Today: the analogue of the surjection / injection factorization of a map in Set ~~in an~~ a category with finite limits.

Set	$\mathcal{C}$ lex category
injections	monomorphisms.
surjections	• (general) epimorphisms • strong epimorphisms • regular ——— • effective ———

coincide under mild hypothesis on $\mathcal{C}$

preliminary on <u>orthogonality of maps</u>

$\mathcal{C}$ arbitrary cat. $u: A \to B$ $\quad f: X \to Y$ in $\mathcal{C}$

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "u"'] & X \arrow[d, "f"] \\ B \arrow[r, "\beta"'] \arrow[ur, dashed, "\exists \delta"] & Y \end{tikzcd}$$

given a commutative square, can ask whether there exist a "<u>diagonal lift</u>", i.e. a map $\delta: B \to X$ such that

$$f\delta = \beta \quad \text{and} \quad \delta u = \alpha$$

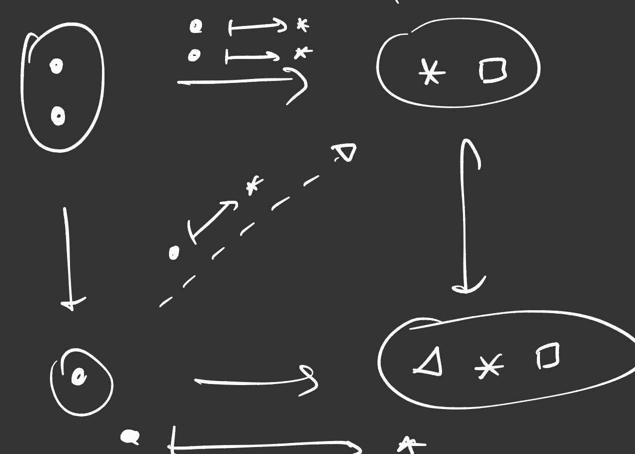
<u>Definition</u> u and f are said to be <u>orthogonal</u> $(u \perp f)$

if for any comm. square $u \downarrow \overset{\longrightarrow}{\underset{\longrightarrow}{}} \downarrow f$,

there <u>exists a</u> <u>unique</u> diagonal lift.

<u>example</u> : in Set u a surjection
f an injection then $u \perp f$

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X \arrow[d, hook] \\
B \arrow[r] \arrow[ur, dashed, "\exists !"] & Y
\end{tikzcd}$$



Reformulation :

the set of commutative squares
$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "u"'] & X \arrow[d, "f"] \\ B \arrow[r, "\beta"'] & Y \end{tikzcd}$$

$$\begin{tikzcd}
\delta \arrow[rrr, mapsto, bend left, "f\circ\delta" description] \arrow[dd, mapsto, bend right] & & & f\circ\delta \\
\mathrm{Hom}(B,X) \arrow[r, dashed, "\sim"'] \arrow[dr, bend right] & \mathrm{Hom}(A,X) \times_{\mathrm{Hom}(A,Y)} \mathrm{Hom}(B,Y) \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner" very near start] & & \mathrm{Hom}(B,Y) \arrow[d] & \beta \arrow[d, mapsto] \\
\delta\circ u & \mathrm{Hom}(A,X) \arrow[rr] & & \mathrm{Hom}(A,Y) & \beta\circ u \\
& \alpha \arrow[rr, mapsto] & & f\circ\alpha &
\end{tikzcd}$$

$u \perp f \iff$ **bijection**

<u>Def</u>: a map $e : A \to B$ (in a cat. C) is an <u>epimorphism</u>

if for any X in C $\quad \mathrm{Hom}_C(B,X) \xrightarrow{\circ e} \mathrm{Hom}_C(A,X)$

is injective.

$$\begin{tikzcd} A \arrow[d, "e"'] \arrow[dr, bend left, "fe = ge"] & \\ B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X \end{tikzcd}$$

such that $f \circ e = g \circ e$

then $f = g$.

(<u>Rem</u> this is dual to the def of a monomorphism) the data of $B \overset{f}{\underset{g}{\rightrightarrows}} X$ s.t. $fe = ge$ is equivalent to a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "e"', "\text{epi}"'] & X \arrow[d, "\Delta_X"] \\ B \arrow[r, "{(f,g)}"'] \arrow[ur, dashed, "\exists!"] & X \times X \end{tikzcd}$$

<u>Rem:</u> $\Delta_X : X \to X \times X$ is always a mono

<u>exercise</u>

$$\boxed{e \text{ epi} \iff e \perp \Delta_X, \text{ for all } X \text{ in } \mathcal{C}} \quad (*)$$

Lemma 1 : if $m : X \to Y$ is a monomorphism.

Then, for any $u : A \to B$, the map

$$\mathrm{Hom}(B, X) \longrightarrow \mathrm{Hom}(A, X) \times_{\mathrm{Hom}(A,Y)} \mathrm{Hom}(B, Y)$$

is <u>injective</u>

proof : exercise.

in other words if m mono

$$\begin{tikzcd} A \arrow[r] \arrow[d, "u"'] & X \arrow[d, "m"] \\ B \arrow[r] \arrow[ur, dashed, "\delta"] & Y \end{tikzcd}$$

δ might not always exist but if it does it is <u>unique</u>

Def a map u is a strong epimorphism if

$u \perp m$ for all monomorphisms m.

Using $(*)$ any strong epi is an epimorphism.

Rem in Set strong epi = epi

but not in other categories.

Consider a diagram

$$\begin{tikzcd} R \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[dr, bend right, "u'"'] & A \arrow[d, "u"] \\ & B \end{tikzcd}$$

colimit is called <u>coequalizer of f and g</u>

$u \circ f = u \circ g = u'$

by def of the colimit.

<u>Def</u>: if $\mathcal{C}$ has coequalizers, a map $u: A \to B$ is called a <u>regular epimorphism</u> if there exist a diagram $R \overset{f}{\underset{g}{\rightrightarrows}} A$ s.t. u is the coequalizer of f, g

given $A \xrightarrow{u} B$ in $\mathcal{C}$ with fiber products

can construct

$$\begin{tikzcd} A \times_B A \arrow[r, "p_2"] \arrow[d, "p_1"'] & A \arrow[d, "u"] \\ A \arrow[r, "u"'] & B \end{tikzcd}$$

$$\underbrace{\begin{tikzcd} A \times_B A \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & A \end{tikzcd}}$$

Kernel pair of $u : A \to B$

the map u defines a cocone

$$\begin{tikzcd} A \times_B A \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[dr, "up_1 = up_2"'] & A \arrow[d, "u"] \\ & B \end{tikzcd}$$

Let $A \xrightarrow{u'} B'$ be the coequalizer of $A \times_B A \rightrightarrows A$

By def of colimits there exist a unique map $B' \xrightarrow{\bar{i}} B$ such that

$$
\begin{tikzcd}
A \times_B A \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] \arrow[ddrr, bend right] & A \arrow[d, "u'"] \arrow[ddr, bend left, "u"] & \\
& B' \arrow[dr, dashed, "\exists ! \ \bar{i}"'] & \\
& & B
\end{tikzcd}
$$

$\bar{i} \circ u' = u$

$$
\begin{tikzcd}
A \arrow[d, "u'"'] \arrow[dr, "u"] & \\
B' \arrow[r, "\bar{i}"'] & B
\end{tikzcd}
$$

commutes.

Def in a cat. with finite limits and coequalizers (of kernel pairs)

a map $u : A \to B$ is called an

effective epimorphism if

u is the coequalizer of its kernel pair

(i.e. if $i : B' \xrightarrow{\sim} B$ is an isomorphism)

Rem Kernel pairs play the role of "equivalence relations"

effective epi $\leftrightarrow$ quotient maps of eq. rel.

regular epi $\leftrightarrow$ __________ of arbitrary relations.

ex in Set, any relation can be completed into an equiv. rel and the quotient stays the same. This is a way to say that regular epi = effective epi in Set.

actually in Set

$$\text{epi} = \text{strong epi} = \text{regular epi} = \text{effective epi}$$

always $\Leftarrow$ lemma 1 $\Leftarrow$ lemma 2 $\Leftarrow$ by def.

Lemma 2 any regular epi is strong. (in a lex cat)

proof

$$\begin{tikzcd}
R \arrow[d, shift left] \arrow[d, shift right] \arrow[rr, dashed, "\text{by univ. prop of } X\times_Y X"] & & X \times_Y X = X \arrow[d, shift right, "id_X"'] \arrow[d, shift left, "id_X"] \\
A \arrow[d, "u"'] \arrow[rr] & & X \arrow[d, "m"] \\
B \arrow[rr] \arrow[urr, dashed, "\text{by univ prop coequalize}"] & & Y
\end{tikzcd}$$

reg epi u

m a mono

cocone

$$\begin{tikzcd}
R \arrow[d, shift left] \arrow[d, shift right] \arrow[r] & X \\
A \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\
& X \arrow[r] & Y
\end{tikzcd}$$