Many categories comes with distinguished structures
akin to algebraic operations (like $+$ $\times$ $\ldots$)
on $\mathbb{N}$

ex Set
- sum of 2 set $A+B$, empty set $\emptyset$ $\emptyset + A = A$
- product of 2 sets $A \times B$, singletons $\{*\}$ $\{*\} \times B = B$

other categories have also these structures
(Monoids, Cat ...)

initial objects

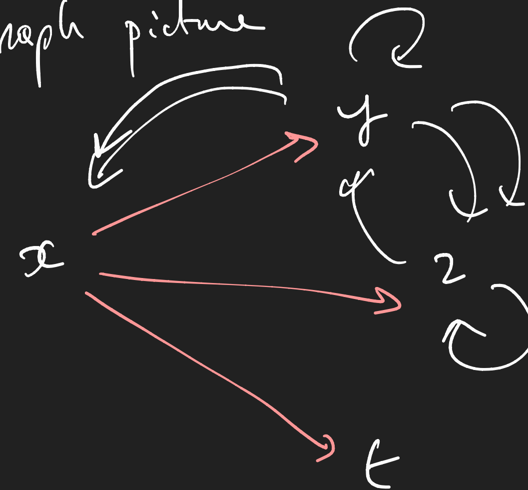
in C cat. an object x is <u>initial</u>

if for any other object y in C

$\mathrm{Hom}_C(x, y)$ is a <u>singleton</u>

(there is one morphism exactly from x to y)

graph picture



matrix picture

	y	x	t
y			
x	$\{*\}$	$\{*\}$	$\{*\}$ ....
z			

only singletons in row x

examples

- Set $\emptyset$ the empty set is initial

$$\mathrm{Hom}(\emptyset, E) = \{*\} \quad \forall E.$$

- Poset $\emptyset$ the empty poset is initial
- Cat $\emptyset$ the empty category is initial
- Mon (the cat of monoid)

$\{1\}$ monoid with one element

is initial.

$$\{1\} \xrightarrow{\text{monoid morphism.}} M$$

image of 1 has to be the unit of

$\mathbb{N}$ = free monoid on one generator.

$\{1\}$ = free monoid on zero generators.

- $\mathcal{C}$ is a poset.

  an initial object corresponds to a <u>minimal element</u> for the order.

- $P(E) = \{\text{subset of } E\}$

  minimal elt is $\emptyset \subset E$

  is an initial object in $(P(E), \leq)$ viewed as a category.

- $(\mathbb{N}, \leq)$ $\quad 0$ is <u>initial</u>

- $(\mathbb{Z}, \leq)$ there is no minimal elt / initial object

<u>Rem</u> not every category admits an initial object.

E set viewed as a cat. has an initial object iff $E = \{*\}$ is a singleton.

Proposition initial objects are unique up to unique isomorphism.

(any two initial objects are canonically isomorphic)

proof: suppose x initial in C

$\mathrm{Hom}(x,x) = \{id_x\}$ is a singleton.

suppose x, x' two initial objects in C

$\mathrm{Hom}(x,x') = \{f\}$ $x \xrightarrow{f} x'$

$\mathrm{Hom}(x',x) = \{g\}$ $x' \xrightarrow{g} x$

the composition $gf : x \to x$ is id_x

similarly $fg = id_{x'}$

so f and g are inverse to each other.

so x and x' are isomorphic.

and this isomorphism between them is unique.

$\square$

ex all empty sets are isomorphic.

$\exists!$ bij. between empty sets

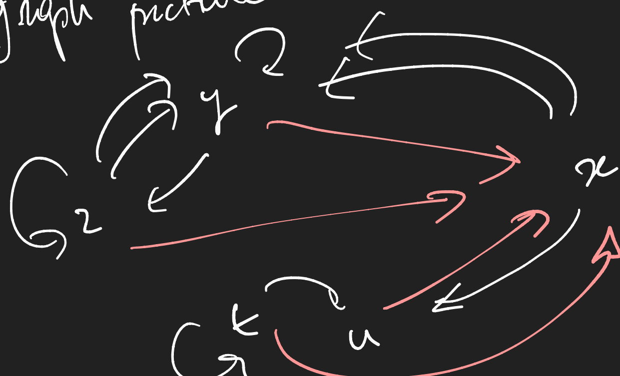
terminal object

in $\mathcal{C}$ cat an object x is *terminal* (or *final*)

if for any y in $\mathcal{C}$

$\mathrm{Hom}_{\mathcal{C}}(y, x)$ is a *singleton*

graph picture



matrix picture

$\nearrow$	y	x	t
y			
x	$\mathrm{Hom}(x,y)$		
t			

Column of singletons.

Recall C^{op} the opposite of C

<u>proposition</u> if x is initial (or terminal) in C, then x^{op} is terminal (or initial) in C.

the two definition of initial/terminal object. are "dual" to each (exchanged by passing to the opposite cat)

<u>examples</u>

- Set any <u>singleton</u> is terminal.
- Poset $\{1\}$ poset with one elt is terminal.
- Cat the <u>punctual</u> cat is terminal.
- Monoids. Groups $\{1\}$ monoid group with single elt

is also <u>terminal</u>

- in a poset P (viewed as a cat.) an object is terminal iff it is a <u>maximal element</u>

• $\mathcal{P}(E) = \{\text{subsets of } E\}$

E is a terminal object

• $(\mathbb{N}, \leq)$ no maximal elt.

no terminal object.

a category $\mathcal{C}$ is called <u>pointed</u>

if it has an initial object 0, a terminal object 1, such that

$$0 \xrightarrow{\sim} 1$$

is an isomorphism.

ex Groups, Monoids

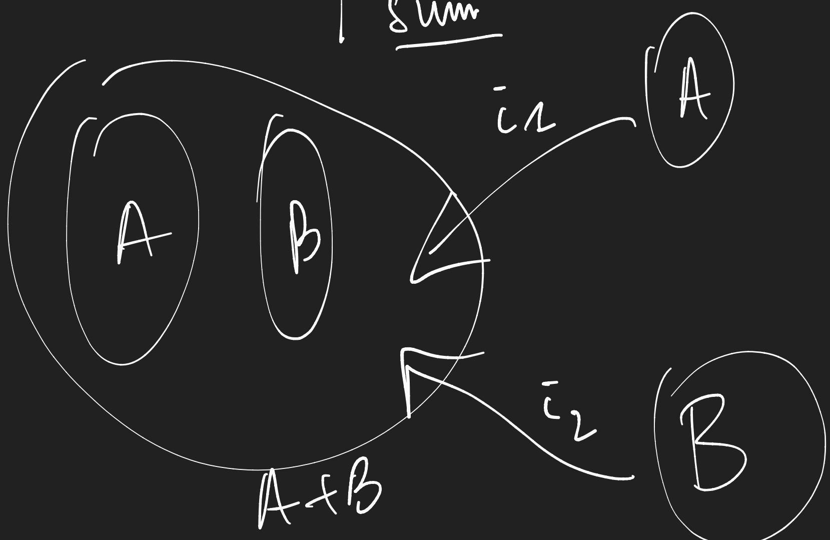
Category with sums

in Set $\ni A, B$

distinguished set $A+B$ ($A \amalg B$)

called the disjoint union of A and B

sum



$A+B$ comes equipped with distinguished inclusions

$$A \xrightarrow{i_1} A+B \xleftarrow{i_2} B$$

$$
\begin{tikzcd}
A \arrow[d, "i_1"'] \arrow[drr, bend left, "j_1"] \arrow[dr, phantom, "1"] & & \\
A+B \arrow[rr, "\exists!\, h"] & & X \\
B \arrow[u, "i_2"] \arrow[urr, bend right, "j_2"'] \arrow[ur, phantom, "2"] & &
\end{tikzcd}
$$

there exists a unique map $A+B \xrightarrow{h} X$ such that

$$h \circ i_1 = j_1$$

$$h \circ i_2 = j_2$$

universal property of $A+B$

the triangle 1 commutes

the triangle 2 commutes.

the whole diagram commutes.

in a category C, given two objects X, Y

the <u>sum of X and Y</u> / <u>coproduct</u> is an object Z

together with maps $X \xrightarrow{i_1} Z \xleftarrow{i_2} Y$

such that, for any other diagram $X \xrightarrow{j_1} T \xleftarrow{j_2} Y$,

there exists a unique map $Z \xrightarrow{h} T$ such that the diagram

$$\begin{tikzcd}
 & X \arrow[dl, "i_1"'] \arrow[dr, "j_1"] & \\
Z \arrow[rr, "h"] & & T \\
 & Y \arrow[ul, "i_2"] \arrow[ur, "j_2"'] &
\end{tikzcd}$$

commutes.

examples

- in Set, get back the usual notion of
  - disjoint union
  - sum
  - coproducts

- in Poset, get the usual sum

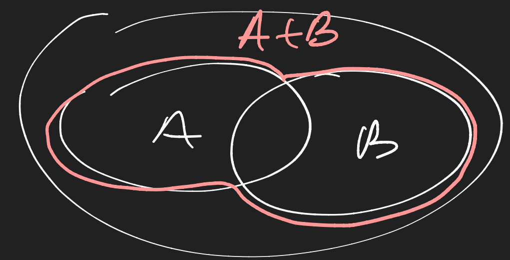
- in Cat, get the sum

- in a poset P viewed as a category this is the notion of join of two elements.

- $\mathcal{P}(E)$

$A \subset E \qquad B \subset E$

$A + B$ is $A \cup B$ the union of A and B in E



- in $(\mathbb{N}, \leq)$

the sum of two element is the max (and not the arithmetic sums)

- Monoids have sums.

  this is the notion of "free product"

  of two monoids.

$\mathrm{List}(E)$ = free monoid on the set E

$$\underbrace{\mathrm{List}(E) + \mathrm{List}(E')}_{\text{sum in the cat. of monoids.}} = \mathrm{List}\,(\underbrace{E + E'}_{\text{sum in the cat of Sets.}})$$

(exo: try to prove this using the univ. prop. of the free monoid)

- E a set viewed as a cat. does not have all sums.

  if $x \neq y$

  $x + y$ does not exist

  but

  $x + x = x$

  (exercise)

$\mathcal{C}$ cat. X, Y objects

__proposition__ if the sum of X and Y exists it is unique up to unique isomorphism.

__proof__ let Z be a sum for X and Y.

$$X \xrightarrow{i_1} Z \xleftarrow{i_2} Y.$$

there is a unique morphism $Z \xrightarrow{1_Z} Z$

s.t.

$$\begin{tikzcd} & X \arrow[dl, "i_1"'] \arrow[dr, "i_1"] & \\ Z \arrow[rr, "1_Z"] & & Z \\ & Y \arrow[ul, "i_2"] \arrow[ur, "i_2"'] & \end{tikzcd}$$

commutes.

has to be the identity of Z!

assume $X \xrightarrow{i_1} Z \xleftarrow{i_2} Y$

and $X \xrightarrow{j_1} Z' \xleftarrow{j_2} Y$

are two sums.

by prop. of Z, there exists a unique map $Z \xrightarrow{f} Z'$

s.t.

$$\begin{tikzcd}
 & X \arrow[dl, "i_1"'] \arrow[dr, "j_1"] & \\
Z \arrow[rr, "f"] & & Z' \\
 & Y \arrow[ul, "i_2"] \arrow[ur, "j_2"'] &
\end{tikzcd}$$

commutes.

by prop of Z', there exists a unique map $Z' \xrightarrow{g} Z$ s.t.

$$\begin{tikzcd}
 & X \arrow[dl, "j_1"'] \arrow[dr, "i_1"] & \\
Z' \arrow[rr, "g"] & & Z \\
 & Y \arrow[ul, "j_2"] \arrow[ur, "i_2"'] &
\end{tikzcd}$$

commute.

hence $fg = id_{Z'}$, $gf = id_Z$

so f, g are inverse of eachother

and Z and Z' are uniquely isomorphic.

It is possible to view <u>sums</u> as <u>initial objects</u>

define a category <u>cocones(X,Y)</u> $\quad$ (X, Y fixed in $\mathcal{C}$)

<u>objects</u> : diagrams $X \xrightarrow{j_1} T \xleftarrow{j_2} Y$

a morphism from $X \xrightarrow{j_1} T \xleftarrow{j_2} Y$ to $X \xrightarrow{l_1} T' \xleftarrow{l_2} Y$

is a diagram

$$
\begin{tikzcd}
& X \arrow[dl, "j_1"'] \arrow[dr, "l_1"] & \\
T \arrow[rr, "f"] & & T' \\
& Y \arrow[ul, "j_2"] \arrow[ur, "l_2"'] &
\end{tikzcd}
$$

(equivalently a morphism is a map $T \xrightarrow{f} T'$ s.t the diagram commute) composition

$$\begin{tikzcd}
& X \arrow[dl] \arrow[d] \arrow[dr] & \\
T \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & T' \arrow[r, "g"] & T'' \\
& Y \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

ex check this is a cat.

proposition

an initial object in cocone(X,Y)

is the same things as

a sum for X and Y.

proof (just unravels the definition of an initial object)

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[dr, bend left] & \\
Z \arrow[rr, "\exists!"] & & T \\
 & X \arrow[ul] \arrow[ur, bend right] &
\end{tikzcd}$$

initial in

cocone(X, Y)