Category Theory

Fall 2020

Homework 6 – Due October 15

Exercises marked 80713 are only for 80713 students.

All answers have to be justified unless otherwise specified.

In all exercises, one can assume a fixed "scale of sizes" $\omega < \alpha < \beta < \gamma$ where ω is the countable ordinal and the other three are inaccessible. An object is call small *it it is α-small. If no adjective is given the object can be assumed to be β-small.*

More limits and colimits

1. (Products as pullbacks) Let C be a category with fiber products and a terminal object, denoted 1. Prove that the fiber product of the cospan $X \to 1 \leftarrow Y$ is the product of X an Y.

2. (Equalizers as pullbacks) In a category with binary products, we can build, for any object B, the *diagonal* of B, which is the map $\Delta = \langle 1_B, 1_B \rangle : B \to B \times B$. Then, given arrows $f, g : A \to B$, we consider the pullback

$$\begin{tikzcd} E \arrow[r] \arrow[d, "e"'] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[d, "\Delta_B"] \\ A \arrow[r, "\langle f,g \rangle"'] & B \times B \end{tikzcd}$$

   (a) Prove that $e : E \to A$ is the equalizer of f and g.

   (b) Using Exercise 1, deduce that a category with fiber products and a terminal object has all equalizers.

3. (Countable sets) Prove that the category $\mathcal{S}\mathrm{et}_{\leq\omega}$ of sets at most countable has products indexed by objects of $\mathcal{S}\mathrm{et}_{<\omega}$ (finite sets) but do not have products indexed by the objects of $\mathcal{S}\mathrm{et}_{\leq\omega}$.

4. (Commutation of colimits) Let I and J be two small categories and $\mathcal{C}$ a category with colimits of shape I and J. We fix a diagram

$$\begin{aligned} X : I \times J &\longrightarrow \mathcal{C} \\ (i,j) &\longmapsto X(i,j). \end{aligned}$$

   (a) Using the universal property of colimits, define diagrams

$$\begin{aligned} X^1 : J &\longrightarrow \mathcal{C} \\ j &\longmapsto \operatorname*{colim}_i X(i,j) \end{aligned} \qquad\qquad \begin{aligned} X^2 : I &\longrightarrow \mathcal{C} \\ i &\longmapsto \operatorname*{colim}_j X(i,j). \end{aligned}$$

(b) If $Z \coloneqq \operatorname{colim}_j X^1(j)$, construct a cocone $X(i,j) \to Z$.

(c) Prove that this cocone $X(i,j) \to Z$ is a colimit cocone.

(d) Prove that $\operatorname{colim}_j X^1(j) = \operatorname{colim}_i X^2(i) = \operatorname{colim}_{i,j} X(i,j)$.

Remark: this last formula is more frequently written

$$\operatorname*{colim}_i \operatorname*{colim}_j X(i,j) = \operatorname*{colim}_j \operatorname*{colim}_i X(i,j).$$

It means that when one can exchange (or commute) the order in which colimits are computing when the indices are independent. This is an analog of the Fubini formula in Calculus. The dual result for limits is also true.

5. (Non-commutation of limits and colimits) Let I and J be two small categories and $\mathcal{C}$ a category with colimits of shape I and limits of shape J. We fix a diagram

$$\begin{aligned} X : I \times J &\longrightarrow \mathcal{C} \\ (i,j) &\longmapsto X(i,j). \end{aligned}$$

   (a) Using the universal properties of limits and colimits, construct a map

$$\operatorname*{colim}_{i\in I} \lim_{j\in J} X(i,j) \to \lim_{j\in J} \operatorname*{colim}_{i\in I} X(i,j)$$

   (b) Find an example of a diagram $X : I \times J \to \mathcal{C}$ for which this map is not an isomorphism (one can assume that I and J are finite sets and $\mathcal{C} = \mathcal{S}\mathrm{et}_{<\omega}$).

6. (80-713 – Functoriality of colimits) Let I be a small category and $\mathcal{C}$ a category with all colimits of shape I. Let $\mathcal{C}^I = Fun(I, \mathcal{C})$ the category of I-diagrams and natural transformations. The purpose of the exercise is to prove that the colimit define a functor $\mathcal{C}^I \to \mathcal{C}$.

   (a) Let $f : X \to Y$ be a natural transformation and $z_i : Y(i) \to Z$ a cocone, prove that $z_i \circ f : X \to Y \to Z$ is a cocone.

   (b) With the same notations, construct a map $\phi : \operatorname{colim}_i X(i) \to \operatorname{colim}_i Y(i)$ such that, for any $u : i \to j \in I$, the following square commutes

$$\begin{tikzcd} X(i) \arrow[r, "inc_i^X"] \arrow[d, "f(i)"'] & \operatorname{colim}_i X(i) \arrow[d, "\phi"] \\ Y(i) \arrow[r, "inc_i^Y"] & \operatorname{colim}_i Y(i) \end{tikzcd}$$

   (where the maps $inc_i^X : X(i) \to \operatorname{colim}_i X(i)$ are the components of the colimit cocone of X).

   (c) Extend the function sending a diagram $X : I \to \mathcal{C}$ to its colimit $\operatorname{colim}_i X(i)$ into a functor $\mathcal{C}^I \to \mathcal{C}$.