Presheaves (C small cat)

a presheaf $F : C^{op} \to Set$ is representable if it is isomorphic to a presheaf $\hat{x} = Hom_C(-, x) : C^{op} \to Set$

$$\hat{x} \xrightarrow{\simeq} F$$

By Yoneda lemma this iso $\hat{x} \xrightarrow{\simeq} F$ corresponds to a unique element ξ in $F(x)$

can reformulate the representability of F in terms of ξ.

Vocabulary

an <u>element</u> of a presheaf $F : C^{op} \to Set$

is a pair (x, ξ) $\Big|$ $x \in C$, $\xi \in F(x)$

an element (x, ξ) of F is called <u>universal</u> if, for any other element (z, ζ) of F, there exists a unique morphism

$u : z \to x$ in C such that

$$F(u) : F(x) \longrightarrow F(z)$$

$$\xi \longmapsto \zeta = F(u)(\xi)$$

equivalently (x, ξ) is universal if the morphism

$$\mathrm{Hom}_{C}(z, x) \xrightarrow{\sim} F(z) \quad \text{is a } \underline{\text{bijection}}$$

$$u \longmapsto F(u)(\xi)$$

since this condition is natural in z this gives an isom.

$$\hat{x} \xrightarrow{\sim} F$$

Morale: being representable $\Longleftrightarrow$ having a universal element.

The object x of a universal element (x, ξ) is called the <u>universe of F</u>

The unique map $z \to x$ associated to an element (z, ζ) is called the <u>characteristic map</u> / <u>classifying map</u> of (z, ζ)

Examples

- $X : I \to C$ diagram

presheaf of cones on X

$$\mathrm{cone}(-,X) : C^{op} \longrightarrow \mathrm{Set}$$

$$Z \longmapsto \mathrm{cone}(Z,X) = \text{set of cones on } X \text{ with apex } Z.$$

is representable by the limit of X (if it exist)

$$\begin{tikzcd} & X_i \arrow[d] \\ Z \arrow[ur] \arrow[r] & X_j \end{tikzcd}$$

The universal element of $\mathrm{cone}(-,X)$ is the limit cone on X.

universal cone

if (x, ξ) is universal for $\mathrm{cones}(-, X)$ then

$x = \lim_i X_i$ is a limit for X = the universe of cones on X.

ξ = limit cone

"

universal cone

$$\begin{tikzcd} & X_i \arrow[d] \\ x \arrow[ur, "\xi_i"] \arrow[dr, "\xi_j"'] & \\ & X_j \end{tikzcd}$$

if (z, ζ) is a cone the characteristic map is the unique u

$z \to x$

such that we have a morphism of cones:

$$\begin{tikzcd} z \arrow[r, "u"] \arrow[drr, "\zeta_i"] \arrow[ddrr, "\zeta_j"'] & x \arrow[dr, "\xi_i"] \arrow[ddr, "\xi_j"] & \\ & & X_i \arrow[d] \\ & & X_j \end{tikzcd}$$

- functor of subsets.

$$P \,/\, \mathrm{sub} : \mathrm{Fin}^{op} \longrightarrow \mathrm{Set}$$

(restrict to Fin = cat of finite sets to have a small cat)

$$\begin{tikzcd}
E \arrow[r, mapsto] \arrow[d, "u"'] & P(E) \\
F \arrow[r, mapsto] & P(F) \arrow[u, "u^{-1}"']
\end{tikzcd}$$

u^{-1} = inverse image

representable by $(\{0,1\}, \{1\})$

$\{1\} \in P(\{0,1\})$

$\{0,1\}$: universe of subsets

$\{1\}$: universal subset.

= universal element for sub / P

$$\begin{tikzcd}
A = \chi_A^{-1}(\{1\}) \arrow[d, phantom, "\cap"] & \{1\} \arrow[d, phantom, "\cap"] \\
E \arrow[r, "\chi_A"] & \{0,1\}
\end{tikzcd}$$

$x \longmapsto 1 \quad x \in A$

$\qquad\quad 0 \quad x \notin A$

χ_A: characteristic map of (E, A)

Remark

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \{1\} \arrow[d, hook] \\
E \arrow[r, "\chi_A"'] & \{0,1\}
\end{tikzcd}
$$

There exist a commutative square which is cartesian.

$\left(A = \chi_A^{-1}(\{1\}) \right)$

an arbitrary element of <u>sub</u>

universal subset inclusion.

any inclusion $A \hookrightarrow E$ is a pullback of $\{1\} \hookrightarrow \{0,1\}$ in a unique way

- related example in Topology

Top = cat of topological spaces. (with size restriction)

$$\mathrm{Op} : \mathrm{Top}^{op} \longrightarrow \mathrm{Set}$$
$$X \longmapsto \mathrm{Op}(X) = \text{set of open subset of } X.$$

universal element for Op : $(S, \{1\})$

S = Sierpinski space = the topology on $\{0,1\}$

$$\mathrm{Op}(S) = \{\emptyset, \{1\}, \{0,1\}\}$$

$$\mathrm{Op}(X) = \mathrm{Hom}_{\mathrm{Top}_{\text{open}}}(X, S) = \text{continuous functions with values in } S$$

- in linear algebra

Vect = the cat of $\mathbb{R}$-vector spaces + $\mathbb{R}$-linear maps.

$$\begin{aligned} \mathrm{Vect}^{op} &\longrightarrow \mathrm{Set} \\ E &\longmapsto E^* \quad \text{dual vector space} \\ &\qquad \| \\ &\quad \mathrm{Hom}_{\mathrm{Vect}}(E, \mathbb{R}) \end{aligned}$$

universal element, $(\mathbb{R}, \xi : \mathbb{R} \xrightarrow{\sim} \mathbb{R})$ linear.

any non zero ξ in $\mathbb{R}\setminus\{0\}$ is universal. $\qquad \xi \in \mathbb{R}\setminus\{0\}$

Suppose $F : C^{op} \to Set$ has two universal elements

(x, ξ) and (z, ζ)

Then there exists a unique isomorphism $x \xrightarrow[\sim]{u} z$ in C

such that $F(u) : F(z) \longrightarrow F(x)$

$$\zeta \longmapsto \xi$$

$$F(u^{-1}) : F(x) \longrightarrow F(z)$$

$$\xi \longmapsto \zeta$$

In this sense universal elements are unique.

the same notion make sense for covariant functors $F : C \longrightarrow Set$

$C = (C^{op})^{op}$

but with some adaptations.

F representable if $\exists x \in C$ and an iso

$$\mathrm{Hom}_C(x, -) \xrightarrow{\sim} F$$

By Yoneda Lemma (applied to C^{op}) this corresponds to an unique element ξ in $F(x)$

the pair (x, ξ) is called a <u>co</u>element

$F : C^{op} \longrightarrow Set$

(x, ξ) $x \in C$ $\xi \in F(x)$

element

universal element (x, ξ)

universe x

characteristic map $z \xrightarrow{u} x$

F is representable

$F : C \longrightarrow Set$

coelement

(co)universal coelement (x, ξ)

(co)universe x

(co)characteristic map $x \longrightarrow z$

F is (co)representable

opposite directions

example of representable covariant functor

- $\mathrm{Fin} \longrightarrow \mathrm{Set}$

$$E \longmapsto 1$$

$$\mathrm{Hom}_{\mathrm{Fin}}(\emptyset, E) = 1$$

representable by $\emptyset$ initial object (empty set)

$\emptyset$ is the <u>couniverse</u> of constant fct of the terminal functor.

- $\mathrm{Fin} \longrightarrow \mathrm{Set}$ (inclusion of finite sets into Sets)

$$E \longmapsto \underset{\mathrm{Hom}_{\mathrm{Fin}}(1, E)}{\overset{E}{\shortparallel}}$$

representable by a singleton

1 is the universe of the inclusion. $\mathrm{Fin} \to \mathrm{Set}$

$(1, id_1)$ is the universal coelement of $\mathrm{Fin} \to \mathrm{Set}$.

- Mon = cat of monoids (+ size restriction)

$U : \mathrm{Mon} \longrightarrow \mathrm{Set}$

$M \longmapsto$ underlying set of M. $= \mathrm{Hom}_{\mathrm{Mon}}(\mathbb{N}, M)$

$(\mathbb{N}, 1)$ is universal coelement for U.

free monoid on one generator

the generator

A counter-example

pb: there is not set of sets.

2 issues: (1) size (can be tamed by inaccessible cardinal)

(2) sets have symmetries. (they live more naturally in a category than a set)

$$\mathrm{Fin}^{op} \longrightarrow \mathrm{Set}$$

$$E \longmapsto \text{set of maps } u : \begin{array}{c} F \\ \downarrow \\ E \end{array} \text{ in Fin}$$

$= \text{set of families of finite sets parametrized by } E.$

would be representable if we had a universe of finite sets

does not exist because issues 1 and 2 before

a universal element is a pair $\left(U, \begin{array}{c} U' \\ \downarrow \\ U \end{array} \right)$ such that

U = universe of set $\begin{array}{c} U' \\ \downarrow \\ U \end{array}$ is a universal family.

Solutions change the problem.

solve 1 : need restrictions of size
(look a families of finite set cardinality $\leq n$)

$\rightsquigarrow$ still problem 2

solve 2 : two ways : a can take the symmetries into account

$$\begin{array}{l} Fin^{op} \longrightarrow \cancel{Set} \\ \phantom{Fin^{op}} \longrightarrow Gpd \end{array} \Bigg) \text{ theory of } \underline{\text{stacks}}$$

b : <u>break</u> the symmetries by <u>adding a structure</u>

$$\mathrm{Fin}^{op} \longrightarrow \mathrm{Set}$$

$$E \longmapsto \text{families } \begin{array}{c} F \\ \downarrow \\ E \end{array}$$

cardinal fiber $\leq n$

+ total order on the fiber.

prevent symmetries.

become representable by

$$\mathbb{N}_{\leq n}$$

total order can be replaced by decorations the elements of the fiber with tree - (ZF)