

Unit and counit of adjunctions

$$L : C \rightleftarrows D : R \quad L \dashv R$$

$$\alpha_{x,y} : \text{Hom}_D(Lx, y) \xrightarrow{\sim} \text{Hom}_C(x, Ry)$$

$$y = Lx \quad \text{Hom}_D(Lx, Lx) \xrightarrow{\sim} \text{Hom}_C(x, RLx)$$

$$\text{id}_{Lx} \longmapsto \eta_x : x \rightarrow RLx \quad \begin{matrix} \text{unit of the} \\ \text{adjunction.} \end{matrix}$$

natural in x

$$\boxed{\eta : \text{id}_C \rightarrow RL}$$

$$x = Ry \quad \text{Hom}(LRy, y) \xrightarrow{\sim} \text{Hom}_C(Ry, Ry)$$

$$\varepsilon_y : LRy \rightarrow y \quad \begin{matrix} \text{counit} \\ \text{nat. transformation.} \end{matrix}$$

$$\varepsilon : LR \rightarrow \text{id}_D$$

any adjunction $L : C \rightleftarrows D : R$

has a unit : $\text{id}_C \xrightarrow{\eta} RL$

Counit : $LR \xrightarrow{\epsilon} \text{id}_D$

the adjunction (the natural isomorphism α)
can be rebuild from η and ϵ

(cf. framework)

examples of unit and counit

- C cat. closed category

$$x \in C \quad X_{x-} \rightarrow (-)^X$$

$$X_{x-} \xrightarrow{id} X_{x-}$$

$$\eta_Y: Y \xrightarrow{\text{"evaluation"}} (X_{x-})^X$$

if $C = \text{Set}$

$$\eta: Y \rightarrow (X_{x-})^X$$

$$y \mapsto f: X \rightarrow X_{x-}$$

$$x \mapsto (x, y)$$

$$\frac{Y^X \xrightarrow{id} Y^X}{X_x(Y^X) \xrightarrow{\varepsilon = \text{evaluation map.}} Y}$$

in $C = \text{Set}$

$$(x, f: X \rightarrow Y) \mapsto f(x)$$

evaluation
of f at x

C cat with an initial object 0_c

$$\{ \xrightarrow{\quad} \} = 1 \quad \begin{matrix} \xleftarrow{L} \\ \xrightarrow{P} \end{matrix} \quad C \quad x \in C$$

the co-unit $\eta_x : 0_c \rightarrow X$

is the canonical map from 0 to X
unique

the unit is the identity of $*$ $\Sigma : * \xrightarrow{\iota} *$

Dually C has a terminal object 1_c $C \begin{matrix} \xrightarrow{P} \\ \xleftarrow{L} \end{matrix} 1 = \{*\}$

the unit $\epsilon_x : X \rightarrow 1_c$

is the unique map from X to 1_c

co-unit $\eta_* = id_*$

- C cat with colimits of shape I

$$C^I \xrightleftharpoons[\Delta]{\text{colim}} C$$

unit $\eta_{(X_i)} : (X_i) \rightarrow \Delta(\text{colim } X_i)$ map of diagram
 $=$ the universal cocone
 colim cocone -

co-unit $\varepsilon_z : \text{colim}(\Delta z) \rightarrow z$ $=$ coproduct of I copies of z
 $z \in C$ if I is a set $\text{colim}_i (\Delta z) = \coprod_{i \in I} z$
 $\downarrow \varepsilon_z$

The situation is dual with limits.

co-unit $\Delta(\lim x_i) \xrightarrow{\epsilon_{(x_i)}} x_i$

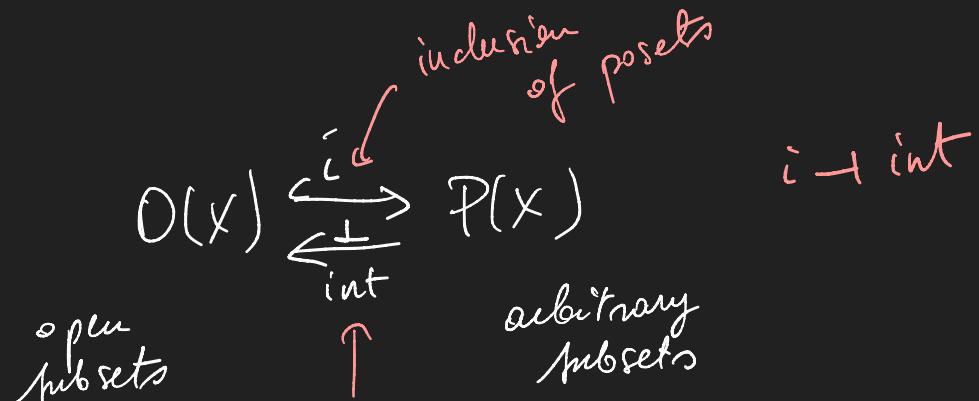
(universal cone
limit cone -

unit $\eta_z : z \rightarrow \lim_i (\Delta z)$

if I set $\lim_i \Delta z = \prod_{i \in I} z$ (product of I copies of z,

$\eta_z : z \rightarrow \prod_{i \in I} z$ "diagonal map"
"constant family map"

• in topology X . top. space



int
er
ior
of a subset.

the unit $u \in O(X)$ $u \xrightarrow{\text{int}} \text{int}(i(u))$ in $O(X)$

the wunit $A \in P(X)$ $i(\text{int}(A)) \subset A$

inclusion of the interior of A into A .

Recall from homework the notion of

fully faithful functor $F: \mathcal{C} \rightarrow \mathcal{D}$

$$\text{Hom}_{\mathcal{C}}(x, y) \xrightarrow{F_{x,y}} \text{Hom}_{\mathcal{D}}(Fx, Fy)$$

$$x \xrightarrow{u} y \longleftrightarrow Fx \xrightarrow{Fu} Fy$$

F is fully faithful if these maps are bijections for any $x, y,$

\mathcal{D} cat. $\text{ob}(\mathcal{D})$ set of objects

\cup

\mathcal{C}_0 = subset of objects.

can restrict the category structure of \mathcal{D} to \mathcal{C}_0 .

\mathcal{C} : $\text{ob}(\mathcal{C}) := \mathcal{C}_0$

$\text{Hom}_{\mathcal{C}}(x, y) := \text{Hom}_{\mathcal{D}}(x, y)$

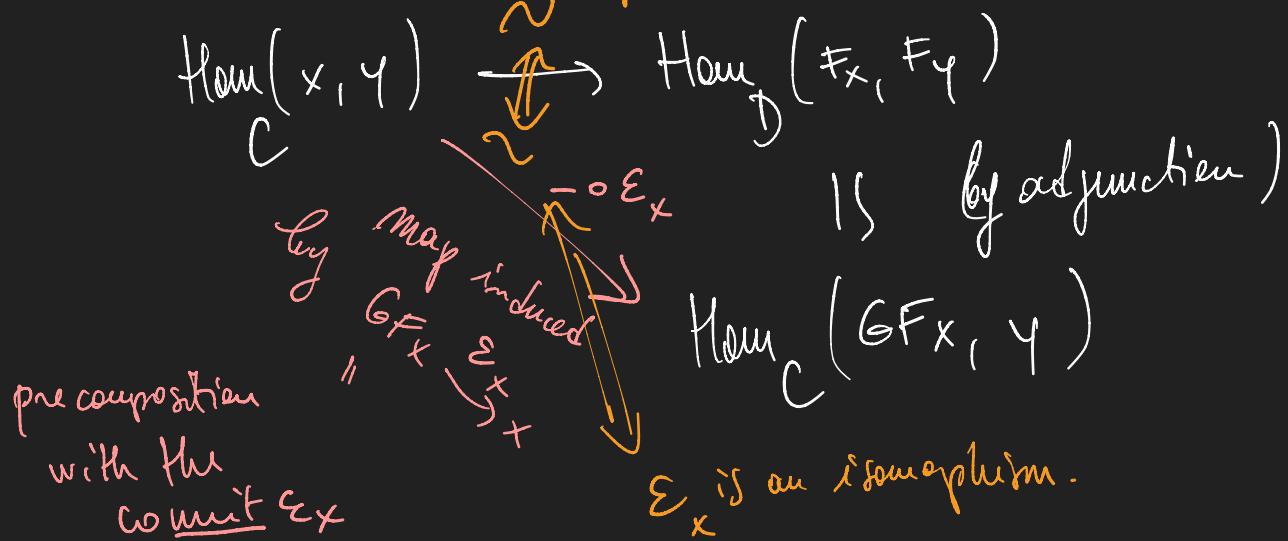
composition = that of \mathcal{D} .

such a \mathcal{C} is called a full subcategory

the obvious functor $\mathcal{C} \rightarrow \mathcal{D}$ is full and faithful.

most examples of full and faithful functors are this way : as inclusion of a full subcategory.

$F: C \rightarrow D$ which is right adjoint (there exist G
 $G \dashv F$)



- Proposition • a right adjoint functor $F: C \rightarrow D$ is fully faithful if and only if the counit ε is a natural isomorphism.
- a left adjoint ————— must η —————

- Definition : $C \hookrightarrow D$ a full subcategory is said
- to be reflective if \bar{C} has a left adjoint
 $(\Rightarrow \bar{C}$ is a right adjoint)
 - to be coreflective if \bar{C} has a right adjoint
 $(\Rightarrow \bar{C}$ is a left adjoint)

examples

- Mon the cat of monoids

an abelian monoid M is such that $ab = ba$ for any a, b in M .

$$\begin{array}{ccc} ab & \swarrow I \\ \text{AbMon} & \subset & \text{Mon} \end{array}$$

(Multiplication is "commutative" or "abelian")

full subcategory of abelian monids is reflective

the inclusion admits a left adjoint: the abelianization

$ab: \text{Mon} \rightarrow \text{AbMon}$ "forces" a monoid to become abelian.

$\text{Free}(n) \hookrightarrow \mathbb{N}^M = \text{free abelian monoid}$

$M \xrightarrow{\sim} M/\sim$ (\sim relation generated by $ab \sim ba$)

• $\text{Mon} \supset \text{Gp}$ full subcategory of groups.

a group is a monoid such that any element has an inverse (x has an inverse if $\exists y$ $xy = 1$ and $yx = 1$)

$\begin{array}{c} \text{ext} \\ \downarrow \\ \text{Gp} \subset \text{Mon} \\ \text{int} \end{array}$ is both reflective and coreflective.

$$\{x \in M \mid \text{invertible}\}_{\text{int}} = M^{\text{int}} \subset M$$

is a group

$$G \xrightarrow{\text{mop of monoid}} G \rightarrow M$$

$$G \rightarrow M^{\text{int}}$$

mop - of groups.

$M \mapsto M^{\text{int}}$
is right adjoint to the
inclusion $Gp \subset \text{Mon}$.

$M \longrightarrow M^{\text{ext}}$ the "external group" of M
 Monoid
 is constructed by adding inverses
 to all elements of M .

example \mathbb{N} = free monoid on 1 generator

$\mathbb{N}^{\text{ext}} = \mathbb{Z}$ group of integers (free group on one generator)

mon. of monoids

$M \xrightarrow{f} iG$ group.
 $x \longmapsto f(x)$ bijection

mon of group.

$M^{\text{ext}} \xrightarrow{\tilde{f}} G$
 $x^{-1} \longmapsto \tilde{f}(x^{-1}) := f(x)^{-1}$ ext $\rightarrow i$

The triple of functors

$$\text{Gp} \begin{array}{c} \xleftarrow{\text{ext}} \\[-1ex] \xrightarrow{\perp} \\[-1ex] \xleftarrow{\text{int}} \end{array}$$

Mon generalizes

to groupoids

$$\begin{array}{ccc} & \xleftarrow{\text{ext}} & \\ & \perp & \\ & \xleftarrow{\text{int}} & \end{array}$$

Categories

any cat C has an internal groupoid C^{int}
 which has the same objects but only the isomorphisms
 as morphisms.

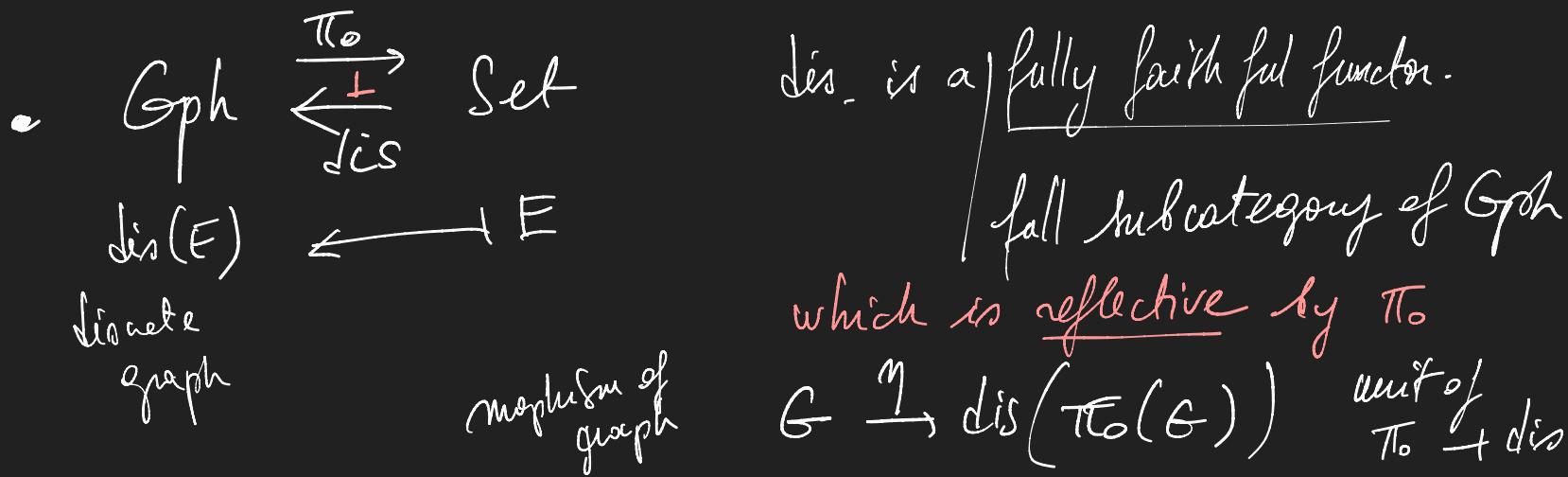
$$C^{\text{int}} \subset C$$

groupoid.

$$C \hookrightarrow C^{\text{ext}} \quad \text{external grayoid}$$

constructed by forcing all morphisms
to become isomorphisms

(\rightarrow theory of localizations of categories)



Morale

full subcat (\rightarrow) property on objects.

reflective subcat \hookleftarrow the property can be forced
 $M \rightarrow M^{\text{ext}}$

coreflective subcat (\rightarrow) the property can be extracted
 $M^{\text{int}} \rightarrow M$