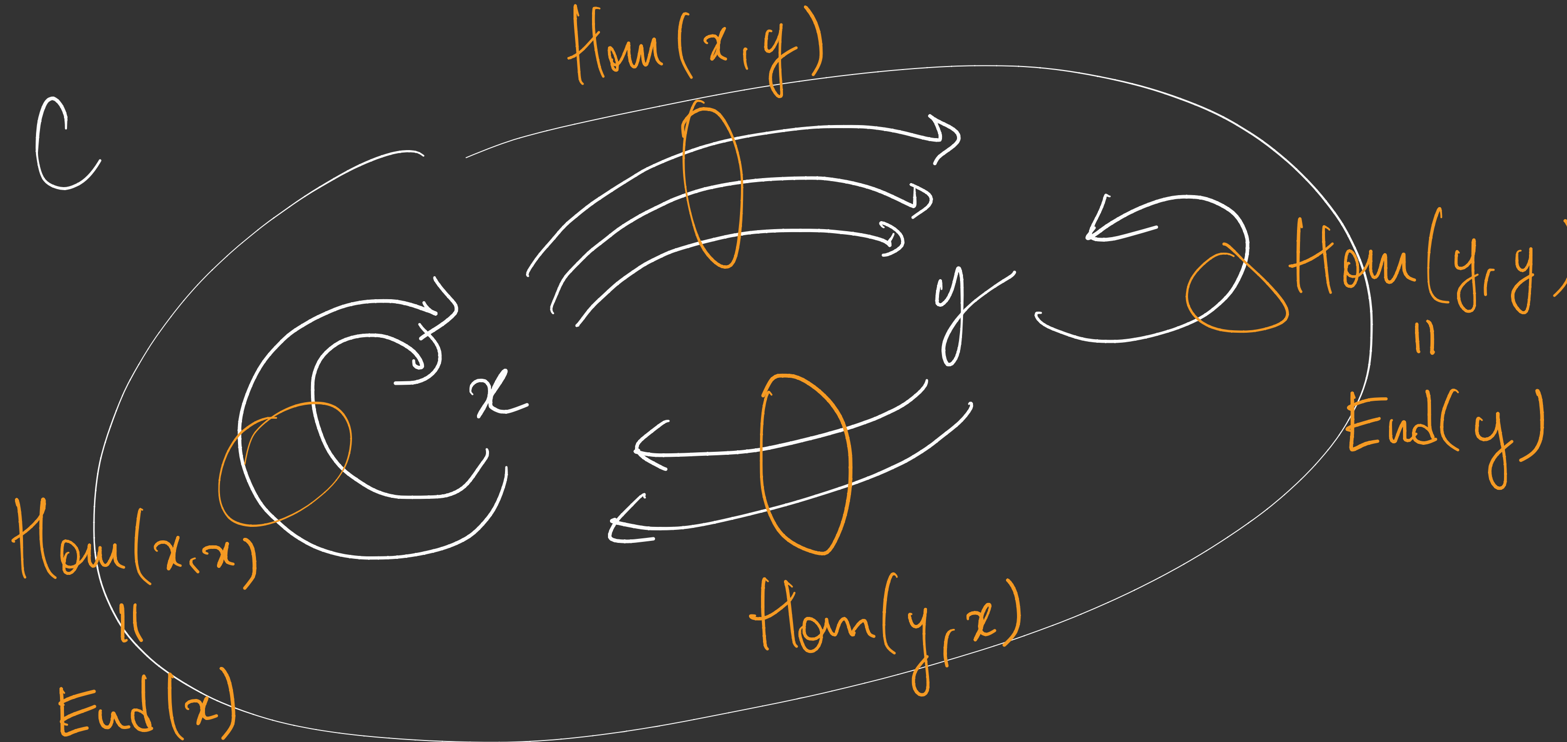


Hom $\rightsquigarrow$ Homomorphism

End $\rightsquigarrow$ Endomorphism.

isomorphisms in a category C

$f : x \to y$ in C is an isomorphism if

there exists $g : y \to x$ such that

$$gf = 1_x \qquad fg = 1_y$$

$$1_x = gf \quad \begin{tikzcd} x \arrow[loop left] \arrow[r, bend left, "f"] & y \arrow[l, bend left, "g"] \arrow[loop right] \end{tikzcd} \quad fg = 1_y$$

$$\{ f : x \to y \mid f \text{ isomorphism} \} =: \mathrm{Iso}(x, y) \subseteq \mathrm{Hom}(x, y)$$

if it exist g can be proven to be unique

it is denoted f^{-1} ("f inverse")

$$\begin{tikzcd} x \arrow[r, bend left, "f"] & y \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

the isomorphisms in $\mathrm{Hom}(x,x) = \mathrm{End}(x)$ are called <u>automorphism</u>

$$\{ f \in \mathrm{End}(x) \mid f \text{ isomorphism} \} =: \mathrm{Aut}(x)$$

$$\mathrm{Aut}(x) \subseteq \mathrm{End}(x)$$

<u>examples</u>. in Set

f is an isomorphism $A \to B$ $\iff$ f is a bijection $A \xrightarrow{\sim} B$

. Set^2 cat of pairs of sets isomorphism $\iff$ pair of isomorphisms.

groupoid

a groupoid is a category where all morphisms are isomorphisms.

$$\mathrm{Hom}(x,y) = \begin{cases} \emptyset & \text{(no isomorphisms between } x \text{ and } y\text{)} \\ \mathrm{Iso}(x,y) & \end{cases}$$

there exists a bijection : $\mathrm{Hom}(x,y) \xrightarrow{\sim} \mathrm{Hom}(y,x)$

$$f \longmapsto f^{-1}$$

Remark $\mathrm{Hom}(x,y)$

in a cat C $\mathrm{Hom}_C(x,y)$ $\mathrm{Iso}_C(x,y)$

$C(x,y)$

Second def of a category

- collection of objects C_0
- for any pair of objects x, y a set $\mathrm{Hom}(x, y)$
- ——— triplet ——— x, y, z

$$\mathrm{Hom}(x, y) \times \mathrm{Hom}(y, z) \longrightarrow \mathrm{Hom}(x, z)$$

$$\left(x \xrightarrow{f} y \;,\; y \xrightarrow{g} z \right) \longmapsto x \xrightarrow{gf} z$$

- for any object x identity arrow

$$1_x \in \mathrm{Hom}(x, x)$$

- associativity for each quatruplet of objects x, y, z, t

$$
\begin{tikzcd}
\overset{1}{\mathrm{Hom}(x,y)} \times \overset{2}{\mathrm{Hom}(y,z)} \times \overset{3}{\mathrm{Hom}(z,t)} \arrow[rr, "\alpha'"] \arrow[d, "\alpha", "\text{compo on } 1,2 \text{ factor}"'] \arrow[drr] & & \mathrm{Hom}(x,y) \times \mathrm{Hom}(y,t) \arrow[d, "\beta'", "\text{composition on factor } 2,3"'] \\
\mathrm{Hom}(x,z) \times \mathrm{Hom}(z,t) \arrow[rr, "\beta"'] & & \mathrm{Hom}(x,t)
\end{tikzcd}
$$

associativity : $\beta'\alpha' = \beta\alpha$

- identity relation...

How to "see" a category

1) graph representation (last time)

2) matrix representation

codomain

$$
\begin{array}{c|cccc}
 & x & y & z & t \\
\hline
x & \mathrm{End}(x) & \mathrm{Hom}(x,y) & \mathrm{Hom}(x,z) & \\
y & \mathrm{Hom}(y,x) & \mathrm{End}(y) & \mathrm{Hom}(y,z) & \cdots \\
z & \mathrm{Hom}(z,x) & \mathrm{Hom}(z,y) & \mathrm{End}(z) & \\
t & & \vdots & & \mathrm{End}(t)
\end{array}
$$

domain

"splitting" of the set $Arr(C)$ of all morphisms

rows and columns are indexed by object of the cat.

diagonal

The matrix of a groupoid is <u>symmetric</u>

(because of the bijection $\mathrm{Hom}(x,y) \xrightarrow{\sim} \mathrm{Hom}(y,x)$)

More examples of categories

- "one arrow" category :

$$0 \xrightarrow{u} 1$$

objects $= \{0, 1\}$

arrow $= \{1_0, 1_1, u\}$

composition $=$ identity relation.

- "commutative triangle"

$$\begin{tikzcd} & 1 \arrow[dr, "g"] & \\ 0 \arrow[ur, "f"] \arrow[rr, "h"'] & & 2 \end{tikzcd}$$

ob : $\{0, 1, 2\}$

arr : $\{1_0, 1_1, 1_2, f, g, h\}$

compo : $h = gf$

+ id. rel.

- "commutative square"

$$\begin{tikzcd} 0 \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, "h"] & 2 \arrow[d, "g"] \\ 1 \arrow[r, "f"'] & 3 \end{tikzcd}$$

$\mathrm{ob} := \{0, 1, 2, 3\}$

$\mathrm{arr} := \{1_0, 1_1, 1_2, 1_3, f, f', g, g', h\}$

composition:

$$gf' = h = fg'$$

Matrix representation

$$0 \xrightarrow{u} 1$$

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & \{1_0\} & \{u\} \\ 1 & \emptyset & \{1_1\} \end{array}$$

$$\begin{tikzcd} 0 \arrow[r, "f"] \arrow[dr, "h"'] & 1 \arrow[d, "g"] \\ & 2 \end{tikzcd}$$

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & \{1_0\} & \{f\} & \{h\} \\ 1 & \emptyset & \{1_1\} & \{g\} \\ 2 & \emptyset & \emptyset & \{1_2\} \end{array}$$

- **Set** is a cat. with only identity morph.

$$Hom(x, y) = \begin{cases} \emptyset & \text{if } x \neq y \\ \{1_x\} & \text{if } x = y \end{cases}$$

- Cat. with 0 or 1 arrow at most between two objects

$$Hom(x, y) = \begin{cases} \emptyset & x \leq y \text{ false} \\ \{*\} & x \leq y \text{ true} \end{cases}$$

$=$ preorder

- category with a single object

$\mathrm{ob} = \{*\}$

arrow $=$ $\mathrm{Hom}(*,*) = \boxed{\mathrm{End}(*) =: M}$

composition : $M \times M \longrightarrow M$

$$\left(* \xrightarrow{f} *,\ * \xrightarrow{g} *\right) \longmapsto * \xrightarrow{gf} *$$

identity $1_* \in M$

M is a monoid

cat with one object $\Longleftrightarrow$ data of the monoid $\mathrm{End}(*)$

in a category C

the set of endomorphism of any object

$\mathrm{End}(x)$ is always a <u>monoid</u>

<u>example</u> of cat with one object

- $(\mathbb{N}, +)$ $\qquad$ $\begin{tikzcd} * \arrow[loop right] \end{tikzcd}$ $\mathbb{N} = \mathrm{End}(*)$

name "$B\mathbb{N}$"

"$\mathbb{N}$ viewed as a category"

- $(\mathbb{R}, \times)$ $\quad * \circlearrowleft \quad \mathbb{R} = \mathrm{End}(*)$

- idempotent monoid $\{1, e\} = M$

monoid product	1	e
1	1	e
e	e	e

$$e^2 = ee = e$$

idempotent

→ category with one object

+ two arrows

for $n \in \mathbb{N}$ $\quad e^n = e$

$$e^n = \underbrace{eee \dots e}_{n \text{ times}}$$

- involution monoid $\{1, \sigma\}$

$$
\begin{array}{c|cc}
 & 1 & \sigma \\
\hline
1 & 1 & \sigma \\
\sigma & \sigma & 1
\end{array}
$$

$\sigma^2 = 1$

involution relation.

$\{1, \sigma\}$ = permutation $\overset{\text{group}}{}$ of two elements

$$\frac{a \quad b}{a \quad b} \qquad \frac{a \quad b}{b \quad a} \qquad \begin{array}{c} a \quad b \\ \hline b \quad a \\ \hline a \quad b \end{array} \begin{array}{c} \sigma \\ \sigma \end{array}$$

$1 \qquad \sigma \qquad \sigma^2 = 1$

- group are monoids where all elements are invertible

cat with one elt $\Leftrightarrow$ monoid

groupoid with elt $\Leftrightarrow$ groups.

- example of monoids

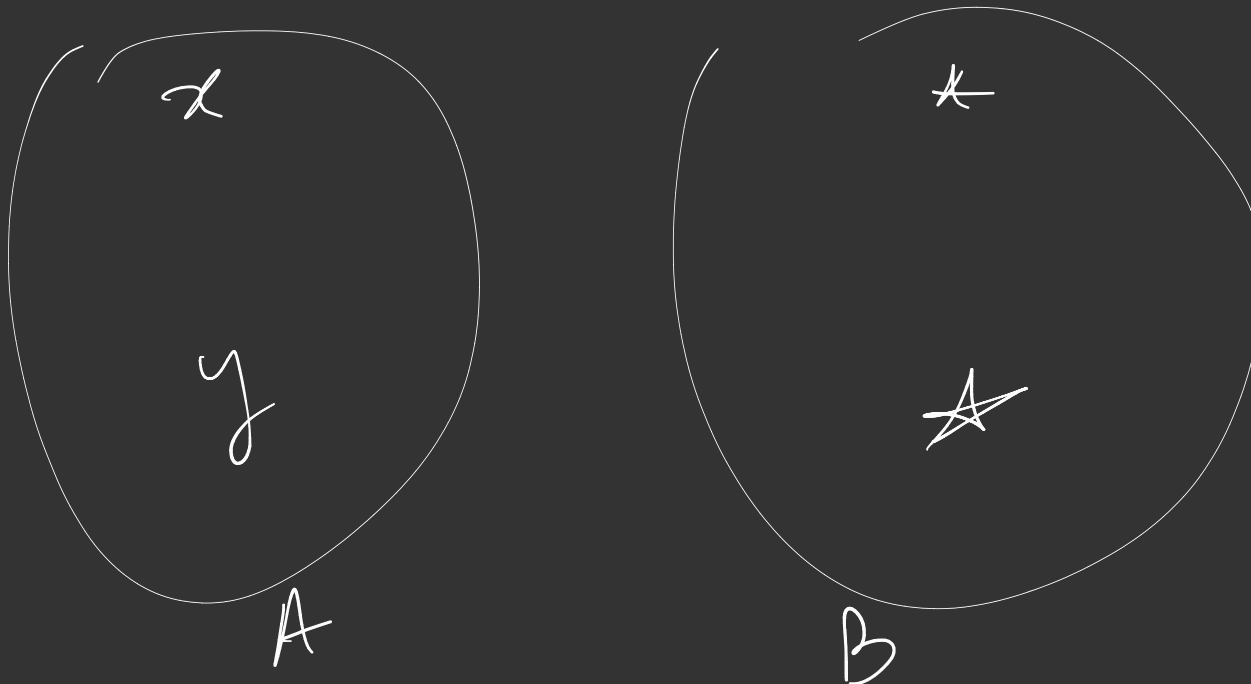
- $(\mathbb{N}, +)$ = free monoid on one generator

$0 \qquad 1 \qquad 1+1 \qquad 1+1+1 \qquad \ldots$

- A set $\quad \mathrm{List}(A)$ = free monoid on the set A

product = concatenation of lists.

$\underbrace{\qquad \text{list } 1 \qquad}\ \underbrace{\qquad \text{list } 2 \qquad}$



$\mathrm{Bij}(A, B)$

no canonical elt

distinguished

$\mathrm{Bij}(A, A)$

canonical elt : identity

distinguished.