<u>Today</u> : coincidence between

strong epi , regular epi , effective epi .

$$
\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "q"] & C \\
A' \arrow[u] \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] & B' \arrow[u] \arrow[r, "q'"] & C' \arrow[u, "u"']
\end{tikzcd}
$$

$A' = C' \times_C A$, $\quad B' = C' \times_C B$

coequalizer

$\downarrow$ pull back along u

need not be a coequalizer

Def in a category C, ~~effective epimorphisms~~ coequalizers of kernel pairs are said to be <u>stable by base change</u> if for any eff. epi q and any map $B' \xrightarrow{u} B$

$$
\begin{tikzcd}
A \times_B A \arrow[r, shift left] \arrow[r, shift right] & A \arrow[r, "q"] & \cancel{B}\, C \arrow[r, dashed] & B \\
A' \times_{B'} A' \arrow[u] \arrow[ur, phantom, "\urcorner"] \arrow[r, shift left] \arrow[r, shift right] & A' \arrow[u] \arrow[ur, phantom, "\urcorner"] \arrow[r, "q'"'] & \cancel{B'}\, C' \arrow[u, "u"'] \arrow[r, dashed] & B'
\end{tikzcd}
$$

then q' is also an effective epimorphism.

exercise: check this.

(use commutation of limits)

example in Set

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "g"] \\ B' \arrow[r] & B \end{tikzcd}$$

is also a surjection $= g'$ $\qquad$ g surjection

<u>Def</u> A <u>regular category</u> is a category with

- finite limits
- coequalizers of kernel pairs exist
- ~~effective epimorphisms~~ are stable by base change.
  coeq of kernel pairs

in a regular category, we can define the <u>image factorisation of a map</u>:

$$\begin{tikzcd} A \times_B A \arrow[r, "p_1"] \arrow[d, "p_2"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

$$\begin{tikzcd} A \times_B A \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & A \arrow[rr, "f"] \arrow[dr] & & B \\ & & C \arrow[ur, dashed] & \end{tikzcd}$$

effective epimorphism

monomorphism

Coequalizer of

$A \times_B A \rightrightarrows A$.

Lemma A a map $m : C \to B$ is a mono iff the two maps $C \times_B C \overset{p_1}{\underset{p_2}{\rightrightarrows}} C$ are equal.

<u>proof</u> $(\Rightarrow)$ by def of kernel pair $C \times_B C \overset{p_1}{\underset{p_2}{\rightrightarrows}} C \xrightarrow{m} B$

$$m \circ p_1 = m \circ p_2$$

by def of mono this implies $p_1 = p_2$.

$(\Leftarrow)$ m mono iff for any pair $X \overset{u}{\underset{v}{\rightrightarrows}} C$ s.t. $m \circ u = m \circ v$, then $u = u$

$$X \overset{u}{\underset{v}{\rightrightarrows}} C \xrightarrow{m} B \iff \begin{tikzcd} X \arrow[r, "u"] \arrow[d, "v"'] & C \arrow[d, "m"] \\ C \arrow[r, "m"'] & B \end{tikzcd} \iff X \xrightarrow{(u,v)} C \times_B C$$

$$
\begin{tikzcd}
X \arrow[rrd, bend left, "u"] \arrow[ddr, bend right, "v"'] \arrow[dr, dashed, "\delta"] & & \\
& C \times_B C \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "m"] \\
& C \arrow[r, "m"'] & B
\end{tikzcd}
$$

$$p_2 \circ \delta = u$$

$$p_1 \circ \delta = v$$

$$
\begin{tikzcd}
X \arrow[r, "\delta"] \arrow[rr, bend left=50, "v = p_1 \circ \delta"] \arrow[rr, bend right=50, "u = p_2 \circ \delta"'] & C \times_B C \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & C
\end{tikzcd}
$$

$p_1 = p_2$ implies

$u = v$.

$\blacksquare$

Lemma B consider
$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & B \\ & C \arrow[ur, "g"'] & \end{tikzcd}$$
$h = gf$.

Then if f and g are epi, so is h.

______ strong epi, so is h.

proof: Homework.

$$\begin{tikzcd} A\times_B A \arrow[r, shift left] \arrow[r, shift right] & A \arrow[rr, "f"] \arrow[dr, "e(f)"'] & & B \\ & & C \arrow[ur, "m(f)"'] & \end{tikzcd}$$

Coequalizer

Proposition $\mathcal{C}$ in a regular category $\mathcal{C}$, the map

$m(f)$ is a monomorphism.

proof

$$
\begin{tikzcd}
A \times_B A \arrow[r, "\varepsilon", "\text{epi}"'] \arrow[d, "e''"', "\text{epi}"] \arrow[dr, dashed, "\text{epi by Lemma B}"] & C \times_B A \arrow[r, "\mu"] \arrow[d, "e'"', "\text{epi}"] & A \arrow[d, "e(f)"', "\text{epi}"] \\
A \times_B C \arrow[r, "\text{epi}"'] \arrow[d, "m''"'] & C \times_B C \arrow[r, "p_2"] \arrow[d, "p_1"'] & C \arrow[d, "m(f)"] \\
A \arrow[r, "e(f)", "\text{epi}"'] & C \arrow[r, "m(f)"'] & B
\end{tikzcd}
$$

by Lemma A enough to prove

$$p_1 = p_2$$

$e(f)$ is a coeq. of kernel pair.

$e'(f)$ is also a coeq of a kernel pair

by regularity of C

the map $A \times_B A \xrightarrow{\delta} C \times_B C$ is ~~epi~~

hence $C \times_B C \underset{p_2}{\overset{p_1}{\rightrightarrows}} C$ are equal iff.

$A \times_B A \xrightarrow{\delta} C \times_B C \underset{p_2}{\overset{p_1}{\rightrightarrows}} C$ $\qquad p_1 \circ \delta = p_2 \circ \delta$. (by def of epi)

$$e(f) \circ m'' \circ e'' = e(f) \circ \mu \circ \varepsilon$$

$A \times_B A \underset{\mu \circ \varepsilon}{\overset{m'' \circ e''}{\rightrightarrows}} A \xrightarrow{e(f)} C$ $\qquad$ by def of C as coequalizer. ∎

$$\begin{tikzcd}
A \times_B A \arrow[r, shift left] \arrow[r, shift right] & A \arrow[rr, "f"] \arrow[dr, "e(f)"'] & & B \\
& & C \arrow[ur, "m(f)"'] &
\end{tikzcd}$$

<u>corollary</u> <u>D</u>: the map $e(f)$ is an <u>effective epi</u>.

<u>proof</u> enough to prove that $A \times_B A$ is $A \times_C A$

$$\begin{tikzcd}
A \times_C A = A \times_B A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\
A \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d] \\
A \arrow[r] & C \arrow[r] & B
\end{tikzcd}$$

in Set

$$\underbrace{A \times_B A}_{\text{equivalence relation of } A.} \rightrightarrows A \xrightarrow{f} B$$

$$A \xrightarrow[\text{surj}]{e(f)} C \xrightarrow[\text{inj}]{m(f)} B$$

" have same image by f "

$\Updownarrow$ same relations.

" have same image by $e(f)$ "

$$\underbrace{A \times_B A}_{\text{only depends on the surjection part of } f} \subset A \times A$$

(does not use the elements of B not in the image)

hence the formula $A \times_B A = A \times_C A$.

Theorem: in a regular category, the notions of

strong epi, regular epi, effective epi

coincide.

proof: eff epi $\Rightarrow$ reg epi $\Rightarrow$ strong epi always.

enough to prove strong epi $\Rightarrow$ effective epi

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "e(f)"'] & & B \\
& C \arrow[ur, "m(f)"'] &
\end{tikzcd}
$$

strong epi

$e(f)$ = eff. epi

$m(f)$ = mono

$$
\begin{tikzcd}
A \arrow[r, "e(f)"] \arrow[d, "f"', "\text{strong epi}"] & C \arrow[d, "m(f)", "\text{mono}"'] \\
B \arrow[r, equal, "\mathrm{id}_B"'] \arrow[ur, dashed, "\exists!\,\delta"] & B
\end{tikzcd}
$$

Lemma: a monomorphism $m : C \to B$ is invertible iff $\exists\ B \xrightarrow{\delta} C$ such that $B \xrightarrow{\delta} C \xrightarrow{m} B$ is 1_B.

proof: exercise.

Hence $m(f)$ is an isomorphism and f is isomorphic to $e(f)$

therefore f is an effective epi.

$\boxtimes$

<u>Morale</u> : in a regular category, we have a

(effective epi, mono) factorization

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\text{eff epi}"'] & & B \\ & C \arrow[ur, "\text{mono}"'] & \end{tikzcd}$$

can be used to define semantics for $\exists$

$$X \times Y \xrightarrow{p} Y$$

$$\begin{tikzcd} \mathrm{Sub}(X\times Y) \arrow[r, dashed, bend left, "p_!"] \arrow[r, phantom, "\perp"] & \mathrm{Sub}(Y) \arrow[l, "p^*"] \end{tikzcd}$$

) exist in a regular category.

$$\begin{tikzcd} {[x,y \mid \varphi]} = \Phi \arrow[r, "e(f)"] \arrow[d] \arrow[dr, "f"] & p_!\Phi = {[y \mid \exists x\, \varphi(x)]} \arrow[d, "m(f)"] \\ X \times Y \arrow[r, "p"'] & Y \end{tikzcd}$$