Plan of the course

I. Algebraic theories (Lawvere theories)

II. Propositional theories (classical & intuitionist)

III. 1st order theories

IV. more ... (λ-calculus ...)

I Algebraic Theories

main idea : theory describing a structure on a set
which is defined by equations

ex monoids, groups, rings, ...

an <u>Algebraic signature</u> is

- a collection S of <u>sorts</u> (indexing variables)
  (most of the time we are gonna have a <u>single</u> sort)

- a collection F of <u>function symbols</u>

  each fct symbol has a <u>arity</u> and a <u>sort</u>

  <u>arity</u> = finite list of sorts

symbolically: $f : (s_1, s_2, \dots s_n) \longrightarrow s_0$

sorts of variables of f — sort of values of f

syntaxically $f(x_1, x_2, \dots x_n)$ — x_i has sort s_i

Examples

- magma = set M with binary composition m no axiom.

$$m : M \times M \longrightarrow M \quad \text{(called multiplication)}$$

Theory of magma : signature : $S = \{ M \}$ one sort

$F : \{ m \}$

when there is only one sort the arities are just natural numbers (number of variable.)

here m has arity 2 , useful to write $m(x, y)$

- pre-ring
    - a single sort $\{S\}$
    - fct symbols : $+$ arity $= 2$

      $\times$ arity $= 2$

      0 arity $= 0$

      1 ———) constant

(fct symbols with arity 0 are called constants)

Structure on a signature $\Sigma = (S, F)$ is the data of
model

- a set M_s for $s \in S$ (each sort is interpreted as a set)

- functions $M_{s_1} \times M_{s_2} \times \dots \times M_{s_n} \xrightarrow{[\![f]\!]} M_{s_0}$

  for any $f \in F$ with arity $(s_1 \dots s_n)$ and sort s_0

example a model of the signature of magma is a magma

- a structure for the signature of pre-rings

a set M with two distinguished element

$$[0] \in M \qquad [1] \in M$$

and two maps $+ : M^2 \longrightarrow M$

$$\times : M^2 \longrightarrow M.$$

to a signature $\Sigma = (S, F)$ is associated a collection $T(\Sigma)$ of terms which are build inductively from some variables and the function symbols

fix infinite set of variables $(x_1, x_2, x_3, \ldots)$

term $\left|\begin{array}{l} \text{a variable} \\ \\ \text{or} \quad f(t_1, \ldots t_n) \end{array}\right.$ for t_i terms of sort s_i, f fct symbol of arity $(s_1 \ldots s_n)$

<u>example</u> Σ = magma signature

$T(\Sigma) = \quad x_1, \; x_2, \; x_3 \; \ldots$ are terms

$m(x_1, x_2) \qquad m(x_2, x_1)$

$m(x_1, x_1)$

$m(m(x_1, x_2), x_3)$ are terms.

$m(x_1, m(x_1, x_1))$

$m(m(x_1, x_2), m(x_2, m(x_3, x_4)))$ ---

- pre-ring signature

terms look like

$$\big((x_1 + x_2) + 1\big) \times x_3$$

$$(0 \times x_1) + x_2$$

...

<u>Context</u> for a signature $\Sigma = (S, P)$

context = list of variables

$\vec{x} = (x_1, x_2, x_3, x_n)$ where x_i has sort S_i

$(x_1 : S_1, x_2 : S_2, \dots)$

<u>term in context</u> = pair (context $\vec{x}$, term t)

such that the variable in t have to be in the context

example: magma :

- $((x_1, x_2), m(x_1, x_2))$ ok
- $((x_1, x_2, x_3), m(x_1, x_3))$
- $((x_1, x_3), m(x_1, x_2))$ not in the context not ok

Σ = signature, M a structure for Σ.

it is possible to interpret all terms in context

$(\vec{x}.\ t)$ as functions

$$M_{s_1} \times \cdots \times M_{s_n} \longrightarrow M_{s_0}$$

build inductively: $(x_1, \ldots, x_n).\ x_i$ $\quad x_i$ is within the context

interpretation is $[\![x_i]\!] : M_{s_1} \times \cdots \times M_{s_n} \xrightarrow{p_i} M_{s_i}$ = projection on the i th factor

$$(m_1, \ldots m_n) \longmapsto m_i$$

Variables in context are interpreted as projections for a term in context $(x_1 \ldots x_n) . f(t_1, \ldots, t_k)$

t_j has sort s_{t_j} $\qquad$ f a fct symbol in Σ.

$$M_{s_1} \times \cdots \times M_{s_n} \xrightarrow{[\![t_i]\!]} M_{s_j}$$

$$\begin{tikzcd}
M_{s_1} \times \cdots \times M_{s_n} \arrow[r, "{([\![t_1]\!], \ldots, [\![t_k]\!])}"] \arrow[dr, "\text{composition}", "{[\![f(t_1, \ldots, t_n)]\!]}"'] & M_{s_{t_j}} \times \cdots \times M_{s_{t_k}} \arrow[d, "{[\![f]\!]}"] \\
 & M_{s_f}
\end{tikzcd}$$

Example magma signature.

$(x_1, x_2) \,.\, m(x_1, x_2)$ $\qquad$ $M^2 \xrightarrow{[\![m]\!]} M$

$(x_1, x_2) \,.\, m(x_1, x_1)$ $\qquad$ $M^2 \xrightarrow{[\![m(x_1, x_1)]\!]} M$

$$(m_1, m_2) \overset{[\![x_1]\!] = P1}{\longmapsto} m_1$$

$$\begin{tikzcd}
M^2 \arrow[r, "{[\![m(x_1,x_1)]\!]}"] \arrow[d, "\text{forget } x_2"'] & M \\
M \arrow[r, "\Delta", "\text{repeat } x_1"'] & M^2 \arrow[u, "{[\![m]\!]}"']
\end{tikzcd}$$

$$m \longmapsto (m, m)$$

$(x_1, x_2) \; . \; m(x_1, m(x_2, x_2))$

$$\begin{tikzcd}
(x_1, x_2) \arrow[d, mapsto] & M^2 \arrow[rrr] \arrow[d, "1_M \times \Delta_M"] & & & M & m(x_1, m(x_2, x_2)) \\
(x_1, x_2, x_2) \arrow[drr, mapsto] & M^3 \arrow[r, "1_M \times [\![m]\!]"] & M^2 \arrow[urr, "[\![m]\!]"'] & & & \\
& & (x_1, m(x_2, x_2)) \arrow[uurrr, mapsto] & & &
\end{tikzcd}$$

Remark: in the interpretation of terms in contexts we use the structure of category of sets (composition of maps, diagonal, product, projections, ...)

We are going to take advantage of this to define a category $\mathrm{Syn}(\Sigma)$ such that structure for Σ are certain functors $\mathrm{Syn}(\Sigma) \longrightarrow \mathrm{Set}$

$\mathrm{Syn}(\Sigma)$ " the syntactic category of Σ "

let's assume that Σ has a <u>single sort</u>

<u>objets</u> : contexts $(x_1, x_2, \dots x_n)$ (sometimes the objets are defined as natural numbers (length of the context)

or $[n]$

<u>morphisms</u> $\mathrm{Hom}\big([m], [n]\big) = ?$

$\mathrm{Hom}\big([m], [1]\big)$ = collection of terms in the context $(x_1, \dots x_m)$

$\Updownarrow$

interpretation as $M^m \longrightarrow M^1$

$$\mathrm{Hom}([m],[n]) = \mathrm{Hom}([m],[1])^n$$

$=$ lists of size n of morphism $[m] \to [1]$

in the interpretation given $M^m \xrightarrow{f_i} M$ $i = 1, \dots n$

can use them to build a map

$$M^m \xrightarrow{(f_1, \dots, f_n)} M^n$$