Syntactic ~~categories~~ poset for propositional logic

How to construct it? $T = (R, A)$ a theory

- first consider only the constructors $\top, \wedge$ for axioms.

$\longrightarrow$ meet semi lattice

i) consider the free meet semi lattice on R order reversed!

$\mathrm{MLat}(R) = \{\text{finite subsets of } R\}^{op} \subset P(R)^{op}$

mor. of MSLat: $\mathrm{MLat}(R) \longrightarrow A$ (A other meet semi lattice)

$\{x_1, \dots x_n\} \longmapsto f(x_1) \wedge \dots \wedge f(x_n)$

fct $R \xrightarrow{f} UA$ (UA underlying set of A)

$x \longmapsto f(x)$

ii) use the axioms to build a "quotient" of MLat(R).
in the category of meet semi-lattice.

each axiom $\varphi_1, \dots, \varphi_n \vdash \psi$ define a relation on MLat(R):

each φ is build from element of R by using $\top, \wedge$

$[\varphi] \in \mathrm{MLat}(R)$ $\quad [\top] = \phi$

$[\varphi \wedge \varphi'] = [\varphi] \wedge [\varphi']$

$[\varphi_1, \dots, \varphi_n]$ interpret as $[\varphi_1] \wedge \dots \wedge [\varphi_n]$

in MLat(R) it might not be true that the axiom is valid:

i.e. that $[\varphi_1] \wedge \dots \wedge [\varphi_n] \leq [\psi]$ Can force this relation to be true.

Recall that in a meet semi lattice

$$a \leq b \iff a \wedge b = a$$

The axiom are going to be taken care of by adding equalities

$$[\varphi_1] \wedge \cdots \wedge [\varphi_n] \wedge [\psi] = [\varphi_1] \wedge \cdots \wedge [\varphi_n]$$

i.e by taking a quotient of MLat(R) for the algebraic structure of meet-semi lattice.

$\sim_A$: minimal equivalence relation on $MLat(R)$

- $[\varphi_1] \wedge \dots \wedge [\varphi_n] \wedge [\psi] \sim_A [\varphi_1] \wedge \dots \wedge [\varphi_n]$

  for each axiom $\varphi_1 \dots \varphi_n \vdash \psi$ in A.

- compatible with the $\wedge$ operation

if $a \sim_A b$, $a' \sim_A b'$ then $a \wedge a' \sim_A b \wedge b'$

<u>Define</u> : $Syn(R, A) = MLat(R) / \sim_A$

if the theory $T = (R, A)$ has more constructor

- Dlattice $\top, \wedge, \bot, \vee$ replace $MLat(R)$ by $DLat(R)$ = the free <u>distributive lattice</u> on R

construction of $DLat(R)$.

X poset $\quad Pr(X) = Hom(X^{op}, \{0<1\})$

$X \longrightarrow Pr(X)$ "Yoneda"

$x \longmapsto \hat{x} : y \mapsto [y \leq x]$

$\{$ finite join of $\hat{x}$ $\}$ $=$ $Pr(X)^{fin} \subset Pr(X)$

$DLat(R) = Pr\left(MLat(R)\right)^{fin}$

- Boole $\top, \wedge, \bot, \vee, \neg$

need to replace $\mathrm{MLat}(R)$ by

$BA(R)$ = free Boolean algebra on R

- Heyting $\top, \wedge, \bot, \vee, \Rightarrow$ need $HA(R)$

free Heyting algebra on R.

Then take quotient by axioms for the appropriate algebraic structure.

Remark : for algebraic theories these two step.

- free MLat/DLat/BA/HA gen by R
- quotient by $\sim_A$ gen by axioms

correspond to the construction of

- $T(\Sigma)$ term for the signature
- $T(\Sigma, A) = T(\Sigma)/\sim_A$ quotient by axioms

Universal property of the syntactic poset (MLat/DLat/BA/HA/Frm ...)

Prop The syntactic poset has a model of the Theory.

Proof $r \in R$ is interpreted as the corresponding elt $[r]$ in $Syn(R, A)$

axioms are validated by definition of $\sim_A$.

$\square$

$\mathrm{Syn}(R, A)$ must have a universal property.

Dlat | BA | HA | Frames

Recall the <u>functor of models</u>: (in the case of Mlat)

C is a Mlat. $T = (R, A)$ a theory

$\mathrm{Mod}(T, C) =$ poset of models of T in C.

Define a functor:

$$\begin{array}{ccc} \overset{\text{BA}|\ldots}{\mathrm{MLat}} & \longrightarrow & \mathrm{Poset} \\ C & \longmapsto & \mathrm{Mod}(T, C) \end{array}$$

the category MLat has Hom enriched over poset

BA | HA | ...

$\mathrm{Hom}_{\mathrm{MLat}}(C, D)$ is a poset

$f \leq g$ if $\forall x \in C \quad f(x) \leq g(x)$

compatible with composition

$\mathrm{Hom}(C, D) \times \mathrm{Hom}(D, E) \longrightarrow \mathrm{Hom}(C, E)$

is a monotone fct.

BA | HA | ...

$\longrightarrow$ any X in MLat defines a functor : $\mathrm{MLat} \longrightarrow \mathrm{Poset}$

"representable"

$Y \longmapsto \mathrm{Hom}(X, Y)$

Thm the functor of models

$$\mathrm{Mod} : \overset{\text{BA/HA/}\cdots}{\mathrm{MLat}} \longrightarrow \mathrm{Poset}$$

$$C \longmapsto \mathrm{Mod}(T, C)$$

is <u>representable by $\mathrm{Syn}(T)$</u>.

i.e. $\mathrm{Mod}(T, C) \simeq \mathrm{Hom}_{\mathrm{MLat}}(\mathrm{Syn}(T), C)$

(natural in C)

proof : same as for alg. theories.

Model of T in C

the interpretation of R gives

a function $R \longrightarrow C$

$\Longleftrightarrow$ $\mathrm{MLat}(R) \longrightarrow C$ morphism of MLat

validity of axioms : factorisation by $\mathrm{Syn}(T) = \mathrm{MLat}(R)/\sim_A$

$$\begin{tikzcd}
\mathrm{MLat}(R) \arrow[rr] \arrow[dr] & & C \\
& \mathrm{Syn}(T) \arrow[ur] &
\end{tikzcd}$$

this builds a map

$$\mathrm{Mod}(T, C) \longrightarrow \mathrm{Hom}(\mathrm{Syn}(T), C)$$

Map the other way obtained from canonical model in $\mathrm{Syn}(T)$

then prove that these two maps are inverse (exercise)

$\square$

examples of syntactic posets

- theory of relation between E and F

  no axioms. $R = E \times F$

  $\mathrm{Syn}(T) =$ free algebra on R.

  $= \mathrm{MLat}(R)$

  or $\mathrm{DLat}(R)$

  or $\mathrm{BA}(R)$

  or $\mathrm{HA}(R)$

  or $\mathrm{Frame}(R)$ ...

- theory of functions $E \to F$

$R = E \times F$

axioms : need $\wedge$, $\perp$ and <u>infinite</u> $\vee$

if F is infinite.

need | <u>complete</u> BA.
HA

Frames

$$\mathrm{Syn}(T) = \mathrm{Free}(R)/\sim_A$$

- real numbers example
  - need infinite $\vee$ $\to$ CBA

    CHA

    Frames.

in the context of Frames

$\mathrm{Syn}(T) = \mathcal{O}(\mathbb{R})$ poset of open subsets of $\mathbb{R}$.

(union of open intervals)

Proof: "Stone spaces" by P. Johnstone.