Category Theory

Fall 2020

Homework 4 – Due October 1

Functors

Exercises marked 80713 are only for 80713 students.

All answers have to be justified unless otherwise specified.

1. (Initial and terminal objects) Do the following categories have an initial or a terminal objet and identify them (give an argument if the answer is negative):
   (a) $[n]$
   (b) $P = \{0 \rightrightarrows 1\}$
   (c) the poset of real numbers $(\mathbb{R}, \leq)$
   (d) the category $x \underset{r}{\overset{s}{\rightleftarrows}} y$ such that $rs = 1_y$.
2. (Maximal elements as colimits) Recall that an order relation $\leq$ on a set P is *total* if, for any x and y in P, it is true that $x \leq y$ or $y \leq x$. By viewing a total order as a category, show that the maximal element of a finite family $(x_1, \ldots, x_n)$ satisfies the definition of the colimit of the family.
3. (Unicity of the product of a family) Let C be a category and $(x_i, i \in I)$ a family of object such that the product of the family exists in C. Prove that this product is unique up to unique isomorphism.
4. (Product as a functor) Let $\mathcal{C}$ be a category in which binary products exist for all pairs A, B of objects. Given $f : A \to C$ and $g : A \to D$, recall that we write $\langle f, g \rangle : A \to C \times D$ for the unique morphism satisfying $p_1 \circ \langle f, g \rangle = f$ and $p_2 \circ \langle f, g \rangle = g$. Given a third morphism $h : B \to D$, we also use the notation

$$f \times h \ = \ \langle f \circ p_1, h \circ p_2 \rangle : A \times B \to C \times D.$$

   (a) Given objects A, B, C and morphisms $f, g : A \to B \times C$ in $\mathcal{C}$, show that we have

$$f = g \quad \Leftrightarrow \quad p_1 \circ f = p_1 \circ g \text{ and } p_2 \circ f = p_2 \circ g.$$

   This is a very handy proof principle that you can use in the following.

   (b) Show that we have

$$\langle g, h \rangle \circ f = \langle g \circ f, h \circ f \rangle$$

   for $f : A \to B$, $g : B \to C$, $h : B \to D$.

(c) Show that we have

$$(h \times k) \circ \langle f, g \rangle = \langle h \circ f, k \circ g \rangle$$

for $f : A \to B$, $g : A \to C$, $h : B \to D$, $k : C \to E$.

(d) Show that we have

$$(h \times k) \circ (f \times g) = (h \circ f) \times (k \circ g)$$

for arrows $A \xrightarrow{f} C \xrightarrow{h} E$ and $B \xrightarrow{g} D \xrightarrow{k} F$.

(e) Show that we have $1_A \times 1_B = 1_{A \times B}$ for all objects A, B in $\mathcal{C}$.

The last two items complete the proof that the assignments

$$(A, B) \mapsto A \times B \qquad \text{and} \qquad (f, g) \mapsto f \times g$$

constitute a functor $(- \times -) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

5. (Bijection formulation of the universal property of colimits) Let $x : I \to C$ be a diagram. Recall that a cocone on x is an object y, called the apex of the cocone, in C together with maps $y(i) : x(i) \to y$ such that, for any map $u : i \to j$ in I, the following triangle commute

$$\begin{tikzcd} x(i) \arrow[r, "y(i)"] \arrow[d, "x(u)"'] & y \\ x(j) \arrow[ur, "y(j)"'] & \end{tikzcd}$$

Let $Cocone(x, y)$ be the set of all cocones on x with apex y. Given a cocone $(y(i) : x(i) \to y)$ on x, we can define a function

$$\begin{aligned} Hom_C(y, z) &\longrightarrow Cocone(x, z) \\ f : y \to z &\longmapsto (z(i) : x(i) \xrightarrow{y(i)} y \xrightarrow{f} z) \end{aligned}$$

Using the universal property of colimits, prove that $(y(i) : x(i) \to y)$ is a colimit cocone if and only if the previous map is a bijection.

6. (80713 – Sums of monoids) Let M and N two monoids, the purpose of the exercise is to define the sum $M + N$ in the category of monoids.

For a monoid M, let UM denote its underlying set, and, for a set E, let $List(E)$ denote the free monoid on E. The underlying set of $M + N$ is the quotient of $List(UM + UN)$ by the relations

- $(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = (x_1, \ldots, (x_i x_{i+1}), \ldots, x_n)$ whenever x_i and x_{i+1} are both in M or both in N (and where the $x_i x_{i+1}$ is their product in M or N).
- $(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ where 1 is the unit of M or N.

The unit of $M + N$ is given by the empty list and the multiplication is induced by the concatenation of lists.

(a) Prove that the inclusion $i_M : M \to M + N$ sending an element x of M to the list with one element (x) is a morphism of monoids.

(b) Using the universal property of the free monoid $List(UM + UN)$, prove that if $M \xrightarrow{f} K \xleftarrow{g} N$ are two morphisms of monoids, there exists a unique morphism of monoids $M + N \xrightarrow{h} K$ such that the following diagram commutes

$$\begin{tikzcd}
& M + N \arrow[dd, "h"] & \\
M \arrow[ur, "i_M"] \arrow[dr, "f"'] & & N \arrow[ul, "i_N"'] \arrow[dl, "g"] \\
& K &
\end{tikzcd}$$

(c) Now, compute the sums of two monoids M and N viewed as categories (draw the matrix representation). Is it again a category associated to a monoid and why?

Morale: The sums of objects depends in which category they are considered.