

Presheaves (C small cat)

a presheaf $F: C^{op} \rightarrow \text{Set}$

is representable if it is isomorphic

to a presheaf $\hat{x} = \text{Hom}_C(-, x): C^{op} \rightarrow \text{Set}$

$$\hat{x} \simeq F$$

By Yoneda lemma this iso $\hat{x} \simeq F$

corresponds to a unique element

$$\xi \text{ in } F(x)$$

can reformulate the
representability of F

in terms of ξ .

Vocabulary

an element of a presheaf $F: C^{op} \rightarrow \text{Set}$

is a pair $(x, \xi) \mid \begin{array}{l} x \in C \\ \xi \in F(x) \end{array}$

an element (x, ξ) of F is called universal if, for any other element (z, ζ) of F , there exists a unique morphism

$u: z \rightarrow x$ in C such that

$$F(u): F(x) \longrightarrow F(z)$$

$$\xi \longmapsto \zeta = F(u)(\xi)$$

equivalently (x, ξ) is universal of the morphism

$$\text{Hom}_{\mathcal{C}}(z, x) \xrightarrow{\sim} F(z) \quad \text{is a bijection}$$
$$u \longmapsto F(u)(\xi)$$

since this condition is natural in z this gives an isom.

$$\hat{x} \xrightarrow{\sim} F$$

Morale : being representable \Leftrightarrow having a universal element.

The object x of a universal element (x, ξ_x) is called
the universe of F

The unique map $z \rightarrow x$ associated to an element (z, ξ_z)
is called the characteristic map of (z, ξ_z)
classifying map

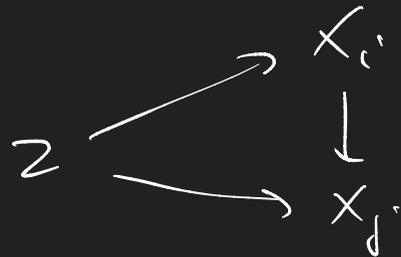
Examples

- $X: I \rightarrow \mathcal{C}$ diagram
presheaf of cones on X

$$\text{cone}(-, X): \mathcal{C}^{\text{op}} \rightarrow \text{Set}$$

$$z \mapsto \text{cone}(z, X) = \text{set of cones on } X \text{ with apex } z.$$

is representable by the limit of X
(if it exist)

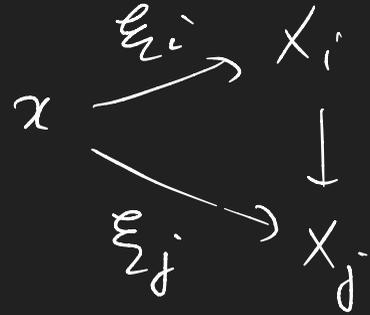


the universal element of $\text{cone}(-, X)$ is the limit cone on X .
universal cone

if (x, ξ) is universal for $\text{cones}(-, X)$ then

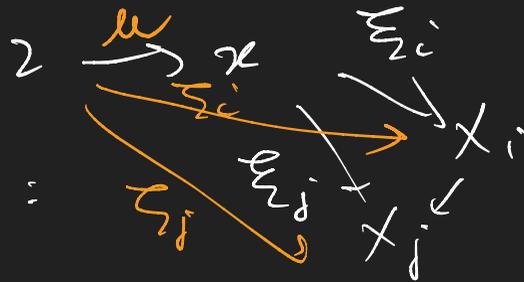
$x = \lim_i X_i$ is a limit for $X =$ the universe of cones on X .

$\xi =$ limit cone
 " universal cone



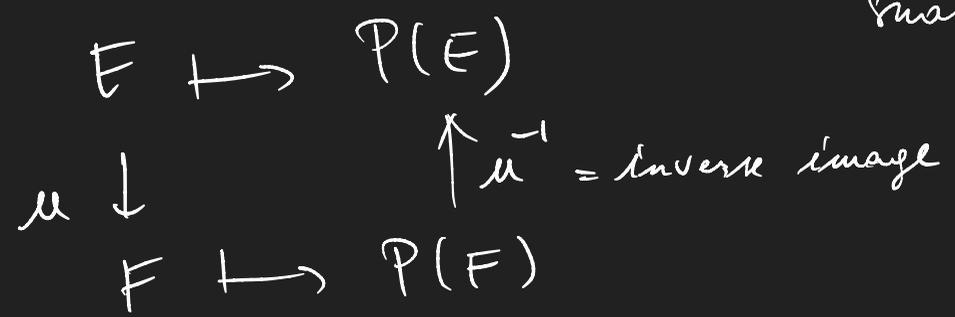
if (z, ζ) is a cone the characteristic map is the unique u
 $z \rightarrow x$

such that
 we have a
 morphism of cones =



• functor of subsets.

$P/sub : Fin^{op} \rightarrow Set$ (restrict to $Fin = cat\ of\ finite\ sets\ to\ have\ a\ small\ cat$)

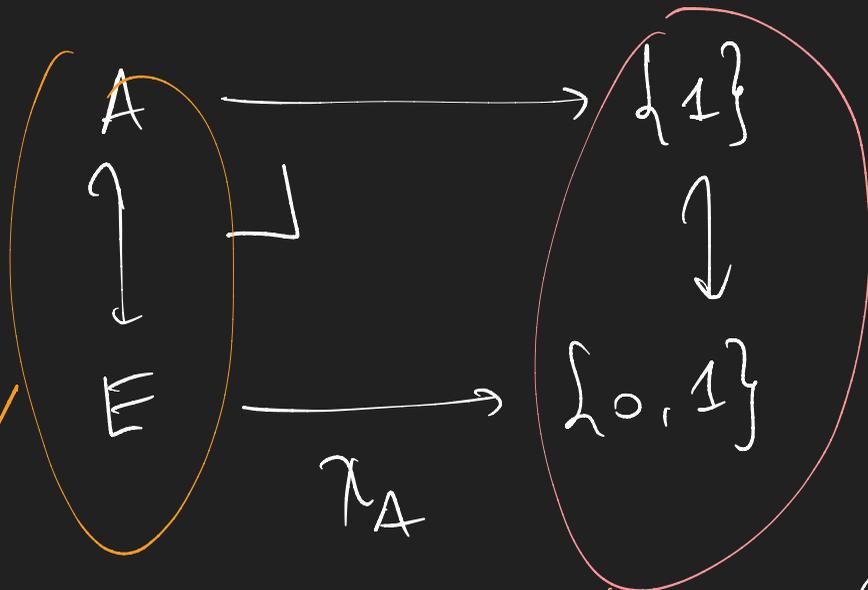


representable by $(\{0,1\}, \{1\})$ $\{1\} \in P(\{0,1\})$

$A = \mathcal{P}_A(\{1\})$
 $\{1\}$ = universal element for sub/P
 $\{1\}$ = universal subset.



Remark



an arbitrary element of sub

there exist a commutative square - which is cartesian.

$$A = \chi_A^{-1}(\{1\})$$

universal subset inclusion -

any inclusion $A \hookrightarrow E$ is a pull back of $\{1\} \hookrightarrow \{0, 1\}$ in a unique way

• related example in Topology

$\text{Top} =$ cat of topological spaces. (with size restriction)

$$\text{Op} : \text{Top}^{\text{op}} \longrightarrow \text{Set}$$

$$X \longmapsto \text{Op}(X) = \text{set of open subset of } X.$$

universal element for $\text{Op} : (S, \mathcal{A}\{1\})$

$S =$ Sierpiński space = the topology on $\{0, 1\}$

$$\text{Op}(S) = \{\emptyset, \mathcal{A}\{1\}, \mathcal{A}\{0, 1\}\}$$

$$\text{Op}(X) = \text{Hom}_{\text{Top}}(X, S)$$

$\text{Top}_{\text{open}} =$ continuous functions with values in S

• in linear algebra

$\text{Vect} =$ the col of \mathbb{R} -vector-spaces + \mathbb{R} -linear maps.

$\text{Vect}^{\text{op}} \longrightarrow \text{Set}$

$E \longmapsto E^*$ dual vector-space

$\text{Hom}_{\text{Vect}}(E, \mathbb{R})$

universal element, $(\mathbb{R}, \xi : \mathbb{R} \xrightarrow{\sim} \mathbb{R})$
linear.

any non zero ξ in $\mathbb{R} \setminus \{0\}$
is universal.

$\xi \in \mathbb{R} \setminus \{0\}$

Suppose $F: C^{\text{op}} \rightarrow \text{Set}$ has two universal elements

$$(x, \xi) \quad \text{and} \quad (z, \zeta)$$

Then there exists a unique isomorphism $x \xrightarrow{\sim} z$
in C

$$\text{such that } F(u): F(z) \rightarrow F(x) \\ \zeta \mapsto \xi$$

$$F(u^{-1}): F(x) \rightarrow F(z) \\ \xi \mapsto \zeta$$

*in this sense
universal elements
are unique.*

the same notion make sense for functors $F: \mathcal{C} \rightarrow \text{Set}$
but with some adaptations. Covariant " $(\mathcal{C}^{\text{op}})^{\text{op}}$

F representable if $\exists x \in \mathcal{C}$ and an iso

$$\text{Hom}_{\mathcal{C}}(x, -) \xrightarrow{\sim} F$$

By Yoneda lemma (applied to \mathcal{C}^{op}) this corresponds to an
unique element ξ_x in $F(x)$

the pair (x, ξ_x) is called a coelement

$$F: C^{\text{op}} \longrightarrow \text{Set}$$

$$(x, \xi) \quad x \in C \quad \xi \in F(x)$$

element

universal element (x, ξ)

universe x

characteristic map $z \xrightarrow{\mu} x$

F is representable

$$F: C \longrightarrow \text{Set}$$

co element

(co)universal co element (x, ξ)

(co)universe x

(co)characteristic map $x \rightarrow z$

F is (co)representable

opposite directions

example of representable covariant functor

• $\text{Fin} \rightarrow \text{Set}$

$E \mapsto 1$
" "

representable by \emptyset initial object
(empty set)

$\text{Hom}_{\text{Fin}}(\emptyset, E)$

\emptyset is the comverse of constant
fct
of the terminal
functor.

• $\text{Fin} \longrightarrow \text{Set}$ (inclusion of finite sets into Sets)

$E \longmapsto E$
"

$\text{Hom}_{\text{Fin}}(1, E)$

representable by a singleton

1 is the universe of the inclusion. $\text{Fin} \rightarrow \text{Set}$

$(1, \text{id}_1)$ is the universal coelement of $\text{Fin} \rightarrow \text{Set}$.

• $\text{Mon} = \text{cat of monoids (+ size restriction)}$

$U: \text{Mon} \rightarrow \text{Set}$

$M \mapsto \text{underlying set of } M = \text{Hom}_{\text{Mon}}(\mathbb{N}, M)$

$(\mathbb{N}, 1)$ is universal coelement for U .

free monoid
on one
generator

the generator

A counter-example

pb: there is not set of sets.

2 issues: (1) size (can be tamed by inaccessible cardinal)

(2) sets have symmetries. (they live more naturally in a category than a set)

$\text{Fin}^{\text{op}} \longrightarrow \text{Set}$

$E \longmapsto$ set of mops $\mu: \begin{array}{c} F \\ \downarrow \\ E \end{array}$ in Fin

= set of families of finite sets parametrized by E .

would be representable if we had a

universe of finite sets

does not exist
because issues
① and ②
before

a universal element is a pair $(U, \begin{array}{c} u' \\ \downarrow \\ u \end{array})$ such that
 $u =$ universe of set $\begin{array}{c} u' \\ \downarrow \\ u \end{array}$ is a universal family.

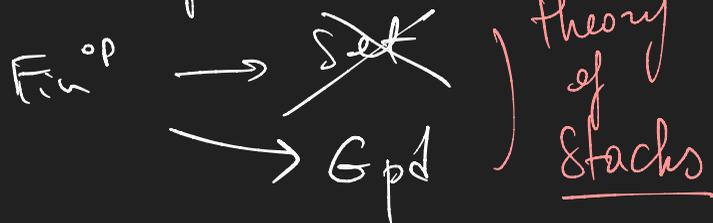
Solutions change the problem.

solve ① : need restrictions of size

(look a families of finite set cardinality $\leq n$)

→ still problem ②

solve ② : two ways : (a) can take the symmetries into account



(b) : break the symmetries by adding a structure

Fiber \rightarrow Set

$E \mapsto$ families

$F \downarrow E$

cardinal fiber $\leq n$

+ total order on the fiber.

become representable by

$\mathbb{N}_{\leq n}$

prevent symmetries.

total order

can be replaced by decorations
the elements of the fiber with
tree - (ZF)