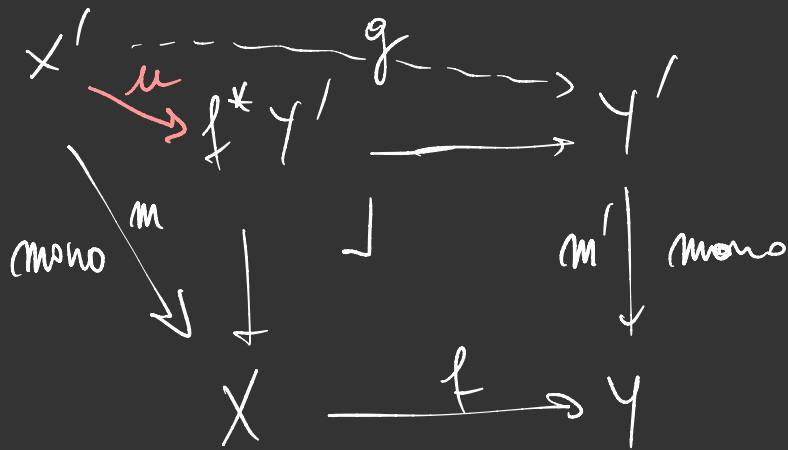
Proposition if \mathcal{C} is regular, for any $f: X \rightarrow Y$ in \mathcal{C}

$$\text{sub}(X) \begin{array}{c} \xrightarrow{f_!} \\ \xleftarrow{f^*} \end{array} \text{sub}(Y)$$

the functor f^* has a left adjoint $f_!$.

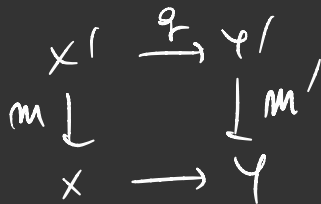
$$\begin{array}{ccc} X' & \xrightarrow{\text{cover}} & f_! X' \\ \text{mono} \downarrow & \searrow & \downarrow \text{mono} \\ X & \xrightarrow{\quad} & Y \end{array} \quad \begin{array}{l} \text{built by} \\ \text{factorisation in } \mathcal{C} \\ \text{of } X' \rightarrow Y \end{array}$$

proof



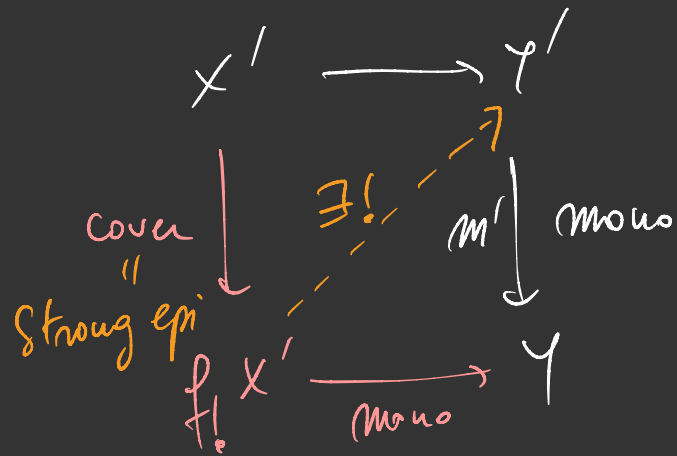
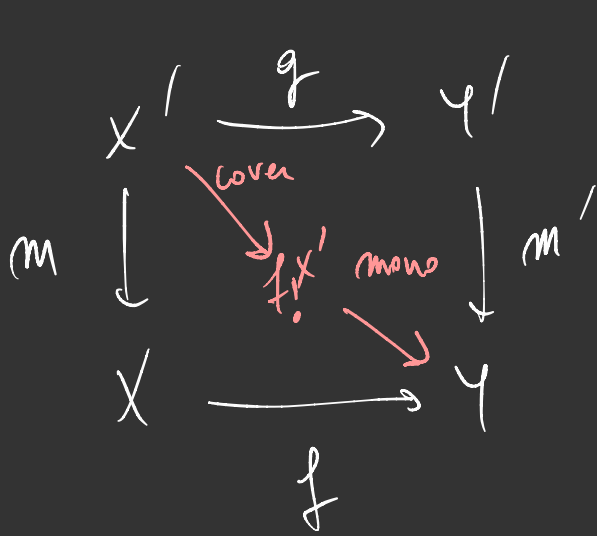
because f^*Y' is a pullback

$\mu: X' \rightarrow f^*Y'$
is equivalent to a
commutative square



9
6

$$\frac{X' \xrightarrow{\mu} f^*Y' \text{ in } \text{Sub}(X)}{f_! X' \rightarrow Y' \text{ in } \text{Sub}(Y)}$$



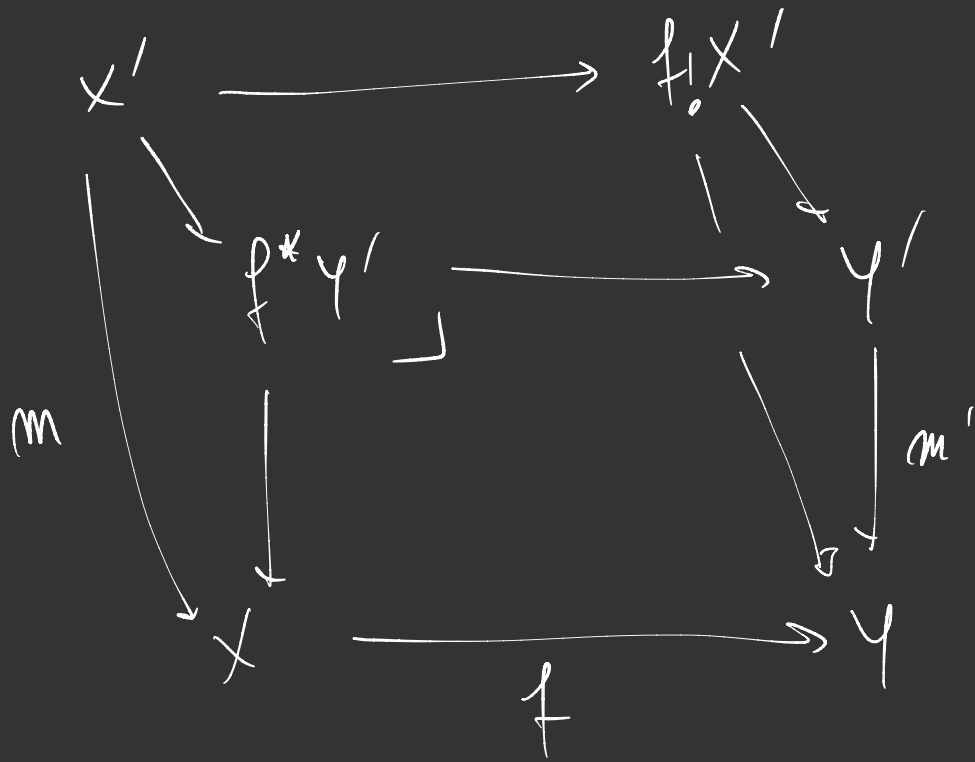
by orthogonality

any $X' \rightarrow f^* Y'$ defines a unique square

$$\begin{array}{ccc}
 X' & \rightarrow & Y' \\
 \downarrow & & \downarrow \\
 X & \rightarrow & Y
 \end{array}$$

in $\text{Sub}(X)$

and each square is associated to a unique map. $f! X' \rightarrow Y'$



intems of sets
 $X' \subset X$ $Y' \subset Y$
 subsets

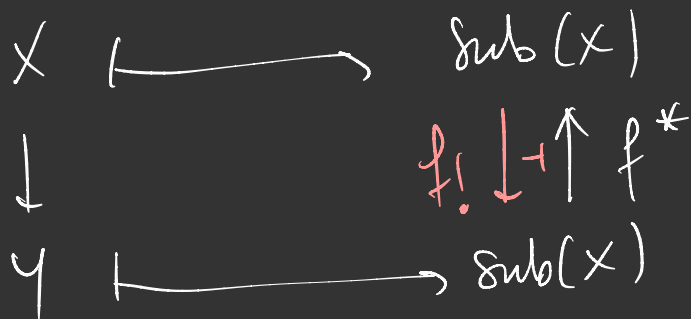
$$f^*(Y')$$

$$X' \subset f^{-1} Y'$$

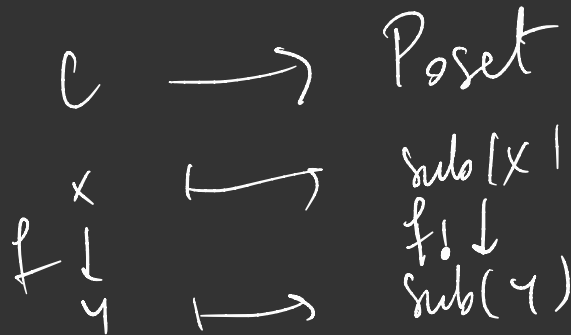
$$f(X') \subset Y'$$

"
 $f_1(X')$

\mathcal{C} regular sub: $\mathcal{C}^{\text{op}} \longrightarrow \text{MLat}$



the f! define a covariant version of sub



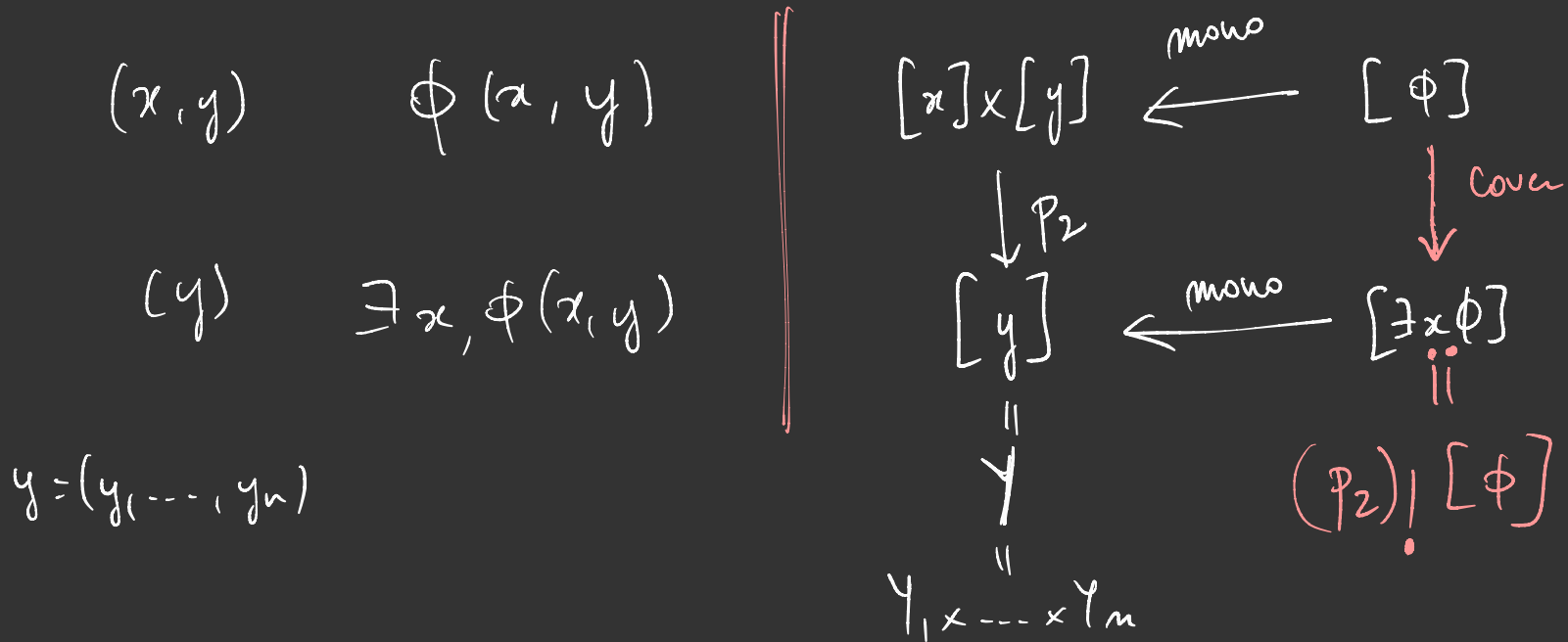
$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$$g! \circ f! = (g \circ f)!$$

because

$$(g \circ f)^* = f^* \circ g^*$$

categorical semantics for \exists quantifier in a regular cat.



in other words, $\exists x$ is the direct image by the projection forgetting/removing the variable x from the context.

$$\begin{array}{ccc}
 [\phi(x, y)] & \xrightarrow{\text{(Cover)}} & [\exists x, \phi(x, y)] = (\exists x) | [\phi(x, y)] \\
 \text{mono} \downarrow & & \downarrow \text{mono} \\
 [x] \times [y] & \xrightarrow{p_2} & [y]
 \end{array}$$

t : family of terms of sort y (with variables not containing x)
 (t_1, \dots, t_n) (y_1, \dots, y_n)

$$(\exists x \phi(x, y)) [t/y] = \exists x (\phi(x, y) [t/y])$$

$$\left(\exists x \phi(x, y) \right) [t/y] = \exists x \left(\phi(x, y) [t/y] \right)$$

$$[t]^* \left(\left[\exists x \phi(x, y) \right] \right)$$

$$(p'_2) \left([t']^* [\phi(x, y)] \right)$$

$$[t]^* (p_2) [\phi]$$

BECK-CHEVALLEY
condition

$$[x] \times [z] \xrightarrow{\text{id}_{[x]} \times [t]} [x] \times [y]$$

$$\begin{array}{ccc} [(x, z) | z] & = p'_2 & \downarrow \\ & & [z] \end{array} \quad \begin{array}{ccc} \downarrow & & \downarrow \\ [z] & \xrightarrow{[z] | t} & [y] \end{array} \quad \begin{array}{ccc} & & p_2 = [(x, y) | y] \\ & & \downarrow \\ & & [y] \end{array}$$

in \mathcal{C} , consider a cartesian square

$$\begin{array}{ccc} T & \xrightarrow{f'} & X \\ g' \downarrow \perp & & \downarrow g \\ Z & \xrightarrow{f} & Y \end{array}$$

$$\begin{array}{ccc} \text{sub}(T) & \xleftarrow{(f')^*} & \text{sub}(X) \\ g' \downarrow \dashv \uparrow (g')^* & & g! \downarrow \dashv \uparrow g^* \\ \text{sub}(Z) & \xleftarrow{f^*} & \text{sub}(Y) \end{array}$$

there exists a canonical nat. transformation.

$$g'! \cdot (f')^* \xrightarrow{\alpha} f^* \cdot g!$$

(because we are in posets
 \Rightarrow equality)

The Beck-Chevalley condition holds if α is an isomorphism
or equality

Construction of α .

$$g'_! (f')^* \xrightarrow{\alpha} f^* g'_!$$

- ingredients
- $g_! \dashv g^*$ \Leftrightarrow $g_! g^* \xrightarrow{\varepsilon} 1 + 1 \xrightarrow{\eta} g^* g_!$
 - $g'_! \dashv (g')^*$ \Leftrightarrow $g'_! (g')^* \xrightarrow{\varepsilon'} 1 + 1 \xrightarrow{\eta'} (g')^* g'_!$

$$\begin{array}{c} (g'_! f'^*) (g^* g'_!) \\ \uparrow \text{id} \times \eta \\ (g'_! f'^*) \circ 1 \end{array}$$

$$f'^* g^* = g'^* f^*$$

$$g'_! f'^* (g^* g'_!) = g'_! g'^* f^* g'_!$$

induced by η



$$g'_! f'^* \xrightarrow{\alpha} f^* g'_!$$

induced by ε'

$g' \rightarrow g'^*$
 adjunction of
 posets

$$g' \circ g'^* \xrightarrow{\varepsilon} \text{id}$$

in $\text{Hom}(\text{sub}(Z), \text{sub}(Z))$
 Poset

$$\begin{array}{ccc} \left(\begin{array}{c} g' \circ g'^* \\ \text{id} \end{array} \right) & \left(\begin{array}{c} f \circ g^* \\ \text{id} \end{array} \right) \\ \varepsilon \downarrow & \downarrow \text{id} \\ \text{id} & \circ f \circ g^* \end{array}$$

Composition is monotone
 (in each variable)

Poset is enriched
 on posets

A, B Posets

$\text{Hom}(A, B)$ is a
 poset

$$f \leq g \quad (\Rightarrow) \quad \forall a \text{ in } A \\ f(a) \leq g(a)$$

Prop : in a regular category \mathcal{C}

the Beck - Chevalley condition holds for

any cartesian square

$$\begin{array}{ccc} T & \xrightarrow{f'} & X \\ g' \downarrow & \lrcorner & \downarrow g \\ Z & \xrightarrow{f} & Y \\ & & \downarrow f \end{array} .$$

proof exercise (hint : use stability by base change of the image factorisation).