Regular category C = category with a good notion of

image factorization

"surjections" and "injections"

"   "

monomorphism

cover $\begin{cases} \text{effective epi} \\ \text{" regular epi} \\ \text{" strong epi} \end{cases}$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\text{cover}"'] & & Y \\ & Z \arrow[ur, "\text{mono}"'] & \end{tikzcd} \quad \text{in } C$$

$\underbrace{Z}_{\text{image of } f}$

$$\begin{tikzcd}
\mathrm{Sub} : C^{op} \arrow[r] & \mathrm{MLat} \ (\longrightarrow \mathrm{Poset}) \\
X \arrow[r, mapsto] \arrow[d, "f"'] & \mathrm{Sub}(X) = \{ X' \xrightarrow[\text{mono}]{} X \} \\
Y \arrow[r, mapsto] & \mathrm{Sub}(Y) \arrow[u, "f^*"']
\end{tikzcd}$$

exist if C is <u>lex</u> and $\mathrm{Sub}(X)$ is a meet semilattice

Proposition if $\mathcal{C}$ is regular, for any $f : X \to Y$ in $\mathcal{C}$

$$
\begin{tikzcd}
\mathrm{sub}(X) \arrow[r, shift left=2, "f_!"] \arrow[r, phantom, "\perp"] & \mathrm{sub}(Y) \arrow[l, shift left=2, "f^*"]
\end{tikzcd}
$$

the functor f^* has a left adjoint $f_!$

$$
\begin{tikzcd}
X' \arrow[r, "\text{cover}"] \arrow[d, "\text{mono}"'] \arrow[dr] & f_! X' \arrow[d, "\text{mono}"] \\
X \arrow[r] & Y
\end{tikzcd}
$$

build by factorisation in $\mathcal{C}$ of $X' \to Y$

proof :

$$\begin{tikzcd}
X' \arrow[rrd, dashed, bend left, "g"] \arrow[rd, "u"] \arrow[rdd, "m"', "\text{mono}"'] & & \\
& f^*Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, "m'"', "\text{mono}"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

because f^*Y' is a pullback

$u : X' \to f^*Y'$

is equivalent to a commutative square

$$\begin{tikzcd}
X' \arrow[r, "g"] \arrow[d, "m"'] & Y' \arrow[d, "m'"] \\
X \arrow[r] & Y
\end{tikzcd}$$

$$\frac{X' \xrightarrow{u} f^*Y' \text{ in } Sub(X)}{f_! X' \longrightarrow Y' \text{ in } Sub(Y)}$$

$$\begin{tikzcd}
X' \arrow[r, "g"] \arrow[d, "m"'] \arrow[dr, "\text{cover}"] & Y' \arrow[d, "m'"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
\qquad
f_! X' \xrightarrow{\text{mono}} Y$$

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "\text{cover}"'] & Y' \arrow[d, "m'"', "\text{mono}"] \\
f_! X' \arrow[r, "\text{mono}"'] \arrow[ur, dashed, "\exists !"] & Y
\end{tikzcd}$$

cover = strong epi

by orthogonality

any $X' \to f^* Y'$ in $\mathrm{Sub}(X)$ defines a unique square $\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$

and each square is associated to a unique map. $f_! X' \to Y'$

$$\begin{tikzcd}
X' \arrow[rr] \arrow[dr] \arrow[ddr, bend right, "m"'] & & f_!X' \arrow[dr] \arrow[ddr] & \\
& f^*Y' \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & Y' \arrow[d, "m'"] \\
& X \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

in terms of sets

$X' \subset X \quad Y' \subset X$

subsets

$$\frac{X' \subset \overbrace{f^{-1}Y'}^{f^*(Y')}}{\underset{f_!(X')}{\underset{\shortparallel}{f(X')}} \subset Y'}$$

$\mathcal{C}$ regular $\quad \mathrm{Sub} : \mathcal{C}^{op} \longrightarrow \mathrm{MLat}$

$$\begin{tikzcd} X \arrow[d] \arrow[r, mapsto] & \mathrm{Sub}(X) \arrow[d, shift right, "f_!"'] \arrow[d, phantom, "\dashv"] \\ Y \arrow[r, mapsto] & \mathrm{Sub}(X) \arrow[u, shift right, "f^*"'] \end{tikzcd}$$

the $f_!$ define a covariant version of Sub

$$\begin{tikzcd} \mathcal{C} \arrow[r] & \mathrm{Poset} \\ X \arrow[d, "f"'] \arrow[r, mapsto] & \mathrm{Sub}(X) \arrow[d, "f_!"] \\ Y \arrow[r, mapsto] & \mathrm{Sub}(Y) \end{tikzcd}$$

$X \xrightarrow{f} Y \xrightarrow{g} Z$

$g_! \circ f_! = (g \circ f)_!$

because

$(g \circ f)^* = f^* \circ g^*$

categorical semantics for $\exists$ quantifier in a regular cat.

(x, y) $\phi(x, y)$

(y) $\exists x, \phi(x, y)$

$y = (y_1, \dots, y_n)$

$$
\begin{tikzcd}
{[x] \times [y]} \arrow[d, "p_2"] & & {[\phi]} \arrow[ll, "\text{mono}"'] \arrow[d, "\text{cover}"] \\
{[y]} & & {[\exists x \phi]} \arrow[ll, "\text{mono}"']
\end{tikzcd}
$$

$[y] = Y = Y_1 \times \dots \times Y_n$

$[\exists x \phi] = (p_2)_! [\phi]$

in other words, $\exists x$ is the direct image by the projection forgetting/removing the variable x from the context.

$$
\begin{tikzcd}
{[\phi(x,y)]} \arrow[r, "(\text{Cover})"'] \arrow[d, "\text{mono}"'] & {[\exists x, \phi(x,y)] = (p_2)_! [\phi(x,y)]} \arrow[d, "\text{mono}"] \\
{[x] \times [y]} \arrow[r, "p_2"'] & {[y]}
\end{tikzcd}
$$

t : family of terms of sort y (with variables z <u>not containing x</u>)

$(t_1, \dots, t_n)$ $(y_1, \dots, y_n)$

$$\left(\exists x\, \phi(x,y)\right)[t/y] = \exists x \left(\phi(x,y)[t/y]\right)$$

$$\left(\exists x\, \phi(x,y)\right)[t/y] = \exists x\left(\phi(x,y)[t/y]\right)$$

$$\begin{array}{ccc}
\left(\exists x\, \phi(x,y)\right)[t/y] & & \exists x\left(\phi(x,y)[t/y]\right) \\
\| & & \| \\
[t]^*\left([\exists x.\, \phi(x,y)]\right) & & (p_2')_!\, [t']^*[\phi(x,y)] \\
\| & & \\
[t]^*(p_2)_!\,[\phi] & &
\end{array}$$

BECK-CHEVALLEY condition

$$\begin{tikzcd}[column sep=huge]
{[x]\times[z]} \arrow[r, "{[t'] \,=\, \mathrm{id}_{[x]}\times[t]}"] \arrow[d, "p_2'"'] \arrow[dr, phantom, "\lrcorner", very near start] & {[x]\times[y]} \arrow[d, "p_2"] \\
{[z]} \arrow[r, "{[t] \,=\, [z \mid t]}"'] & {[y]}
\end{tikzcd}$$

$[(x,z) \mid z] = p_2'$

$p_2 = [(x,y) \mid y]$

in C, consider a cartesian square

$$
\begin{tikzcd}
T \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "g"] \\
Z \arrow[r, "f"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{sub}(T) \arrow[d, shift right, "g'_!"'] \arrow[d, phantom, "\dashv"] & \mathrm{sub}(X) \arrow[l, "(f')^*"'] \arrow[d, shift right, "g_!"'] \arrow[d, phantom, "\dashv"] \\
\mathrm{sub}(Z) \arrow[u, shift right, "(g')^*"'] & \mathrm{sub}(Y) \arrow[l, "f^*"] \arrow[u, shift right, "g^*"']
\end{tikzcd}
$$

there exists a canonical nat. transformation.

$$g'_! \circ (f')^* \xrightarrow{\alpha} f^* \circ g_!$$

(because we are in posets iso $\Leftrightarrow$ equality)

The Beck-Chevalley condition holds if α is an isomorphism
or equality

Construction of α.

$$g'_! (f')^* \xrightarrow{\alpha} f^* g_!$$

ingredients

- $g_! \dashv g^* \iff g_! g^* \xrightarrow{\varepsilon} 1 \quad + \quad 1 \xrightarrow{\eta} g^* g_!$
- $g'_! \dashv (g')^* \iff g'_! g'^* \xrightarrow{\varepsilon'} 1 \quad + \quad 1 \xrightarrow{\eta'} g'^* \circ g'_!$
- $f'^* g^* = g'^* f^*$

$(g'_! f'^*)(g^* g_!)$

$\uparrow \mathrm{id} \times \eta$

$(g'_! f'^*) \circ 1$

$$g'_! f'^* (g^* g_!) = g'_! g'^* f^* g_!$$

$\uparrow (\mathrm{id}, \eta)$ $\quad$ $(\varepsilon, \mathrm{id}) \downarrow$

induced by η $\quad$ induced by ε'

$$g'_! f'^* \xRightarrow{\alpha} f^* g_!$$

$g'_! \dashv g'^*$

adjunction of posets

$g'_! \circ g'^* \xrightarrow{\varepsilon} id$

$\leq$

in $\mathrm{Hom}_{\mathrm{Poset}}(\mathrm{sub}(Z), \mathrm{sub}(Z))$

$$\begin{array}{ccc} (g'_! \circ g'^*) & \circ & (f_! \circ g^*) \\ \varepsilon \downarrow & & \downarrow id \\ id & \circ & f_! g^* \end{array}$$

Composition is monotone (in each variable)

Poset is enriched on posets

A, B Posets

$\mathrm{Hom}_{\mathrm{Poset}}(A, B)$ is a poset

$f \leq g \iff \forall a$ in A $f(a) \leq g(a)$

Prop: in a regular category C the Beck–Chevalley condition holds for any cartesian square

$$\begin{tikzcd} T \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, "g"] \\ Z \arrow[r, "f"'] & Y \end{tikzcd}$$

.

Proof exercise (Hint: use stability by base change of the image factorisation).