Unit and counit of adjunctions

$RL \circlearrowright$ $\quad$ $\circlearrowleft LR$

$$L : C \rightleftarrows D : R \qquad L \dashv R$$

$$\alpha_{x,y} : \operatorname{Hom}_D(Lx, y) \xrightarrow{\sim} \operatorname{Hom}_C(x, Ry)$$

$y = Lx$

$$\operatorname{Hom}_D(Lx, Lx) \xrightarrow{\sim} \operatorname{Hom}_C(x, RLx)$$

$$\begin{array}{ccc} \psi & & \\ id_{Lx} & \longmapsto & \eta_x : x \longrightarrow RLx \end{array}$$

unit of the adjunction.

natural in x

$$\boxed{\eta : id_C \longrightarrow RL}$$

$x = Ry$

$$\operatorname{Hom}(LRy, y) \xrightarrow{\sim} \operatorname{Hom}_C(Ry, Ry)$$

$$\varepsilon_Y : LRY \longrightarrow Y \longleftarrow\!\!\mid id_Y$$

counit nat. transformation.

$$\varepsilon : LR \longrightarrow id_D$$

any adjunction $L : C \rightleftarrows D : R$

has a <u>unit</u> : $id_C \xrightarrow{\eta} RL$

<u>Counit</u> : $LR \xrightarrow{\varepsilon} id_D$

the adjunction (the natural isomorphism α)

can be rebuild from η and ε

(cf. Homework)

examples of unit and counit

- $\mathcal{C}$ cart. closed category

$x \in \mathcal{C}$ $\qquad X \times - \dashv (-)^X$

$$\underbrace{X \times}_{\text{"evaluation"}} Y \xrightarrow{id} X \times Y$$

"coevaluation"

$$\eta_Y : Y \longrightarrow (X \times Y)^X$$

if $\mathcal{C} = \mathrm{Set}$ $\qquad \eta : Y \longrightarrow (X \times Y)^X$

$$y \longmapsto f : X \longrightarrow X \times Y$$

$$x \longmapsto (x, y)$$

$$\frac{Y^X \xrightarrow{id} Y^X}{X \times (Y^X) \xrightarrow{\varepsilon = \text{evaluation map.}} Y}$$

in $\mathcal{C} = \mathrm{Set}$

$$(x, f : X \to Y) \longmapsto \underbrace{f(x)}_{\text{evaluation of } f \text{ at } x}$$

• C cat with an initial object 0_C

$$\{*\} = 1 \underset{p}{\overset{0_C}{\underset{\perp}{\leftrightarrows}}} C \qquad X \in C$$

the counit $\eta_X : 0_C \longrightarrow X$

is the ~~canonical~~ unique map from 0 to X

the unit is the identity of $*$ $\quad \varepsilon : * \xrightarrow{id} *$

Dually C has a terminal object 1_C $\qquad C \underset{1_C}{\overset{p}{\underset{\perp}{\rightleftarrows}}} 1 = \{*\}$

the unit $\varepsilon_X : X \longrightarrow 1_C$

is the unique map from X to 1_C

counit $\eta_* = id_*$

- $\mathcal{C}$ cat with colimits of shape I

$$\mathcal{C}^I \underset{\Delta}{\overset{\mathrm{colim}}{\rightleftarrows}} \mathcal{C} \quad (\perp)$$

<u>unit</u> $\eta_{(X_i)} : (X_i) \longrightarrow \Delta(\mathrm{colim}\, X_i)$ map of diagrams

= the universal cocone / colim cocone.

<u>counit</u> $\varepsilon_Z : \mathrm{colim}(\Delta Z) \longrightarrow Z$

$Z \in \mathcal{C}$

(if I is a set $\mathrm{colim}_i(\Delta Z) = \coprod_{i \in I} Z \xrightarrow{\varepsilon} Z$ = coproduct of I copies of Z)

the situation is dual with <u>limits</u>.

<u>colimit</u> $\Delta(\lim X_i) \xrightarrow{\varepsilon_{(X_i)}} X_i$

universal cone
limit cone.

<u>limit</u> $\eta_Z : Z \longrightarrow \lim_i (\Delta Z)$

if I set $\lim_i \Delta Z = \prod_{i \in I} Z$ (product of I copies of Z.

$\eta_Z : Z \longrightarrow \prod_{i \in I} Z$ "diagonal map"
"constant family map"

- in topology X. top. space

inclusion of posets

$$\begin{tikzcd} O(X) \arrow[r, shift left, hook, "i"] \arrow[r, phantom, "\perp"] & P(X) \arrow[l, shift left, "\mathrm{int}"] \end{tikzcd}$$

$i \dashv \mathrm{int}$

open subsets

arbitrary subsets

interior of a subset.

the unit $U \in O(X)$ $\quad U \xrightarrow{=} \mathrm{int}(i(U))$ in $O(X)$

the counit $A \in P(X)$ $\quad i(\mathrm{int}(A)) \subset A$

inclusion of the interior of A into A.

Recall from Homework the notion of <u>fully faithful functor</u> $F : C \longrightarrow D$

$$\mathrm{Hom}_C(X,Y) \xrightarrow{F_{X,Y}} \mathrm{Hom}_D(FX, FY)$$

$$X \xrightarrow{u} Y \longmapsto FX \xrightarrow{Fu} FY$$

F is <u>fully faithful</u> if these maps are <u>bijections</u> for any X, Y.

D cat. $\quad ob(D)$ set of objects

$\cup$

C_0 = subset of objects.

can restrict the category structure of D to C_0.

C : $ob(C) := C_0$

$\mathrm{Hom}_C(x,y) := \mathrm{Hom}_D(x,y)$

composition = that of D.

such a C is called a <u>full subcategory</u>

the obvious functor $C \to D$ is full and faithful.

most examples of full and faithful functors arise this way: as inclusion of a full subcategory.

$F : C \longrightarrow D$ which is <u>right adjoint</u> (there exist G, $G \dashv F$)

bijection

$$\begin{tikzcd}
\mathrm{Hom}_C(x,y) \arrow[r, "\sim"] & \mathrm{Hom}_D(Fx,Fy) \arrow[d, phantom, "\wr"] \\
 & \mathrm{Hom}_C(GFx,y) \arrow[ul, "-\circ\varepsilon_x"]
\end{tikzcd}$$

(by adjunction)

map induced by $GFx \xrightarrow{\varepsilon_x} x$

= precomposition with the <u>counit</u> ε_x

ε_x is an isomorphism.

Proposition

- a right adjoint functor $F: C \to D$ is *fully faithful* if and only if the counit ε is a natural *isomorphism*.

- a left adjoint ———— unit η is ———— —

Definition: $C \xhookrightarrow{i} D$ a full subcategory is said

- to be *reflective* if i has a left adjoint ($\Leftrightarrow$ i is a right adjoint)

- to be *coreflective* if i has a right adjoint ($\Leftrightarrow$ i is a left adjoint)

examples

- Mon the cat of monoids

an abelian monoid M is such that $ab = ba$ for any a, b in M.

(multiplication is "commutative" or "abelian")

$$\begin{tikzcd} \mathrm{AbMon} \arrow[r, phantom, "\subset"] & \mathrm{Mon} \arrow[l, bend right, "ab"'] \arrow[l, phantom, bend right=20, "\perp"] \end{tikzcd}$$

full subcategory of abelian monoids. is reflective

the inclusion admits a left adjoint: the abelianization

"forces" a monoid to become abelian.

$ab : \mathrm{Mon} \longrightarrow \mathrm{AbMon}$

$\mathrm{Free}(n) \longmapsto \mathbb{N}^n$ = free abelian monoid

$M \longmapsto M/\sim$ ($\sim$ relation generated by $ab \sim ba$)

- $\mathrm{Mon} \supset \mathrm{Gp}$ full subcategory of groups.

a group is a monoid such that any element has an inverse (x has an inverse if $\exists y$ $xy = 1$ and $yx = 1$)

$$\begin{tikzcd} \mathrm{Gp} \arrow[r, hook, "i"] & \mathrm{Mon} \arrow[l, bend right, "ext"', "\perp"] \arrow[l, bend left, "int", "\perp"'] \end{tikzcd}$$

is both reflective and coreflective.

$\{x \in M \mid \text{invertible}\} = M^{int} \subset M$

is a group

$M \longmapsto M^{int}$

is right adjoint to the inclusion $\mathrm{Gp} \overset{i}{\hookrightarrow} \mathrm{Mon}$.

G a group

$iG \longrightarrow M$ map of monoid

$\updownarrow$ bijection.

$G \longrightarrow M^{int}$ map of groups.

$$M \longrightarrow M^{ext}$$

monoid

the "external group" of M is constructed by adding inverses to all elements of M.

<u>example</u> $\mathbb{N}$ = free monoid on 1 generator

$\mathbb{N}^{ext} = \mathbb{Z}$ group of integers (free group on one generator)

mor. of monoids $\quad M \xrightarrow{f} iG$ group.

$$x \longmapsto f(x)$$

bijection

$$ext \dashv i$$

mor of group. $\quad M^{ext} \xrightarrow{\tilde{f}} G$

$$x^{-1} \longmapsto \tilde{f}(x^{-1}) := f(x)^{-1}$$

- the triple adjunction $\mathrm{Gp} \begin{array}{c} \xleftarrow{\mathrm{ext}} \\ \perp \\ \xhookrightarrow{i} \\ \perp \\ \xleftarrow[\mathrm{int}]{} \end{array} \mathrm{Mon}$ generalizes

to $\text{Groupoids} \begin{array}{c} \xleftarrow{\mathrm{ext}} \\ \perp \\ \hookrightarrow \\ \perp \\ \xleftarrow[\mathrm{int}]{} \end{array} \text{Categories}$

any cat C has an $\left|\begin{array}{l}\text{internal groupoid } C^{\mathrm{int}} \\ \text{core groupoid } C^{\mathrm{core}}\end{array}\right.$

which has the same objects but only the isomorphisms as morphisms.

$$\underset{\text{groupoid.}}{C^{\mathrm{int}}} \subset C$$

$C \longrightarrow C^{ext}$ external groupoid

constructed by forcing all morphisms to become isomorphisms

($\longrightarrow$ theory of localizations of categories)

- $$\begin{tikzcd} Gph \arrow[r, shift left=2, "\pi_0"] \arrow[r, phantom, "\perp"] & Set \arrow[l, shift left=2, "dis"] \end{tikzcd}$$

$dis(E) \longleftarrow\!\shortmid E$

discrete graph

dis is a fully faithful functor.

full subcategory of Gph

which is reflective by π_0

morphism of graph $\quad G \xrightarrow{\eta} dis(\pi_0(G))$ unit of $\pi_0 \dashv dis$.

<u>morale</u>

full subcat $\longleftrightarrow$ property on objects.

reflective subcat $\longleftrightarrow$ the property can be <u>forced</u>

$M \longrightarrow M^{ext}$

coreflective subcat $\longleftrightarrow$ the property can be <u>extracted</u>

$M^{int} \longrightarrow M$