Bits of topology

XIX c : two problems → convergence of sequence
computation of limits) Analysis

↘ patching spaces to get new ones
(Manifolds ...) geometry

these differents motivation ended up producing the same structure :

" algebras of open subspaces "

⇓

" Frames "

What is a space?

→ set X with distinguished subsets. called open subsets.

$$\mathcal{O}(X) \subset \mathcal{P}(X)$$

axiom:

- stable by finite intersection ($\Rightarrow X \in \mathcal{O}(X)$)

  the patching of two open is done along an open

- stable by arbitrary unions

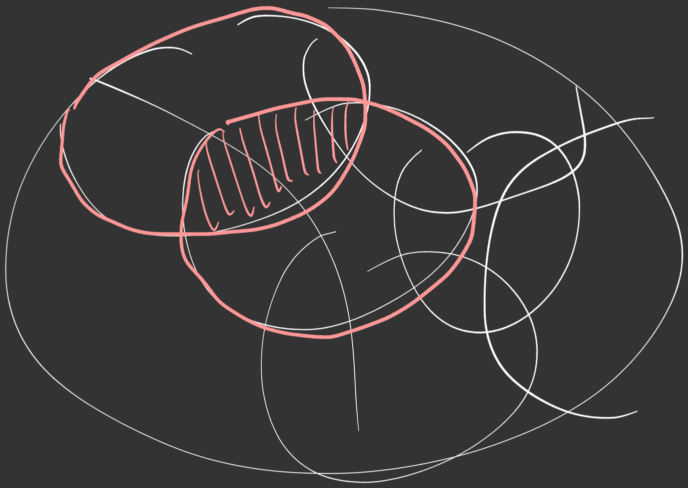
  ($\Rightarrow \emptyset \in \mathcal{O}(X)$)

  can patch as many open as we want and still be open.

} sub frame of $\mathcal{P}(X)$

open subspaces are meant to be suitable pieces for patching.

- can break a space in open pieces (covering) and patch it back



a continuous map between to topological spaces

function $X \xrightarrow{f} Y$

$$\begin{tikzcd}
P(X) \arrow[d, phantom, "\cup"] & P(Y) \arrow[l, "f^{-1}"'] \arrow[d, phantom, "\cup"] & \text{] frame morphism} \\
O(X) & O(Y) \arrow[l, dashed, "f^{-1}"'] & \text{] frame morphism.}
\end{tikzcd}$$

continuity = inverse image of open is open.

close relationship between top. space and frames.

Cat of Top. sp
and continuous maps

Cat of frames
and frames morphism

$$\begin{tikzcd}
\mathrm{Top}^{op} \arrow[r, shift left, "\text{forgetting set of points}"] & \mathrm{Frames}. \arrow[l, shift left, "\text{points}"] \\
(\underbrace{X}_{\text{set of points}}, \underbrace{O(x)}_{\text{frame}}) \arrow[r, mapsto] & O(X) \\
\mathrm{points}(F) & F \arrow[l, mapsto]
\end{tikzcd}$$

Frame = poset with arbitrary joins
finite meets
+ distributivity cdt.

$\{0<1\}$ is a frame: $\{0<1\} = P(\{*\})$

This the frame of open subspaces of a point

$\{*\}$ has a unique topology: $O(*) = P(\{*\})$

$(X, O(X))$ Space   a point of X is a continuous map

$$\begin{tikzcd}
\{*\} = 1 \arrow[r] & X \\
* \arrow[r, mapsto] & x
\end{tikzcd}$$

$$\begin{tikzcd}
P(1) & P(X) \arrow[l, "f^{-1}"'] \\
O(1) \arrow[u, equal] & O(X) \arrow[u, phantom, "\cup"] \arrow[l]
\end{tikzcd}$$

frame morphism

f is always continuous!

points a frame F =

frame morphism $F \longrightarrow \{0 < 1\} = \underline{2} = O(1)$

$pt(F) = \mathrm{Hom}_{\mathrm{Frame}}(F, \underline{2})$ = set of points of F.

$pt(F)$ has always a topology :

$$\mathrm{Hom}_{\mathrm{Frames}}(F, \underline{2}) \times F \xrightarrow{ev} \{0, 1\}$$

$$(\varphi : F \to \underline{2} \ , \ A) \longmapsto \varphi(A)$$

$$F \longrightarrow \mathrm{Hom}_{\mathrm{Set}}\left(\mathrm{Hom}_{\mathrm{Frames}}(F, \underline{2}), \{0,1\} \right)$$

$$\cup$$

$$P(\mathrm{pt}(F))$$

$$A \longmapsto \{ \varphi : F \to \underline{2} \mid \varphi(A) = 1 \}$$

fact (exercise)

$$\begin{tikzcd} F \arrow[rr] \arrow[dr, two heads, "\text{surjection}", "\text{frame morph.}"'] & & P(pt(F)) \\ & \begin{array}{c}\text{image}\\ \shortparallel \\ \text{a frame}\end{array} \arrow[ur, hook, "\text{injection}"', "\text{frame morph.}"] & \end{tikzcd}$$

is a frame morphism.

open subsets of a topology on $pt(F)$

if we're lucky $F \longrightarrow P(pt(F))$ is already injective

F is said to "have enough points" if the case.

F has enough points precisely when it is in the image of the functor

$$\mathrm{Top}^{op} \longrightarrow \mathrm{Frames}$$

unfortunately this functor is not surjective.

not all frames have enough points.

References

- "Stone Space"
- Anel – Joyal "Topo-logy"

last course we saw the syntactic frames for a propositional theory.

Powerposet

$$\begin{tikzcd}
\mathrm{Set} & & \\
\mathrm{Poset} \arrow[u] \arrow[rr, shift left, "P(X) = \mathrm{Hom}(X^{op}, 2)"] \arrow[d, shift right, "M"'] & & \mathrm{CLat} \arrow[ll, shift left] \arrow[d, shift right] \\
\mathrm{MLat} \arrow[u, shift right] \arrow[r, shift left, "P_{fin}"] & \mathrm{DLat} \arrow[l, shift left] \arrow[r, shift left, "\mathrm{Idl}"] \arrow[d, shift right] \arrow[dd, bend left] & \mathrm{Frames} \arrow[l, shift left] \arrow[u, shift right] \arrow[ddl, shift left, "\mathrm{clopen}"] \\
& (\mathrm{HAlg}) \arrow[u, shift right] \arrow[d, shift right] & \\
& \mathrm{BAlg}. \arrow[u, shift right] \arrow[uur, shift left, "\mathrm{Idl}"] &
\end{tikzcd}$$

(commutative)

forgetful functor

left adjoint or "free" structures

right adjoint

The notion of Frame is (in a sense) the "most powerful" for syntactic posets.

Propositional theory → Syntactic Frame

Space of models

(underlying set is $\mathrm{Mod}(T, 2)$)

The set of models $\mathrm{Mod}(T, 2)$ has a canonical topology!

T a prop. theory.

$\mathrm{Syn}(T)$ the syntactic frame.

F frame $\qquad \mathrm{Mod}(T, F) \simeq \mathrm{Hom}_{\mathrm{Frame}}(\mathrm{Syn}(T), F)$

$2 = \overset{\shortparallel}{\{0 < 1\}} = \mathcal{O}(pt) \qquad \underset{(\text{models in Set})}{\mathrm{Mod}(T, 2)} \simeq \underset{pt(\mathrm{Syn}(T))}{\underset{\shortparallel}{\mathrm{Hom}(\mathrm{Syn}(T), 2)}}$

has a topology!

More generally

$$\begin{tikzcd}
\mathrm{Mod}(T, \underset{\substack{\| \\ \mathcal{O}(X)}}{F}) \arrow[r, phantom, "\simeq"] & \mathrm{Hom}_{\mathrm{Frame}}(\mathrm{Syn}(T), \underset{\substack{\| \\ \mathcal{O}(X)}}{F}) \arrow[d] \\
 & \mathrm{Hom}_{\mathrm{Top}}(\mathrm{pt}(F) \underset{\substack{\| \\ X}}{} \underset{\|}{,} \mathrm{pt}(\mathrm{Syn}(T)))
\end{tikzcd}$$

continuous maps

models in $\mathcal{O}(X)$ $\longleftrightarrow$ continuous function $X \longrightarrow \mathrm{pt}(\mathrm{Syn}(T))$

__example__ T = subset of a fix set E.

$\mathrm{Syn}(T)$ = free frame on E = $P(M(E))$

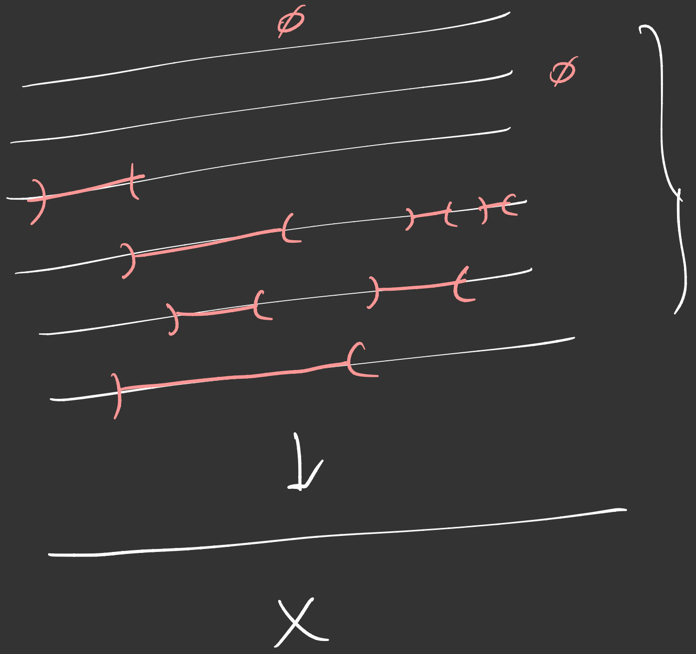
$M(E) = P_{fin}(E)^{op}$

$$\mathrm{Mod}(T, O(X)) = \underset{\mathrm{Frame}}{\mathrm{Hom}}\big(P(M(E)), O(X)\big)$$

$$= \underset{\substack{\mathrm{Poset} \\ (\mathrm{set})}}{\mathrm{Hom}}\big(E, O(X)\big)$$

a __model__ is a map $E \longrightarrow O(X)$

$e \longmapsto U_e \subset X$



$$X \times E = \coprod_{E} X = \text{topological space}$$

Model $E \longrightarrow O(X)$ $\Longleftrightarrow$ a single element in $O(X \times E)$

$\Longleftrightarrow$ open subset of $X \times E$.

if $X = pt$

model = subset of E.