Category Theory

Fall 2020

Homework 1 – Due September 10

Categories or not

Exercises marked 80713 are only for 80713 students.

1. (An example of a non-category) A function $f : A \to B$ between to sets is called *n-bounded* if any element b in B has at most n preimages by f (that is the set $\{a|f(a) = b\}$ has cardinal less than n). Prove that, if $n \geq 2$, the collection of sets and n-bounded functions do not form a category for the composition of functions (idea: find two n-bounded functions whose composition is not n-bounded).

2. (Isomorphisms in a category) In a category C, a map $f : x \to y$ is an *isomorphism* if there exists a map $g : y \to x$ such that $gf = 1_x$ and $fg = 1_y$. The map g is called an *inverse map* of f.

   (a) Prove that the map g, if it exists is unique (suppose there are two of them and show they must be equal).

   (b) Prove that if g is an inverse map of f then f is an inverse map of g.

   (c) Prove that the function $inv_{x,y} : Iso(x, y) \to Iso(y, x)$ sending an isomorphism f to its inverse is a bijection (prove that $inv_{y,x}$ is the inverse function of $inv_{x,y}$).

   As a consequence of this bijection, the matrix representation of a groupoid is always symmetric.

3. (Pre-orders are categories) Recall that a preorder on a set A is a relation $\leq$ which is

   - reflective ($a \leq a$, for all a in A)
   - and transitive (if $a \leq b$ and $b \leq c$, then $a \leq c$)

   It is possible to define a category C from a preorder in the following way: the objects are the elements of A and for any two objects

   $$Hom(x, y) = \begin{cases} \varnothing & \text{if } x \leq y \text{ is false} \\ \{*\} & \text{if } x \leq y \text{ is true} \end{cases}$$

   The reflectivity gives $Hom(x, x) = \{*\}$ and the unique element it the identity arrow of x. The transitivity gives composition maps

   $$Hom(x, y) \times Hom(y, z) \to Hom(x, z).$$

   (a) Describe the isomorphisms in C.

   (b) Recall that an *equivalence relation* is a preorder which is *symmetric* (if $a \leq b$ then $b \leq a$). Prove that the corresponding category is a groupoid.

4. ("3 for 2" property of isomorphisms) In a category C, consider three functions f, g and h such that $h = gf$

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "h=gf"'] & & C \end{tikzcd}$$

Show that if any two of these three arrows are isomorphisms, then the third must be as well (make three cases).

5. (80713 – Monoid actions) Let M be a monoid, E a set and $End(E)$ the monoid of functions $E \to E$. An *action of the monoid M on E* is a morphism of monoids $\mu : M \to End(E)$.

   Given such an action, we define, for any x and y in E, the set

$$Hom(x, y) = \{m \in M \mid \mu(m)(x) = y\};$$

   We use this to define a category:

   (a) Define identity maps for each x in E.
   (b) Define a composition $Hom(x, y) \times Hom(y, z) \to Hom(x, z)$ involving the composition of the monoid.
   (c) Prove the associativity of the composition (using the monoid axioms).
   (d) Prove the identity relations (using the monoid axioms).
   (e) Describe the isomorphisms of this category.