I small category

C category

$C^I = \mathrm{Fun}(I, C)$

$=$ cat. of I-diagrams in C

<u>Constant diagram functor</u>

$$\Delta : C \longrightarrow C^I$$

$$x \longmapsto \Delta(x) : I \longrightarrow C$$

$$\begin{array}{ccc} i & \longmapsto & X \\ u \downarrow & & \downarrow 1_X \\ j & \longmapsto & X \end{array}$$

let $Y : I \longrightarrow C$, $X \in C$

$i \longmapsto Y(i)$

$\mathrm{Hom}_{C^I}(Y, \Delta X) =$ nat. transf. $\alpha : Y \to \Delta X$

$= \forall i,\ Y(i) \xrightarrow{\alpha_i} \Delta(X)(i) = X$

such that $\forall u : i \to j \in I$

$$\begin{array}{ccc} Y(i) & \xrightarrow{\alpha_i} & \Delta(X)(i) = X \\ {\scriptstyle Y(u)} \downarrow & & \downarrow {\scriptstyle \Delta(X)(u)} \quad \downarrow 1_X \\ Y(j) & \xrightarrow{\alpha_j} & \Delta(X)(j) = X \end{array}$$

<u>commutes</u>

this last condition can be written as a commutative triangle

$$\begin{tikzcd} Y(i) \arrow[r, "\alpha_i"] \arrow[d, "Y(u)"'] & X \\ Y(j) \arrow[ur, "\alpha_j"'] & \end{tikzcd}$$

this is exactly the definition of a cocone $Y(i) \longrightarrow X$:

there exists a bijection

$$\mathrm{Hom}_{C^I}(Y, \Delta(X)) \simeq \mathrm{Cocones}(Y(i), X)$$

we have by definition of $\operatorname{colim} Y(i)$

$$\mathrm{Cocone}(Y(i), X) \simeq \mathrm{Hom}_C(\operatorname*{colim}_i Y(i), X)$$

we get a bijection

$$\mathrm{Hom}_C(\operatorname{colim} Y(i), X) \cong \mathrm{Hom}_{C^I}(Y, \Delta(X))$$

similarly for limits we get a bijection

$$\mathrm{Hom}_C\left(X, \lim_i Y(i)\right)$$
$$\cong$$
$$\mathrm{Hom}_{C^I}\left(\Delta(X), Y\right)$$

$$\Delta_I : C \longrightarrow C^I$$

the construction of the colimit / limit

define functor

$$\mathrm{colim}_I : C^I \longrightarrow C$$
$$Y(i) \longmapsto \mathrm{colim}_{i \in I} Y(i)$$

$$\lim_I : C^I \longrightarrow C$$
$$Y(i) \longmapsto \lim_{i \in I} Y(i)$$

We have 3 functors

$$\begin{tikzcd} C^I \arrow[r, shift left=2, "\mathrm{colim}"] \arrow[r, shift right=2, "\lim"'] & C \arrow[l, "\Delta"'] \end{tikzcd}$$

having relations between them

$$\mathrm{Hom}_C(\mathrm{colim}\, Y(i), X) \simeq \mathrm{Hom}_{C^I}(Y, \Delta(X))$$

$$\mathrm{Hom}_C(X, \lim Y(i)) \simeq \mathrm{Hom}_{C^I}(\Delta(X), Y)$$

this are examples of the notion of <u>adjunction</u>

the pair colim, Δ

the pair Δ, lim

are each adjunction.

we have bijections

Colim is on the left side

set of morphisms $\overbrace{\mathrm{Colim}}\, Y(i) \longrightarrow X$

set of morphisms $Y \longrightarrow \underbrace{\Delta(X)}$

Δ is on the right side

Colim is gonna be called a <u>left adjoint</u> functor

Δ ______ <u>right adjoint</u> ______

Definition given C, D categories

an adjunction between C and D

is the data of

- a functor $F : C \to D$
- ________ $G : D \to C$
- for any $x \in C$, $y \in D$ a bijection

$$\alpha_{x,y} : \mathrm{Hom}_D(Fx, y) \simeq \mathrm{Hom}_C(x, Gy)$$

which is natural in x and y

a natural isomorphism between

$\mathrm{Hom}_D(F(-), -) :$

$$C^{op} \times D \longrightarrow \mathrm{Set}$$

$$(x, y) \mapsto \mathrm{Hom}_D(Fx, y)$$

and

$\mathrm{Hom}_C(-, G(-)) :$

$$C^{op} \times D \longrightarrow \mathrm{Set}$$

$$(x, y) \mapsto \mathrm{Hom}_C(x, G(y))$$

bijection $\alpha_{x,y}$ between

the morphisms $Fx \longrightarrow y$

and
the morphisms $x \longrightarrow Gy$

F is called the left adjoint

G ———— right adjoint

in the literature the notation

$F \underset{(\alpha)}{\dashv} G$

$\longrightarrow$ usually dropped from the notation.

for an adjunction
where F is left adjoint
an G is the right adjoint

Examples of adjunctions

- $\mathrm{colim}_I \dashv \Delta_I$

$$\mathrm{colim}_I : C^I \underset{\perp}{\rightleftarrows} C : \Delta_I$$

$$\alpha_{x,y} : \mathrm{Hom}_C(\mathrm{colim}\, Y(i), X) \simeq \mathrm{Hom}_{C^I}(Y, \Delta(X))$$

- $\Delta_I \dashv \lim_I$

$$\Delta_I : C \underset{\perp}{\rightleftarrows} C^I : \lim_I$$

$$\alpha_{x,y} : \mathrm{Hom}_C(X, \lim Y(i)) \cong \mathrm{Hom}_{C^I}(\Delta(X), Y)$$

<u>Remark</u> that Δ is a right adjoint in the colim $\dashv \Delta$ adjunction but is a left adjoint in the $\Delta \dashv \lim$ adjunction.

$I = \emptyset$

$C^I = 1 = \{*\}$ punctual category

(there is a single functor $\emptyset \longrightarrow C$)

$$\begin{array}{ccccc}
\mathrm{colim}_I : & C^I & \underset{\perp}{\rightleftarrows} & C & : \Delta_I \\
& \| & & & \| \\
\text{initial object } 0_C & : 1 & \underset{\perp}{\overset{0_C}{\rightleftarrows}} & C & : \text{unique functor } C \to 1
\end{array}$$

the adjunction reads

$$\mathrm{Hom}_C(\mathrm{colim}\, Y(i), X) \simeq \mathrm{Hom}_{C^I}(Y, \Delta(X))$$

$$\mathrm{Hom}_C(\mathrm{colim}_{\emptyset}\, *, X) \simeq \mathrm{Hom}_1(*, *) = \{id_*\}$$

for any X there is a unique morphism

$$0_C = \underbrace{\mathrm{colim}_{\emptyset}\, *}_{\text{initial object in } C} \longrightarrow X$$

on the limit side $(I = \emptyset)$

$$\begin{array}{ccc}
\Delta_I : C & \underset{\perp}{\rightleftarrows} & C^I : \lim_I \\
\| & & \| \\
\text{unique functor } C \to 1 : C & \underset{1_C}{\overset{\perp}{\rightleftarrows}} & 1 : \text{terminal object.}
\end{array}$$

the adjunction reads

for any X in C there exists a unique morphism

$$X \longrightarrow \lim_{\emptyset} * = \underbrace{1_C}_{= \text{ definition of terminal object of } C.}$$

notation

$\operatorname{colim}_I \dashv \Delta_I \quad \Delta_I \dashv \lim_I$

is abbreviated

$$\operatorname{colim}_I \dashv \Delta_I \dashv \lim_I$$

generally

$$\overbrace{F \dashv G}^{\text{adjoint pair}} \dashv \overbrace{H \dashv K}^{\text{adjoint pair}} \dashv L$$

$$F \dashv \underbrace{G \dashv H}_{\text{adjoint pair}} \dashv \underbrace{K \dashv L}_{\text{adjoint pair}}$$

$$\begin{tikzcd}
C \arrow[r, shift left=8, "F"] \arrow[r, phantom, shift left=6, "\perp"] & D \arrow[l, shift left=-4, "G"'] \\
\end{tikzcd}$$

$$\begin{tikzcd}
C \arrow[r, shift left=8, "F"] & D \arrow[l, shift right=-4, "G"] \arrow[l, shift left=-4, "K"] & \\
C \arrow[r, shift left=2, "H"] \arrow[r, shift left=-8, "L"'] & D
\end{tikzcd}$$

if C has both initial and terminal objects

$$
\begin{tikzcd}
\Delta_\phi : C \arrow[r, phantom, "\perp"] \arrow[r, "\perp"] & 1 \arrow[l, bend right, "0_C"'] \arrow[l, bend left, "1_C"]
\end{tikzcd}
$$

$$0_C \dashv \Delta_\phi \dashv 1_C$$

example : I = set with 2 element $= \{0, 1\}$

$$
\begin{tikzcd}
C^I = C \times C \arrow[r, bend left, "\operatorname{colim}_I = + = \text{binary sums}"] \arrow[r, bend right, "\lim_I = \times = \text{binary product}"'] & C \arrow[l, "\Delta_I = \text{diagonal}"']
\end{tikzcd}
$$

if C has binary sums and product we can write

$$+ \dashv \Delta \dashv \times$$

More examples of adjunction

- $$\begin{tikzcd} \mathrm{Set} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathrm{Mon} \arrow[l, shift left, "U"] \end{tikzcd}$$

F = free monoid functor

U = underlying set of a monoid.

we have $F \dashv U$

for a set E and a monoid M
the universal property of the free monoid on E builds a bijection between

the set of morphism of monoids

$$F(E) \longrightarrow M$$

and the set of maps

$$E \longrightarrow U(M)$$

$$\alpha_{E,M} : \mathrm{Hom}_{\mathrm{Mon}}(F(E), M) \simeq \mathrm{Hom}_{\mathrm{Set}}(E, U(M))$$

- $\mathrm{Gph} \underset{U}{\overset{F}{\rightleftarrows}} \mathrm{Cat}$

F = free cat. on a graph

U = underlying graph of a cat.

$F \dashv U$

the universal prop. of $F(G)$:

$$\alpha_{G,C} : \mathrm{Hom}_{\mathrm{Cat}}(F(G), C) \simeq \mathrm{Hom}_{\mathrm{Gph}}(G, U(C))$$

- $\mathrm{Set} \underset{i}{\overset{\pi_0}{\leftrightarrows}} \mathrm{Gph}$

π_0 = connected components

i = a set viewed a graph.

$\pi_0 \dashv i$

$$\exists\, \alpha_{G,E}\ \mathrm{Hom}_{\mathrm{Set}}(\pi_0(G), E) \cong \mathrm{Hom}_{\mathrm{Gph}}(G, i(E))$$