Where are we?

cat theory is a <u>synthesis</u> for many structures. ✓

<u>clarification</u> tool ✓

<u>language</u> ← here

<u>calculus</u> (later)

the "Hom" functor (the most important functor of all)

$\mathcal{C}$ a category $\quad x \in ob(\mathcal{C})$ an object

$$\mathrm{Hom}_{\mathcal{C}}(x, -) : \mathcal{C} \longrightarrow \mathrm{Set}$$

$$\begin{tikzcd}
y \arrow[r, mapsto] \arrow[d, "f"'] & \mathrm{Hom}_{\mathcal{C}}(x, y) \arrow[d, dashed, "{\mathrm{Hom}(x, f)}"] \\
z \arrow[r, mapsto] & \mathrm{Hom}_{\mathcal{C}}(x, z)
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, "g"] & y \arrow[d, dashed, "f"] \\
x \arrow[r, "fg"'] & z
\end{tikzcd}$$

(ex: check this is a functor)

<u>example</u> : $\mathcal{C} = Set$ $\qquad E$ a set

$$\mathrm{Hom}(E,-) : Set \longrightarrow Set$$
$$A \longmapsto \mathrm{Hom}(E,A) = A^E$$

if $E = \{0,1\}$ has two elt. $\quad \mathrm{Hom}(E,A) = A^2$

The "other" Hom functor

$\mathcal{C}$ cat $\qquad x \in ob(\mathcal{C})$

$$\mathrm{Hom}(-, x) : \mathcal{C} \longrightarrow \mathrm{Set}$$

$$\begin{tikzcd}
y \arrow[r, mapsto] \arrow[d, "f"'] & \mathrm{Hom}(y, x) \arrow[d, phantom, "\times"] \\
z \arrow[r, mapsto] & \mathrm{Hom}(z, x) \arrow[u, bend right, "\mathrm{Hom}(f, x)"']
\end{tikzcd}$$

going the opposite way of f

$$\begin{tikzcd}
y \arrow[r, "g"] \arrow[d, "f"'] & x \\
z \arrow[r, "h"'] & x
\end{tikzcd}$$

strictly speaking this is <u>not a functor</u>

even if send identities to identities and respect composition.

$$
\begin{tikzcd}
\mathrm{Hom}(x,y) \arrow[d, "f\circ -"] \\
\mathrm{Hom}(x,z)
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r] \arrow[dr, bend right] & y \arrow[r] \arrow[d, "f"] & x \\
& z \arrow[ur, "g"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
gf & \mathrm{Hom}(y,x) \\
g \arrow[u, mapsto] & \mathrm{Hom}(z,x) \arrow[u, "-\circ f"']
\end{tikzcd}
$$

This motivates the definition of <u>contravariant functors</u>

$f : C \xrightarrow[\text{contra}]{} D$ is the data

$f_0 : ob(C) \longrightarrow ob(D)$

$f_1 : Arr(C) \longrightarrow Arr(D)$

such that for any $u : x \to y$ in C

$dom(f_1(u)) = f_0(cod(u))$

$cod(f_1(u)) = f_0(dom(u))$

+ sends id. to id.

+ respects compositions

$$\begin{tikzcd} C \arrow[r, "\text{contra}"'] & D \\ x \arrow[d, "u"'] \arrow[r, mapsto] & f_0(x) \\ y \arrow[r, mapsto] & f_0(y) \arrow[u, "f_1(u)"'] \end{tikzcd}$$

in CT. there is two kinds of functors

the usual ones (called "covariant")

and the contravariant ones

not so convenient to have two kinds of functors

There's a trick to describe a contravariant functor

by means of a covariant one :

use the opposite category

The opposite of a category

C is a cat. its opposite is denoted C^{op} is the cat.

- $ob(C^{op}) = ob(C)$ (same objects)

$x^{op} \longleftrightarrow x$

the opposite of an object x is denoted x^{op}

- $\mathrm{Hom}_{C^{op}}(x^{op}, y^{op}) = \mathrm{Hom}_C(y, x)$

$f^{op} \longleftrightarrow f$

(in other terms $x^{op} \xrightarrow{f^{op}} y^{op}$ is the same thing as $y \xrightarrow{f} x$

- identity arrows: $1_{x^{op}} = (1_x)^{op}$

- composition:

$$\begin{tikzcd} x^{op} \arrow[r, "f^{op}"] \arrow[rr, bend right, "g^{op} \circ f^{op}"'] & y^{op} \arrow[r, "g^{op}"] & z^{op} \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} x & y \arrow[l, "f"'] & z \arrow[l, "g"'] \arrow[ll, bend left, "fg"] \end{tikzcd}$$

$$g^{op} \circ f^{op} = (fg)^{op}$$

proposition

a contravariant functor $C \xrightarrow[\text{contra}]{} D$ is the same thing as a covariant functor $C^{op} \longrightarrow D$

ex : ~~$\mathrm{Hom}_C(-,x) : C \xrightarrow[\text{contra}]{} \mathrm{Set}$~~

$\mathrm{Hom}_C(-,x) : C^{op} \longrightarrow \mathrm{Set}$ (the most important contravariant functor) example $\quad \mathcal{C} = \mathrm{Set}$

$$\mathrm{Set}^{op} \longrightarrow \mathrm{Set}$$

$$E \longmapsto P(E) = \text{set of subsets of } E$$

$$= \mathrm{Hom}_{\mathrm{Set}}(E, \{0,1\})$$

= example of a contravariant Hom functor.

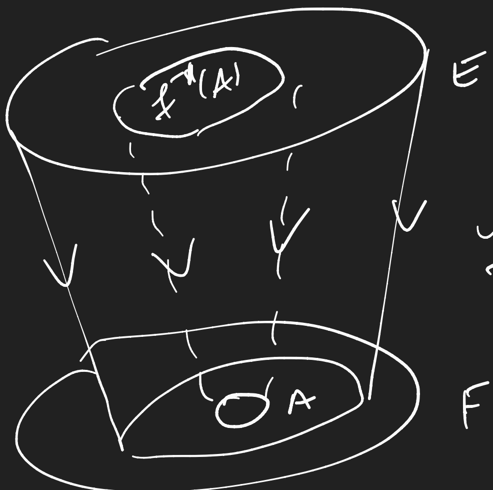
bijection $\quad P(E) \xrightarrow{\sim} \mathrm{Hom}(E, \{0,1\})$

$A \subset E \mapsto$ characteristic function

$E \to \{0,1\}$

$x \mapsto \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if not} \end{cases}$

Subsets and functions



$$\begin{tikzcd} E \arrow[d, "f"] \\ F \supset A \end{tikzcd} \qquad f^{-1}(A) = \{x \in E \mid f(x) \in A\}$$

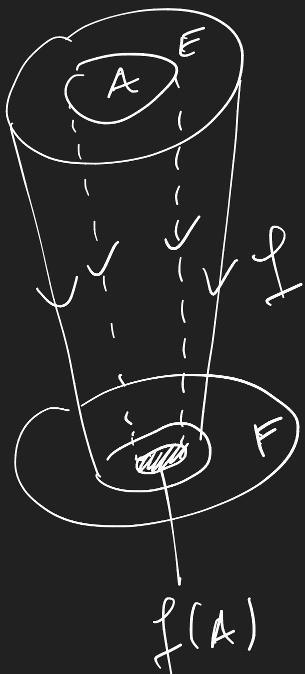
$$\begin{tikzcd} P(E) & f^{-1}(A) \\ P(F) \arrow[u] & A \arrow[u, mapsto] \end{tikzcd}$$

<u>exercise</u>: check it.

char. fct of $f^{-1}(A)$ = composition of char fct of A with f.

$$\begin{tikzcd} E \arrow[r, "\chi_{f^{-1}(A)}"] \arrow[d, "f"'] & \{0,1\} \\ F \arrow[ur, bend right, "\chi_A"'] & \end{tikzcd}$$

<u>Remark</u> : there is another way to transport subsets along functions $f : E \to F$: The <u>image</u> of a subset



$$\begin{array}{ccccc} E & \supset & A & \qquad & P(E) \\ \downarrow & & & & \downarrow \\ F & \supset & \tilde{f}(A) & & P(F) \\ & & \shortparallel & & \end{array}$$

$$\{ y \in F \mid \exists \underline{x \in A} \text{ st. } f(x) = y \}$$

<u>image</u> of A by f

the association of $P(E)$ to E defines <u>two</u> functors

a contravariant one

$$\mathrm{Set}^{op} \longrightarrow \mathrm{Set}$$

$$\begin{tikzcd} E \arrow[r, mapsto] \arrow[d, "f"'] & P(E) \\ F \arrow[r, mapsto] & P(F) \arrow[u, "f^{-1}"'] \end{tikzcd}$$

inverse image

a covariant one

$$\mathrm{Set} \longrightarrow \mathrm{Set}$$

$$\begin{tikzcd} E \arrow[r, mapsto] \arrow[d, "f"'] & P(E) \arrow[d, "\tilde{f}"] \\ F \arrow[r, mapsto] & P(F) \end{tikzcd}$$

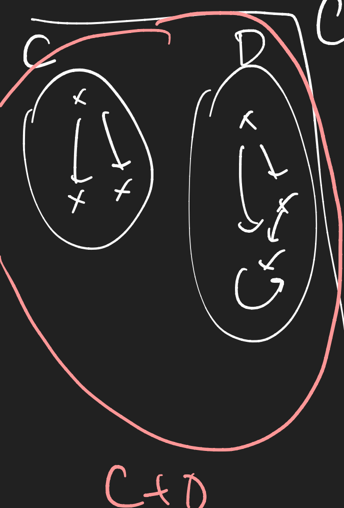
direct image

a functor is not determined by what it does on objects only
(these two functors have the same action on objets)
but are very different.

The opposition is a way to produce a cat. from a cat.

other ways to produce a cat from some categories:

- the sum of categories. $\mathcal{C}$, D are 2 cat.



$C+D$ $(C \amalg D)$ is the cat. s.t.

$$ob(C+D) = ob(C) + ob(D)$$

$$Arr(C+D) = Arr(C) + Arr(D)$$

(no arrow from an object of C to an object of D and vice-versa)

id, compo. are defined by those of C and D

identity arrows

if $x \in C$

use 1_x in C

if $x \in D$

use $1_x \in D$.

- the product of categories $\mathcal{C}, \mathcal{D}$ 2 cat

$\mathcal{C} \times \mathcal{D}$ (the product of $\mathcal{C}$ and $\mathcal{D}$) is the cat:

- $ob(\mathcal{C} \times \mathcal{D}) = ob(\mathcal{C}) \times ob(\mathcal{D})$

  $=$ pairs (x, y) $\quad x \in ob(\mathcal{C})$, $y \in ob(\mathcal{D})$

- $Arr(\mathcal{C} \times \mathcal{D}) = Arr(\mathcal{C}) \times Arr(\mathcal{D})$

$$(x, y) \longrightarrow (x', y')$$

$$\Big(\underset{\substack{\text{in} \\ \mathcal{C}}}{x \xrightarrow{f} x'} \;,\; \underset{\substack{\text{in} \\ \mathcal{D}}}{y \xrightarrow{g} y'} \Big)$$

- identity of (x, y)

  is $(1_x, 1_y)$

- composition as expected..

example

$$[1] = \{ 0 \xrightarrow{u} 1 \}$$

$$ob([1]) = \{0, 1\} \qquad Arr([1]) = \{ \overset{0}{1_0}, \overset{1}{1_1}, u \}$$

$$[1] \times [1] \qquad ob = \{0,1\} \times \{0,1\} = \{(0,0), (0,1), (1,0), (1,1)\}$$

Arr

	00	01	10	11
00	$(1_0, 1_0)$	$(0, u)$	$(u, 0)$	(u, u)
01	$\emptyset$	$(0, 1)$	$\emptyset$	$(u, 1)$
10	$\emptyset$	$\emptyset$	$(1, 0)$	$(1, u)$
11	$\emptyset$	$\emptyset$	$\emptyset$	$(1, 1)$

$$\begin{tikzcd}
00 \arrow[r, "{(0,u)}"] \arrow[d, "{(u,0)}"'] \arrow[dr, "{(u,u)}"] & 01 \arrow[d, "{(u,1)}"] \\
10 \arrow[r, "{(1,u)}"'] & 11
\end{tikzcd}$$

this is the commutative square category!

<u>example</u> $1 = \{*\}$ the punctual category

C is any cat. $1 \times C = C$

(isomorphic to C.)

(for the <u>sum</u> $\emptyset$ = the empty category

$\emptyset + C = C$

(isomorphic categories))

we saw two Hom functors $\qquad \mathcal{C}$ cat, $x \in \mathcal{C}$

$y \in \mathcal{C}$

$$\mathrm{Hom}_{\mathcal{C}}(x, -) : \mathcal{C} \longrightarrow \mathrm{Set}$$

$$\mathrm{Hom}_{\mathcal{C}}(-, y) : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$$

can be put together in a single functor :

$$\mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathrm{Set}$$

$$(x^{op}, y) \longmapsto \mathrm{Hom}_{\mathcal{C}}(x, y)$$

if $\mathcal{C}_1, \mathcal{C}_2, \mathcal{D}$ are categories.

a functor of two variables is a functor

$$\mathcal{C}_1 \times \mathcal{C}_2 \longrightarrow \mathcal{D}$$

ex $\mathrm{Hom}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathrm{Set}$

is a functor of 2 variable. which is contravariant in the first variable and covariant in the second var.