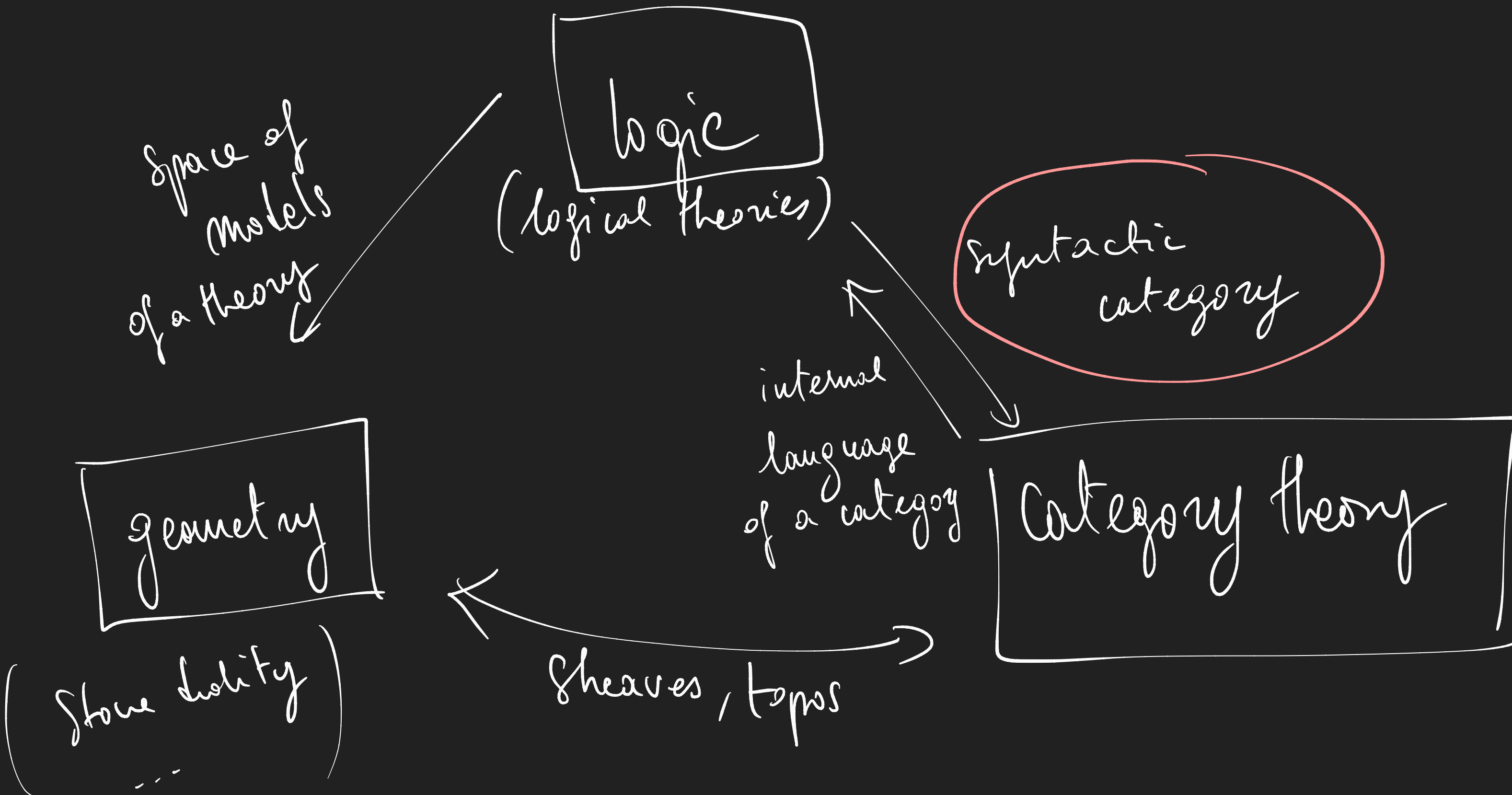
CATEGORICAL LOGIC



Zoology of logical theories

What is a logical theory ?

- <u>sorts</u> / <u>types</u> classifying <u>variables</u> ($x : \mathbb{R}$ $n : \mathbb{N}$)

- <u>term</u> of a given type (can depend on some variable
  (fct symbols)
  <u>ex</u> : a fct $f(x) : \mathbb{R}$ $x : \mathbb{R}$
  $\mathbb{R} \to \mathbb{R}$

- <u>propositions</u> / <u>formula</u> $\varphi(x)$ thing which is either <u>true</u> or <u>false</u>
  (can depend on variables)

- <u>dependent types</u>

— a classification of theories —

	without dependent types or relations	with dependent types: dependent formula only	with dependent types: general dependent types
without variables	×	propositional theories type op. : 1 form op. : $=, \top, \wedge, \bot, \vee, \Rightarrow, \neg$	<u>dependent type theories</u> $=, 1, \times, 0, +, \to, \Sigma, \Pi,$ $\mathbb{N}, \mathcal{U}$
with variables 1st order	algebraic (Lawvere) theories type op. : $1, \times$ form. op : $=, \top$	1st order theories type op. : $1, \times$ form op. : $=, \top, \wedge, \bot, \vee, \Rightarrow, \neg,$ $\exists, \forall$	
with variables higher order	<u>λ-calculus (typed)</u> type op. : $1, \times, \to, \mathbb{N}$ form op : $=, \lambda$	<u>Higher order theories</u> type op. : $1, \times, \to, \Omega, \mathbb{N}$ form op. : $=, \top, \wedge, \bot, \vee, \Rightarrow, \neg,$ $\exists, \forall$	

in this course.

type op. = operators to construct types

form. op. = ——— formulas

— Corresponding categorical semantics —

	without dependent types or relations	with dependent types: dependent formula only	with dependent types: general dependent types
without variables	×	propositional theories Boolean alg, Frames Heyting alg ...	dependent type theories
with variables 1st order	algebraic (Lawvere) theories categories with products or finite limits	1st order theories regular cat. coherent cat, pretopoi, Heyting cat ...	
with variables higher order	λ-calculus (typed) cartesian closed cat	Higher order theories	locally cartesian closed categories, topoi

examples of theories

algebraic theories / equational theories

monoids, groups, rings, ...

1 sort/type M (underlying set)

fct symbol $m(x, y)$ of type M

x, y are variables in M.

$m : M \times M \to M$

unit axiom

1st order axiom $\left(\begin{array}{l} \cancel{\exists e : M} \\ \quad m(x, e) = x \end{array} \right.$

to go around it
introduce constant $e : M$
(fct symbol)
+ eqt $m(x, e) = x$

cdt of associativity : $m(x, m(y, z)) = m(m(x, y), z)$

" equality.

x, y, z variable in M.

- <u>propositional theory</u>

  no variable, no fct symbols (terms)
  (no sorts)

  formula symbols $\varphi, \psi \ldots$

<u>examples</u> E, F are fixed <u>sets</u>

- there is a prop. theory of function $E \to F$
- ________________________ of bijections between $\mathbb{N}$ and $\mathbb{R}$
  (non trivial but has <u>no models</u>)

- 1^{st} order theory

alg theory + extra axiom involving $\exists/\forall$ quantifier or a negation $\neg$

divisible groups, local rings, fields ...

ex monoid with unit

field : ring + axiom $\neg(x=0) \vdash \exists y, xy = 1$

Semantics of theories

logical theory = something akin to a language
(there is a notion of syntax to build the terms, formulas of the theory)

have interpretation (semantic)
= object oriented

- prop. theory : semantic is $\{$true, false$\}$ for any formula
  $\{1, \emptyset\}$
- alg theory : semantic = a set M + fct $M^2 \to M$ satisfying equations
- 1^{st} order theory : semantic = a set X for sorts + subset $Y \subset X$ for formula

all done in the world of sets.

from the point of view of category theory

<u>Set</u> (the cat of sets) is one among many categories.

e.g. $Set/_I$ (families of sets)

Set^C (diagrams of sets)

Top (topological spaces)

Man (manifolds)

$Sh(C, \tau)$ cat of <u>sheaves</u>

⋮

<u>Categorical semantics</u> = replace Set by another category (with the proper structure)

• Alg theory   monoid.

M ~~set~~ object of a category $\mathcal{C}$

$m : M \times M \to M$ multiplication
$\underline{\text{cart product}}$: need $\mathcal{C}$ to have finite product ($\mathcal{C}$ = cartesian category)

+ condition

$m(x, m(y,z)) = m(m(x,y), z) \iff$ commutative square in $\mathcal{C}$.

$$\begin{tikzcd}
M \times M \times M \arrow[r, "m \times 1_M"] \arrow[d, "1_M \times m"'] \arrow[dr, phantom, "="] & M \times M \arrow[d, "m"] \\
M \times M \arrow[r, "m"'] & M
\end{tikzcd}$$

possible to make sense of a <u>model</u> of the <u>algebraic theory</u> of monoids in

<u>any category with cartesian products</u>

theories	categorical semantics are possible in
algebraic	cartesian categories. or categories with finite limits ...
propositional	Boolean algebra (viewed as posets) or Distributive lattices or Frames or Heyting algebras ...
1st order	regular categories or pretopoi or Heyting categories ...