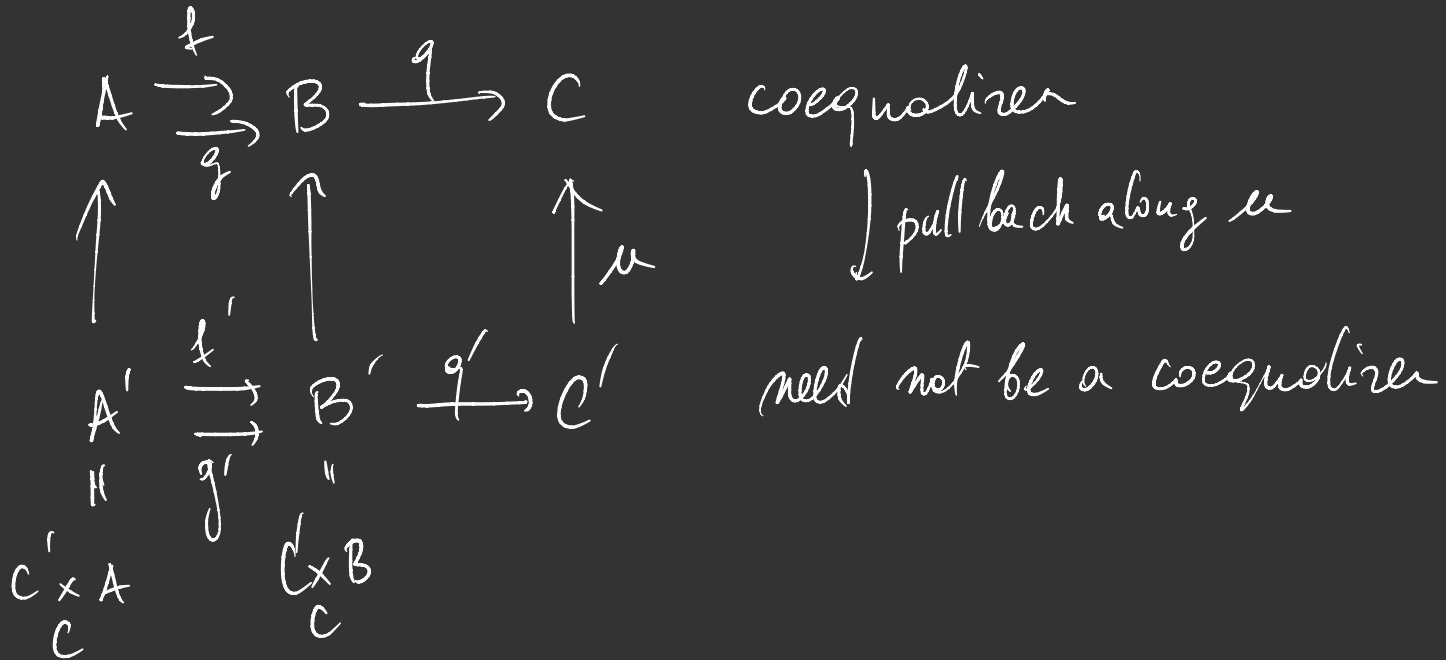


Today: coincidence between

strong epi, regular epi, effective epi.



Def in a category \mathcal{C} , ^{coequalizers of kernel pairs} ~~effective epimorphisms~~ are said to be stable by base change if for any eff-epi q and any map $B' \xrightarrow{u} B$

$$\begin{array}{ccccc}
 A \times A & \rightrightarrows & A & \xrightarrow{q} & B & \dashrightarrow & B \\
 \uparrow & \lrcorner & \uparrow & \lrcorner & \uparrow u & & \\
 A' \times A' & \rightrightarrows & A' & \xrightarrow{q'} & B' & \dashrightarrow & B' \\
 \uparrow & & & & & & \\
 B' & & & & & &
 \end{array}$$

then q' is also an effective epimorphism.

exercise: check this.

(use commutation of limits)

example in Set

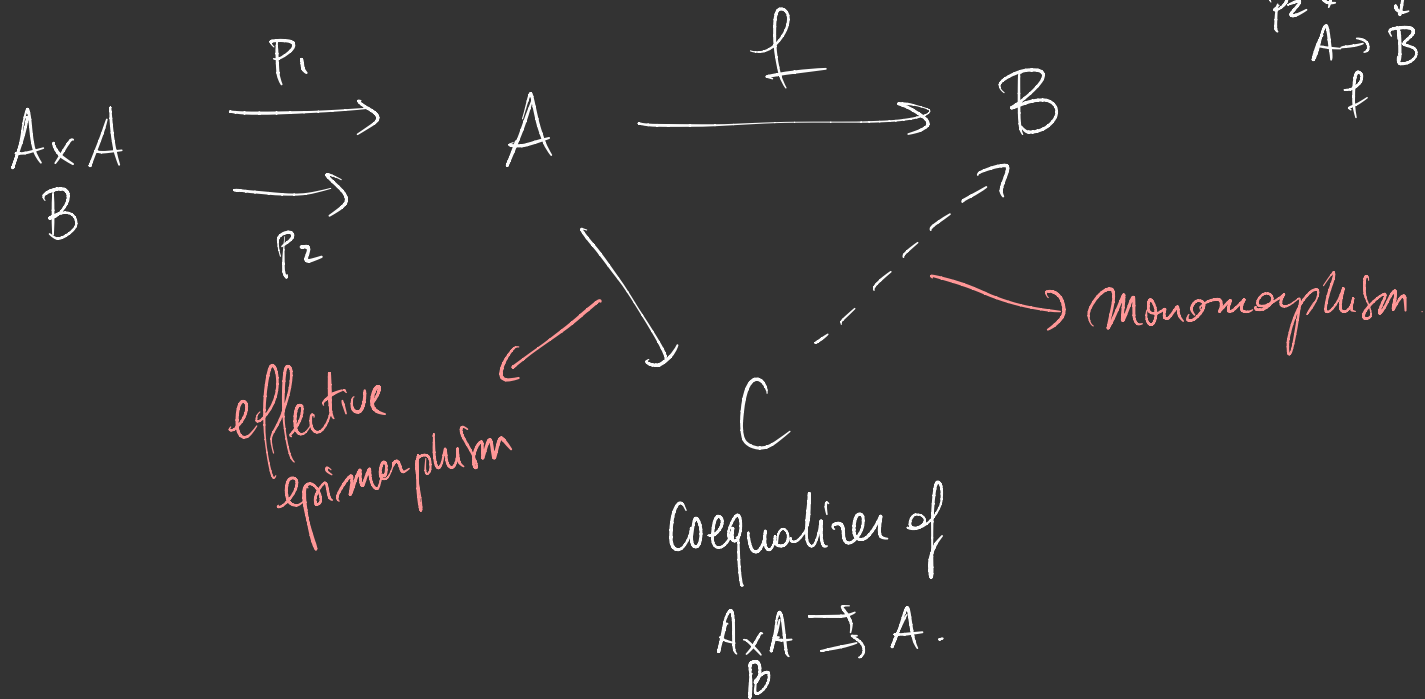
$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow q' & \lrcorner & \downarrow q \\ B' & \longrightarrow & B \end{array}$$

is also a q' q surjection
surjection

Def A regular category is a category with

- finite limits
- coequalizers of kernel pairs exist
- ~~effective epimorphisms~~ are stable by base change.
coeq of kernel pairs

In a regular category, we can define the image
factorisation of a map:



Lemma A a map $m: C \rightarrow B$ is a mono iff the
 \top two maps $C \times_C C \begin{matrix} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{matrix} C$ are equal.

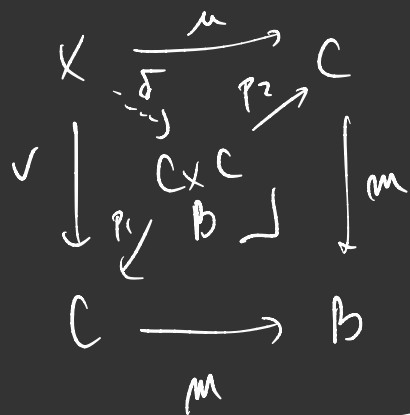
Proof (\Rightarrow) by def of kernel pair $C \times_C C \begin{matrix} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{matrix} C \xrightarrow{m} B$

$$m \circ p_1 = m \circ p_2$$

by def of mono this implies $p_1 = p_2$.

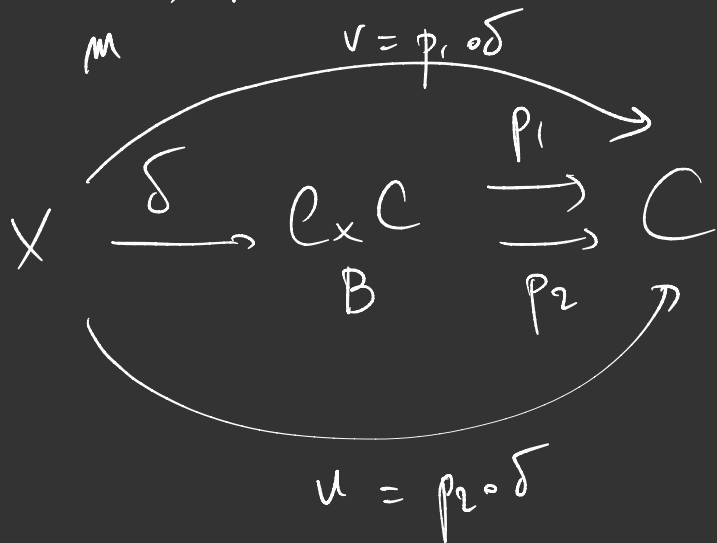
(\Leftarrow) m mono iff for any pair $X \begin{matrix} \xrightarrow{u} \\ \xrightarrow{v} \end{matrix} C$ s.t. $m \circ u = m \circ v$, then $u = v$

$$X \begin{matrix} \xrightarrow{u} \\ \xrightarrow{v} \end{matrix} C \xrightarrow{m} B \Leftrightarrow \begin{matrix} X \xrightarrow{u} C \\ v \downarrow \quad \downarrow m \\ C \xrightarrow{m} B \end{matrix} \Leftrightarrow X \xrightarrow{(u,v)} C \times_C C$$



$$p_2 \circ \delta = u$$

$$p_1 \circ \delta = v$$



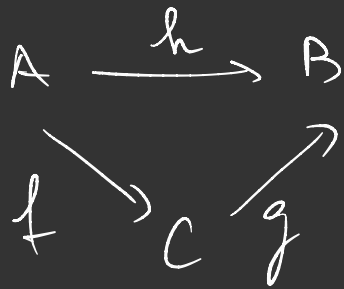
$p_1 = p_2$ implies

$$u = v.$$



Lemma B

consider



$$h = g \circ f.$$

then if f and g are epi, so is h .

strong epi, so is h .

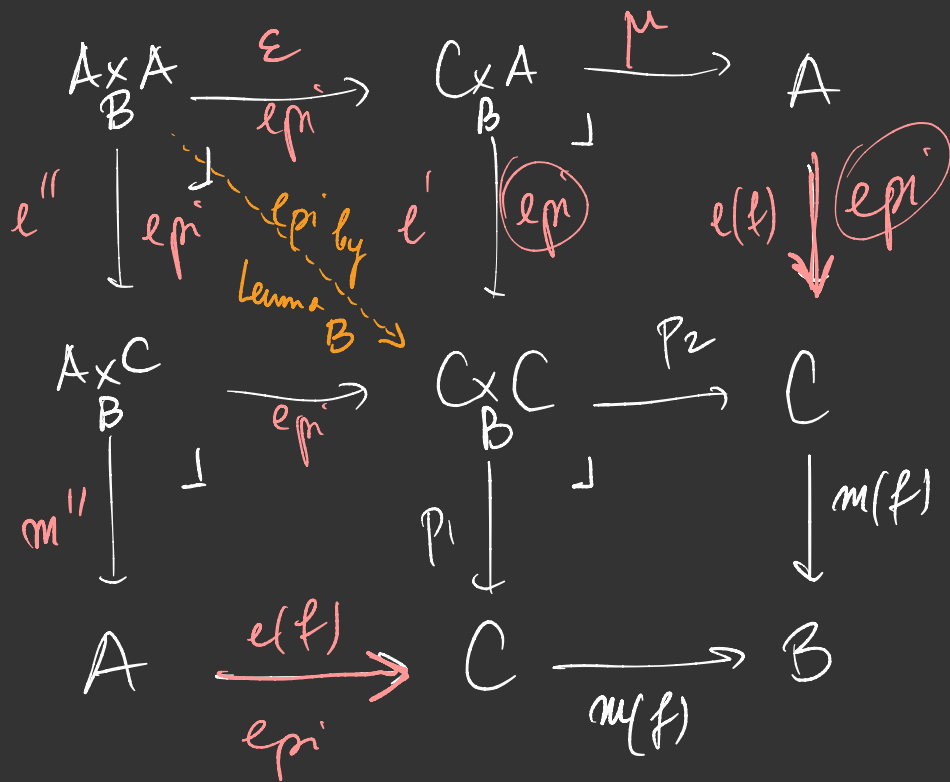
proof: Homework

$$\begin{array}{ccc}
 A \times A & \rightrightarrows & A \xrightarrow{f} B \\
 \downarrow B & & \downarrow e(f) \quad \uparrow m(f) \\
 & & C
 \end{array}$$

Cocoulires

Proposition C in a regular category \mathcal{C} , the map $m(f)$ is a monomorphism.

proof



by lemma A
 enough to prove
 $P_1 = P_2$

$e(f)$ is a coeq.
 of kernel pair.

$e'(f)$ is also a
 coeq of a kernel
 pair

by regularity of C

The map $A \times_B A \xrightarrow{\delta} C \times_B C$ is epi

hence $C \times_B C \xrightleftharpoons[p_2]{p_1} C$ are equal iff.

$$A \times_B A \xrightarrow{\delta} C \times_B C \xrightleftharpoons[p_2]{p_1} C \quad p_1 \circ \delta = p_2 \circ \delta \quad (\text{by def of epi})$$

\parallel
 \parallel

$$A \times_B A \xrightleftharpoons[\mu \circ \varepsilon]{m'' \circ e''} A \xrightarrow{e(f)} C$$

$e(f) \circ m'' \circ e'' = e(f) \circ \mu \circ \varepsilon$

by def of C as coequalizer -

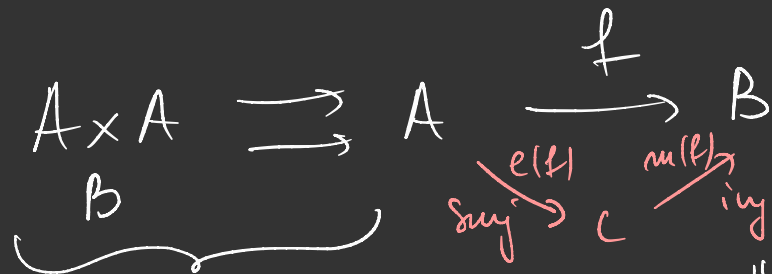
$$\begin{array}{ccc}
 A \times_B A & \rightrightarrows & A \xrightarrow{f} B \\
 & & \searrow e(f) \quad \nearrow m(f) \\
 & & C
 \end{array}$$

Corollary D: the map $e(f)$ is an effective epi.

Proof enough to prove that $A \times_B A$ is $A \times_C A$.

$$\begin{array}{ccccc}
 A \times_C A = A \times_B A & \rightarrow & A & \rightrightarrows & A \\
 \downarrow & & \downarrow & \lrcorner & \downarrow \\
 A & \rightarrow & C & \xlongequal{\quad} & C \\
 \parallel \lrcorner & & \parallel \lrcorner & & \downarrow \\
 A & \rightarrow & C & \rightarrow & B
 \end{array}$$

in Set



equivalence relation of A.

"have same image by f"

⇔ same relations.

"have same image by e(f)"

$$\underbrace{A \times A}_B \subset A \times A$$

only depends on the surjection part of f

(does not use the elements of B not in the image)

hence the formula $A \times_B A = A \times_C A$.

Theorem : in a regular category, the notions of

strong epi, regular epi, effective epi

coincide.

proof : eff epi \Rightarrow reg epi \Rightarrow strong epi always.

enough to prove strong epi \Rightarrow effective epi

$$A \xrightarrow{f} B \quad \text{strong epi.}$$

$$\begin{array}{ccc}
 & & \\
 e(f) & \searrow & \\
 \text{"} & & C & \nearrow & m(f) = \text{mono} \\
 \text{eff. epi.} & & & &
 \end{array}$$

$$\begin{array}{ccc}
 A & \xrightarrow{e(f)} & C \\
 \downarrow f & \dashrightarrow \exists! \delta & \downarrow m(f) \\
 B & \xlongequal{\text{id}_B} & B
 \end{array}
 \quad \text{strong epi.} \quad \text{mono}$$

lemma : a monomorphism $m: C \rightarrow B$ is invertible iff
 $\exists B \xrightarrow{\delta} C$ such that $B \xrightarrow{\delta} C \xrightarrow{m} B$ is 1_B .

proof : exercise.

Hence $m(f)$ is an isomorphism and f is isomorphic to $e(f)$

therefore f is an effective epi.

□

Morale : in a regular category, we have a
(effective epi, mono) factorization

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \text{eff epi} \searrow & & \nearrow \text{mono} \\ & C & \end{array}$$

can be used to define semantics for \exists

$$X \times Y \xrightarrow{P} Y$$

$$\begin{array}{ccc} & \xrightarrow{P!} & \\ \uparrow \perp * & & \\ \text{Sub}(X \times Y) & \xleftarrow{P} & \text{Sub}(Y) \end{array} \text{ exist in a regular category.}$$

$$\begin{array}{ccc} [x, y | \varphi] = \overline{\Phi} & \xrightarrow{e(\varphi)} & P! \overline{\Phi} = [y | \exists x \varphi(x)] \\ \downarrow & \searrow \varphi & \downarrow m(\varphi) \\ X \times Y & \xrightarrow{P} & Y \end{array}$$