Adjunctions

Let $F : C \rightleftarrows D : G$

an <u>adjunction</u> between F and G

if isomorphisms

$$\alpha_{x,y} : \mathrm{Hom}_D(Fx, y) \simeq \mathrm{Hom}_C(x, Gy)$$

natural in x and y.

More examples

- E, F sets.

$$p : E \times F \longrightarrow E \quad \text{(projection)}$$
$$(x, y) \longmapsto x$$

$$P(E \times F) \xleftarrow{p^{-1}} P(E)$$

(power set viewed as categories)

inverse image

$$\{(x,y) \in E \times F \text{ s.t. } p(x,y) \in A\} \longleftarrow A \subset E$$

$$A \times F \subset E \times F$$

$$p^{-1}(A) = A \times F$$

p^{-1} has both a left and a right adjoint.

$$p_! : P(E \times F) \longrightarrow P(E)$$
$$B \longmapsto p_!(B) = \{ a \in E \mid \exists y \in F,\ (a,y) \in B \subset E \times F \}$$

$$p_* : P(E \times F) \longrightarrow P(E)$$
$$B \longmapsto p_*(B) = \{ a \in E \mid \forall y \in F,\ (a,y) \in B \subset E \times F \}$$

We have $p_! \dashv p^{-1} \dashv p_*$

let's see what this means

for any $A \subset E$

$B \subset E \times F$

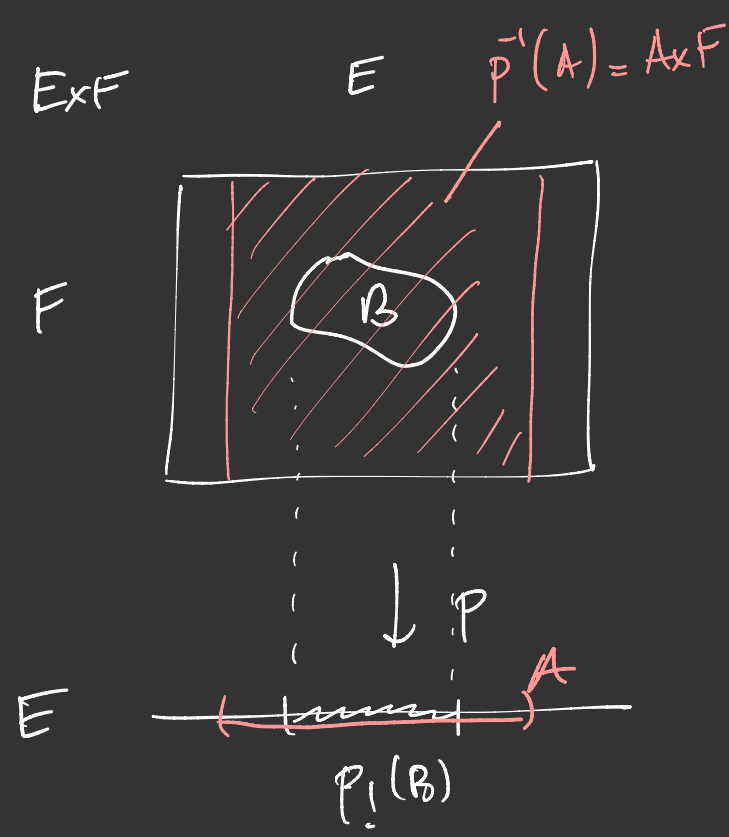
we have

$$p_!(B) \subset A \iff B \subset p^{-1}(A)$$

$$\Updownarrow$$

$$\mathrm{Hom}(p_! B, A) \simeq \mathrm{Hom}(B, p^{-1}(A))$$

$$p_! \dashv p^{-1}$$



$p_!(B)$ is the image of B by p

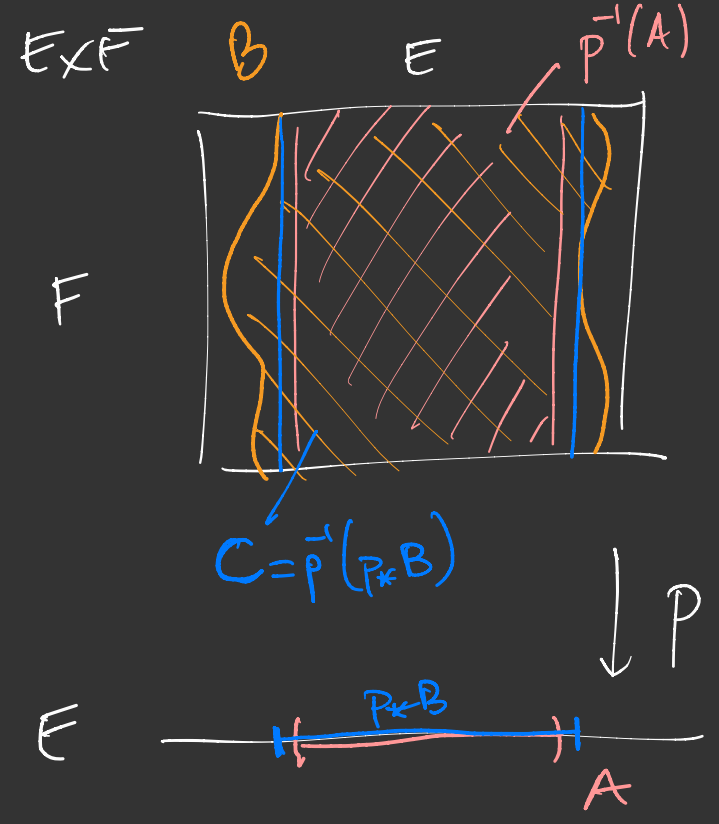
$A \subset E$

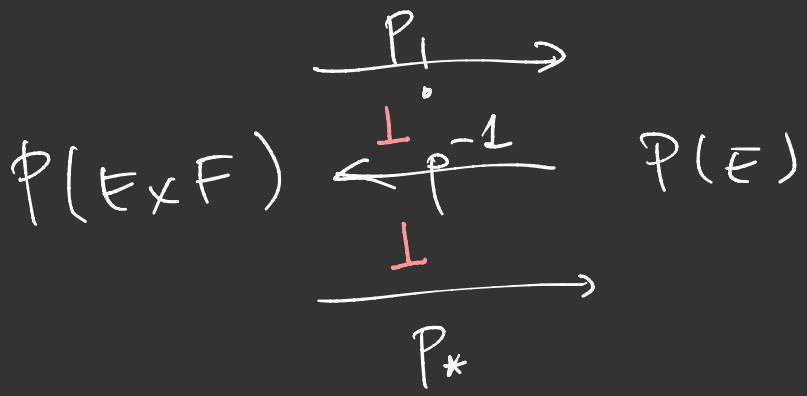
$B \subset E \times F$

$$p^{-1}(A) \subset B \iff A \subset p_*(B)$$

$p_*(B)$ is the largest subset C of E such that

$$p^{-1}(C) \subset B$$





this setting is related to logic

if $B \subset E \times F$ is thought as a formula $\phi(x,y)$

then $p_!(B)$ corresponds to the formula $\exists y\, \phi(x,y)$

and $p_*(B)$ corresponds to

$\forall y\, \phi(x,y)$

<u>morale</u>: quantifiers $\exists, \forall$ in logic are instances of adjoint functors.

(discovery from W. Lawvere)

example in topology

if X is a topological space

$$O(X) \subset P(X) \supset Cl(X)$$

" poset of open subsets of X " " poset of all subsets of X " " poset of closed subsets of X.

these two inclusions of posets have adjoints.

$$O(X) \underset{\text{interior}}{\overset{i}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} P(X) \underset{\text{closure}}{\overset{\bar{i}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} Cl(X)$$

$U \in O(X)$

$A \in P(X)$

$j(U) \subset A$ " $\Leftrightarrow$ " $U \subset \mathring{A}$

if $Z \in Cl(X)$

$A \subset i(Z)$ " $\Leftrightarrow$ " $\bar{A} \subset Z$

$cl(A) \subset Z$

Characterizations of adjunctions

$$F : C \rightleftarrows D : G$$

in terms of universal properties

colim $\quad \Delta$

$F \dashv G$ iff there exists a natural transformation

colim ⊣ cocone

$$\eta : id_C \longrightarrow G \circ F$$

$\Delta \circ$ colim

(eta)

called the unit of the adjunction.

such that

for any $x \in C \quad y \in D$

for any $\quad u = $ arbitrary cocone

$$x \xrightarrow{u} Gy \quad \text{in } C$$

there exists a unique map

colim x_i ''

$$Fx \xrightarrow{v} y \quad \text{in } D$$

such that the composition

$$x \xrightarrow{\eta_x} GFx \xrightarrow{G(v)} Gy$$

is $u : x \longrightarrow Gy$.

Universal property of the colimit cocone.

example : colim.

I small cat.

C is a cat with I-colim

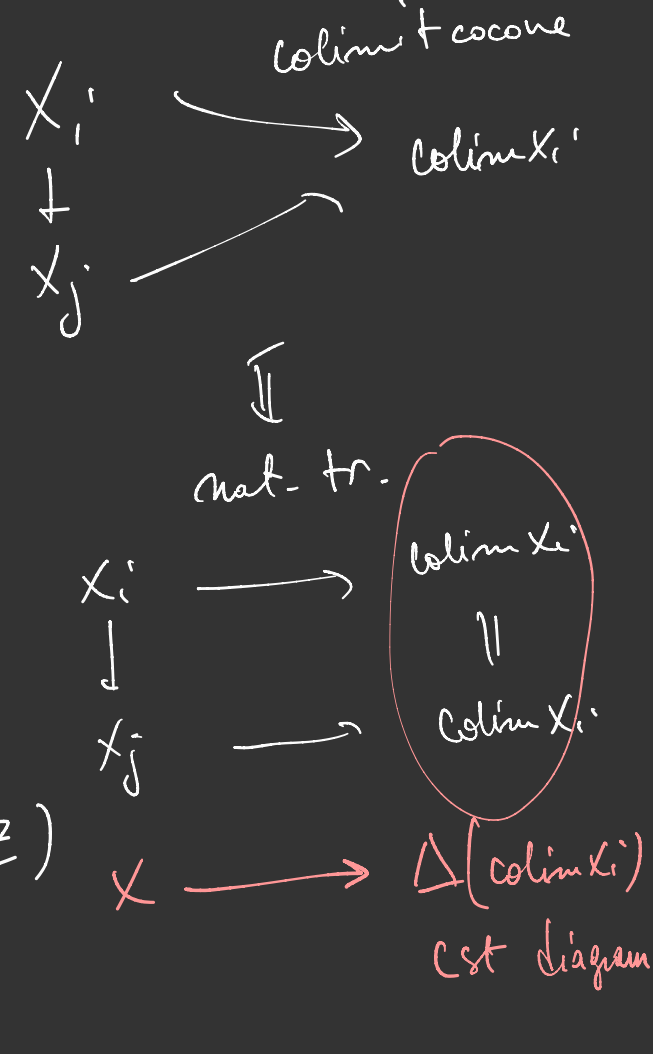
$$C^I \underset{\Delta}{\overset{\text{colim}}{\rightleftarrows}} C \quad (\text{colim} \dashv \Delta)$$

$X : I \to C$ a diagram

$Z := \operatorname{colim}_I X_i$ $\qquad \Delta(Z) : I \to C$ constant diagram.

recall $\operatorname{Hom}_{C^I}(X, \Delta(Z)) = \operatorname{cocone}(X, Z)$

$\Downarrow$ colimit cones $\qquad \Downarrow$ colim cone



other example

$F : Set \rightleftarrows Mon : U$ ($F \dashv U$)

free monoid

underlying set

The universal property of $F(E)$ is exactly the property in the characterization of adjunctions 2 slides back.

exercise

Take any univ. prop. and translate it in term of the characterization of adjunction.

Characterization of adjunction in terms of initial objects

colim $C^{\downarrow} \rightleftarrows C$ $\quad \Delta$

$F : C \rightleftarrows D : G$

"common category of G under x"

$=$

$x \downarrow G$

introduce the following category: fix $x \in C$ (e.g. a diagram)

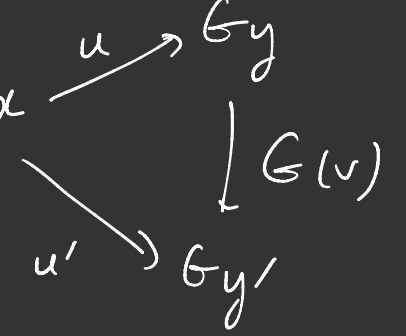
objects: pairs $(y ; x \xrightarrow{u} Gy)$ $\quad$ $= $ cocone with apex y

where $y \in D$

morphism: $(y, x \xrightarrow{u} Gy) \longrightarrow (y', x \xrightarrow{u'} Gy')$

is a map $y \xrightarrow{v} y'$ in D

such that the triangle



commute.

$=$ morphism of cocone.

$x \downarrow G$ generalizes the construction of the category of cocones.

$F : C \rightleftarrows D : G$

$F \dashv G$ iff for any $x \in C$

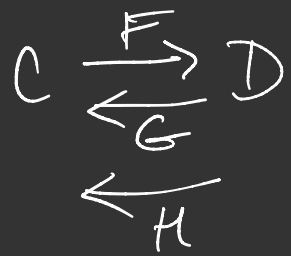
the category $x \downarrow G$ has an

initial object.

Remarks the two characterization of adjunctions in terms of universal property and initial objects are not symmetric in F and G. They come with dual statement in terms of (co-) universal properties (= univ prop of the limit of a diagram) and terminal objects

There exist a fourth characterization of adjunctions in terms a unit $\eta : 1_C \longrightarrow G \circ F$ and a counit $\varepsilon : F \circ G \longrightarrow 1_D$

$F \dashv G$ if there exist η and ε satisfying some condition (see Homework)



Proposition

Suppose $F \dashv G$ and $F \dashv H$ — G and H are both right adjoint to F

then there exists a canonical natural isomorphism of functors

$$G \simeq H.$$

need a lemma

Yoneda principle

C cat $\quad x, y \in C$

$\mathrm{Hom}_C(-, x) : C^{op} \longrightarrow Set$

$\mathrm{Hom}_C(-, y) : C^{op} \longrightarrow Set$

recall that a ~~map~~ isomorphism $x \xrightarrow{\sim} y$ induces a natural ~~transformation~~ isomorphism

$\mathrm{Hom}_C(-, x) \xrightarrow{\text{to}-}_{\sim} \mathrm{Hom}_C(-, y)$

Yoneda principle

given a natural isomorphism

$$\alpha : \mathrm{Hom}_{\mathcal{C}}(-, x) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(-, y)$$

there exists a unique isomorphism $f : x \xrightarrow{\sim} y$ in $\mathcal{C}$ such that

$$\alpha = f \circ -$$ (post composition with f)