## Category Theory Fall 2020 Homework 4 – Due October 1

## Functors

Exercises marked 80713 are only for 80713 students.

All answers have to be justified unless otherwise specified.

- 1. (Initial and terminal objects) Do the following categories have an initial or a terminal objet and identify them (give an argument if the answer is negative):
  - (a) [n]
  - (b)  $P = \{0 \Rightarrow 1\}$
  - (c) the poset of real numbers  $(\mathbb{R}, \leq)$
  - (d) the category  $x \xrightarrow[r]{s} y$  such that  $rs = 1_y$ .
- 2. (Maximal elements as colimits) Recall that an order relation  $\leq$  on a set *P* is *total* if, for any *x* and *y* in *P*, it is true that  $x \leq y$  or  $y \leq x$ . By viewing a total order as a category, show that the maximal element of a finite family  $(x_1, \ldots, x_n)$  satisfies the definition of the colimit of the family.
- 3. (Unicity of the product of a family) Let C be a category and  $(x_i, i \in I)$  a family of object such that the product of the family exists in C. Prove that this product is unique up to unique isomorphism.
- 4. (Product as a functor) Let  $\mathcal{C}$  be a category in which binary products exist for all pairs A, B of objects. Given  $f: A \to C$  and  $g: A \to D$ , recall that we write  $\langle f, g \rangle : A \to C \times D$  for the unique morphism satisfying  $p_1 \circ \langle f, g \rangle = f$  and  $p_2 \circ \langle f, g \rangle = g$ . Given a third morphism  $h: B \to D$ , we also use the notation

$$f \times h = \langle f \circ p_1, h \circ p_2 \rangle : A \times B \to C \times D.$$

(a) Given objects A, B, C and morphisms  $f, g: A \to B \times C$  in  $\mathcal{C}$ , show that we have

$$f = g \iff p_1 \circ f = p_1 \circ g \text{ and } p_2 \circ f = p_2 \circ g.$$

This is a very handy proof principle that you can use in the following.

- (b) Show that we have
- $\langle g,h\rangle \circ f = \langle g\circ f,h\circ f\rangle$
- for  $f: A \to B, g: B \to C, h: B \to D$ .

(c) Show that we have

$$(h \times k) \circ \langle f, g \rangle = \langle h \circ f, k \circ g \rangle$$

- for  $f: A \to B, g: A \to C, h: B \to D, k: C \to E$ .
- (d) Show that we have

$$(h \times k) \circ (f \times g) = (h \circ f) \times (k \circ g)$$

for arrows  $A \xrightarrow{f} C \xrightarrow{h} E$  and  $B \xrightarrow{g} D \xrightarrow{k} F$ .

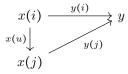
(e) Show that we have  $1_A \times 1_B = 1_{A \times B}$  for all objects A, B in  $\mathcal{C}$ .

The last two items complete the proof that the assignments

$$(A, B) \mapsto A \times B$$
 and  $(f, g) \mapsto f \times g$ 

constitute a functor  $(-\times -)$ :  $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ .

5. (Bijection formulation of the universal property of colimits) Let  $x : I \to C$  be a diagram. Recall that a cocone on x is an object y, called the apex of the cocone, in C together with maps  $y(i) : x(i) \to y$  such that, for any map  $u : i \to j$  in I, the following triangle commute



Let Cocone(x, y) be the set of all cocones on x with apex y. Given a cocone  $(y(i): x(i) \rightarrow y)$  on x, we can define a function

$$Hom_C(y, z) \longrightarrow Cocone(x, z)$$
$$f: y \to z \longmapsto (z(i): x(i) \xrightarrow{y(i)} y \xrightarrow{f} z)$$

Using the universal property of colimits, prove that  $(y(i): x(i) \rightarrow y)$  is a colimit cocone if and only if the previous map is a bijection.

6. (80713 – Sums of monoids) Let M and N two monoids, the purpose of the exercise is to define the sum M + N in the category of monoids.

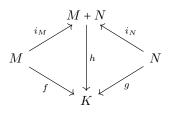
For a monoid M, let UM denote its underlying set, and, for a set E, let List(E) denote the free monoid on E. The underlying set of M + N is the quotient of List(UM + UN) by the relations

- $(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = (x_1, \ldots, (x_i x_{i+1}), \ldots, x_n)$  whenever  $x_i$  and  $x_{i+1}$  are both in M or both in N (and where the  $x_i x_{i+1}$  is their product in M or N).
- $(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$  where 1 is the unit of M or N.

The unit of M + N is given by the empty list and the multiplication is induced by the concatenation of lists.

(a) Prove that the inclusion  $i_M : M \to M + N$  sending an element x of M to the list with one element (x) is a morphism of monoids.

(b) Using the universal property of the free monoid List(UM + UN), prove that if  $M \xrightarrow{f} K \xleftarrow{g} N$  are two morphisms of monoids, there exists a unique morphism of monoids  $M + N \xrightarrow{h} K$  such that the following diagram commutes



(c) Now, compute the sums of two monoids M and N viewed as categories (draw the matrix representation). Is it again a category associated to a monoid and why?

Morale: The sums of objects depends in which category they are considered.