Completeness results for propositional theories

various kind of prop. theories.

$F \hookrightarrow P(pt(F))$

Prop. th.	Syntactic Poset	Frame	Space of models	enough points
Coherent $(\top, \wedge, \bot, \vee)$	Distributive lat. D	$Idl(D)$	Coherent space	yes Coherent duality
Boolean $(\top, \wedge, \bot, \vee, \neg)$ (+ excluded middle rule)	Boolean alg. B	$Idl(B)$	Stone Space	yes Stone duality
"Geometric" $(\top, \wedge, \bot, \bigvee)$	Frame F	F	Topological space	no
general $(\top, \wedge, \bot, \vee, \Rightarrow)$	Heyting alg. H	—	—	—

if T geometric theory $Syn(T)$ is a Frame

$$\mathrm{TopMod}(T,-): \mathrm{Top}^{op} \longrightarrow \mathrm{Set} \quad (\text{not Poset})$$

$$X \longmapsto \mathrm{Mod}(T, \mathcal{O}(X)) = \mathrm{Hom}_{\mathrm{Frames}}(Syn(T), \mathcal{O}(X))$$

=

The functor of "topological semantics" of T.
is representable by the space $pt(Syn(T)) =: pt(T)$

$$\mathrm{Hom}_{\mathrm{Frame}}(Syn(T), \mathcal{O}(X)) = \mathrm{Hom}_{\mathrm{Top}}(X, pt(T))$$

(continuous maps)

T could be called the "syntactic space" of T.

† Boolean $\mathrm{Syn}(T)$ = Boolean alg.

Stone $\mathrm{Mod}(T)$: $\mathrm{Stone}^{op} \simeq \mathrm{BAlg} \longrightarrow \mathrm{Set}$ (we could take values in Poset)

$X \;/\; B \longmapsto \mathrm{Mod}(T, B)$

$\| $

$\mathrm{Hom}_{\mathrm{BAlg}}(\mathrm{Syn}(T), B)$

representable by | the Stone space associated to $\mathrm{Syn}(T)$ | dual

Rem : the equivalence $\mathrm{Stone}^{op} \simeq \mathrm{BAlg}$ is the content of Stone duality.

Remark if T Boolean theory.

we get 2 spaces : the Stone space dual to $Syn(T)$

the top. space dual to $Idl(Syn(T))$

they are the same space.

Examples of Frames / theories without enough points.

a Frame F has enough point if the canonical frame morphism

$$F \longrightarrow \underbrace{P}_{\text{power set}}(\underbrace{pt(F)}_{\text{set of point of } F})$$

models of T if $F = \mathrm{Sym}(T)$

is injective

$\iff \forall A, B \in F$, $\exists\, x : F \longrightarrow 2$ a point such that

$$x(A) \neq x(B)$$

in general a frame does not have enough points.

- + theory of surjections $\mathbb{N} \to \mathbb{R}$ (= geometric theory)

it is possible to prove that the frame $\mathrm{Syn}(T)$ is not trivial
(not the zero frame, with a single element)

= where $\bot = \top$

this means the theory of surjections $\mathbb{N} \to \mathbb{R}$ is not contradictory

even though the theory has no models in $\{0 < 1\}$

$\Longleftrightarrow$ the frame $\mathrm{syn}(T)$ has no points at all !

$$\mathrm{Syn}(T) \longrightarrow P\big(pt(T)\big) = P(\emptyset) = 0$$

not injective !

examples of theories with enough points

- the geometric theory of real numbers has enough points.

$$Syn(T) = \mathcal{O}(\mathbb{R}) \subset \mathcal{P}(\mathbb{R}) \quad (= \mathbb{R} = pt(T) = \text{models of } T \text{ in } \{0,1\})$$

can inclusion

some geom. th. (which are not coherent or Boolean) do have enough points.

- theory of fct $E \to F$ for $E \neq \emptyset = F$

theory with no models / points but it is contradictory

$\bot = \top$ in $Syn(T) = 0$

$Syn(T) = 0 \simeq \mathcal{P}(\emptyset)$ injective! has enough points.

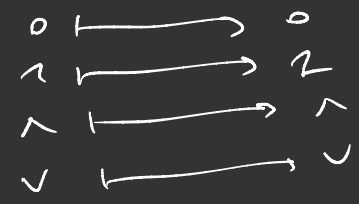
Points of coherent/Boolean theories

point of a dist. lattice D

$\mathrm{Idl}(D) \xrightarrow{\text{frame morphism}} \{0 < 1\} = \underline{2}$

a prime filter in D $\Longleftrightarrow$ $x : D \xrightarrow{\text{Dlatt.}} \underline{2}$

$D \supset \varphi = \{ d \mid x(d) = 1 \}$



φ filter : $x \in \varphi$, $x \le y$ then $y \in \varphi$

$x, y \in \varphi$ then $x \wedge y \in \varphi$

prime $x \vee y \in \varphi$ then $x \in \varphi$ or $y \in \varphi$.

points of a Boolean algebra B

$\mathrm{Idl}(B) \xrightarrow{\text{frame}} \underline{2}$

$B \supset \varphi = \{ b \mid x(b) = 1 \}$

$\Longleftrightarrow$

$x : B \xrightarrow{\text{BAlg}} \underline{2}$

1	1
0	0
$\vee$	$\vee$
$\wedge$	$\wedge$
$\neg$	$\neg$

ultrafilter of B.

= filter such that

for any $b \in B$ either $b \in \varphi$

or $\neg b \in \varphi$

(in part any ultrafilter is prime)

= maximal element in the poset of filters of B.

Thm Any distributive lattice (in particular any Boolean algebra) has enough points.

$$D \xrightarrow{\text{can.}} P(pt(D))$$

is always injective

$pt(D) = \mathrm{Hom}(D, \underline{2})$
$\mathbf{Dlat}$

($\Leftrightarrow$ for any $A, B \in D$ there exists an $x : D \to \underline{2}$ $x(A) \neq x(B)$)

This result has an interpretation in terms of completeness of coherent and Boolean logic.

Recall T alg theory M : model of T in some cat C

$M \models t_1 = t_2$ (t_1, t_2 terms) $\iff M(t_1) = M(t_2)$ in Arrow(C)

" M validates the formula $t_1 = t_2$ "

$T \vdash t_1 = t_2 \iff [t_1] = [t_2]$ in $Syn(T)$

" T proves $t_1 = t_2$ "

Can introduce the same notions for a prop. theories - T

M model $\quad Syn(T) \longrightarrow P \qquad P = \begin{vmatrix} \text{Frame} \\ \text{Dlatt} \\ \text{BAlg} \end{vmatrix}$

$M \models \varphi_1, \dots, \varphi_n \vdash \psi$ if $M(\varphi_1) \wedge \dots \wedge M(\varphi_n) \leq M(\psi)$ in P.

validate the sequent

$T \mid \varphi_1, \dots, \varphi_n \vdash \psi$ if $[\varphi_1] \wedge \dots \wedge [\varphi_n] \leq [\psi]$ in $Syn(T)$

T proves the sequent

Thm (Strong completeness) for any prop. theory T

$T \mid \varphi_1, \dots, \varphi_n \vdash \psi \iff$ for all models M: $M \models \varphi_1, \dots, \varphi_n \vdash \psi$

$\iff$ the universal model $U \models \varphi_1, \dots, \varphi_n \vdash \psi$

Thm (classical completeness) for any coherent or Boolean theory T

$T \mid \varphi_1, \dots, \varphi_n \vdash \psi \iff$ for all models in $\underline{\mathbf{Set}}$ i.e. in $\{0 < 1\}$

$M \models \varphi_1, \dots, \varphi_n \vdash \psi$

$\iff$ $\mathrm{Syn}(T)$ has enough points

proof (sketch)

$T \mid \varphi_1 \dots \varphi_n \vdash \psi \iff [\varphi_1] \wedge \dots \wedge [\varphi_n] \leq [\psi]$ in $Syn(T)$

if for all $Syn(T) \xrightarrow{M} \{0 < 1\}$ $\quad M(\varphi_1) \wedge \dots \wedge M(\varphi_n) \leq M(\psi)$

$M \models \varphi_1 \dots \varphi_n \vdash \psi$

$\Updownarrow$

$Syn(T) \hookrightarrow \mathcal{P}(pt(T))$

injective.