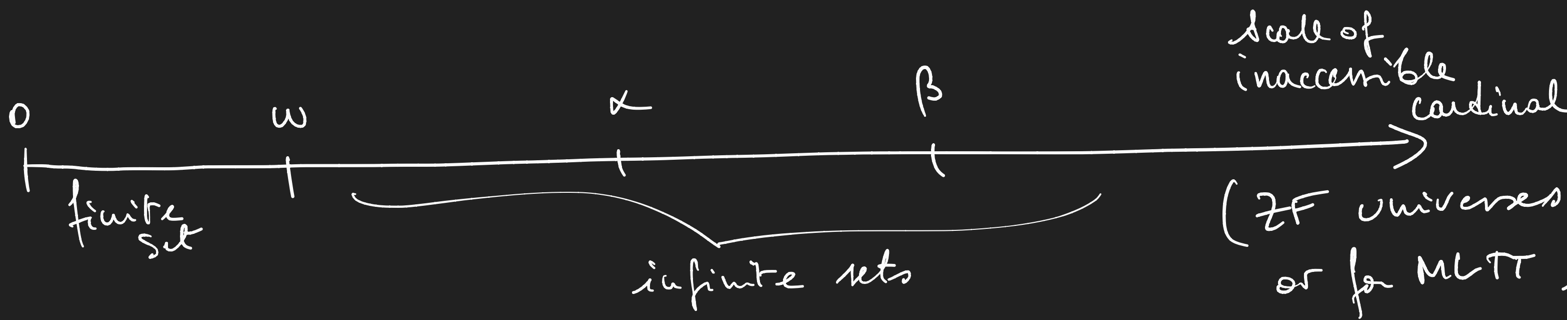


Set theory is fine (no contradiction) if we bound the size of set by some inaccessible cardinal.

κ cardinal   a set E is κ-small if $|E| < \kappa$

the category of κ-small sets is denoted $\mathrm{Set}_{<\kappa}$

ex : $\kappa = \omega =$ countable cardinal $\mathrm{Set}_{<\kappa} = \mathrm{FinSet}$

a category C is <u>κ-small</u> if

$$|Arr(C)| < \kappa$$

since $ob(C) \subset Arr(C)$

$$|ob(C)| \leq |Arr(C)| < \kappa$$

<u>example</u> $\kappa = \omega$

a category is <u>finite</u> (ω-small)

if it has a finite set of arrows. (this implies that there's a finite set of objects)

the category of κ-small categories is denoted

$$Cat_{<\kappa}$$

Other small objects

- a monoid is κ-small if its underlying set is κ-small

$\mathrm{Mon}_{<\kappa}$ cat of κ-small monoids.

- a group ——————————

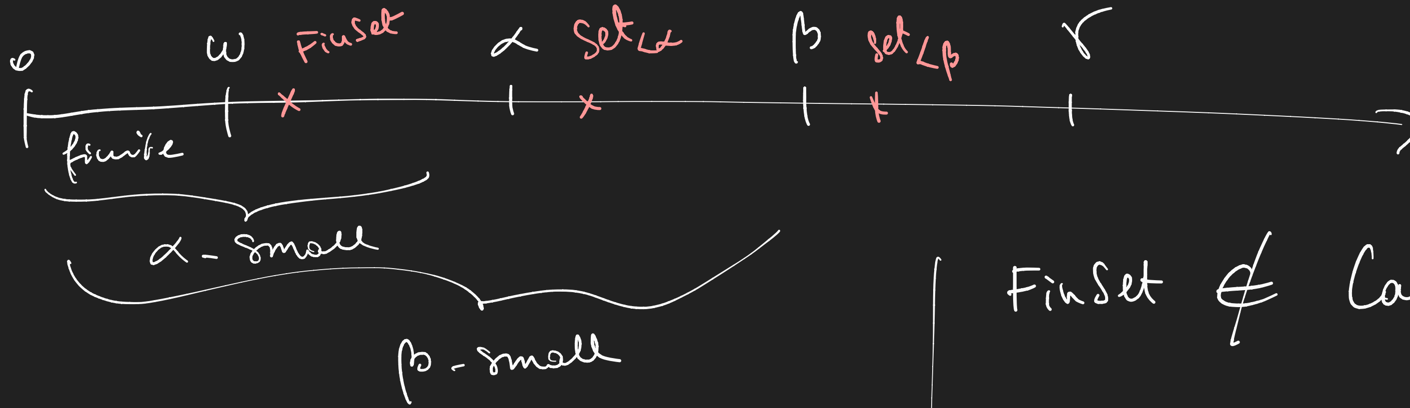
$\mathrm{Gp}_{<\kappa}$

- a poset ——————————

$\mathrm{Poset}_{<\kappa}$

...

Remark the notion of being small is more complex for a topological space



FinSet is not finite.

but it's α-small

for α any inaccessible

cardinal $\omega < \alpha$

$\text{FinSet} \notin \text{Cat}_{<\omega}$

$\text{FinSet} \in \text{Cat}_{<\alpha}$

$\text{Set}_{<\alpha} \notin \text{Cat}_{<\alpha}$

but if $\beta > \alpha$ another in. card.

$\text{Set}_{<\alpha} \in \text{Cat}_{<\beta}$.

Vocabulary for size

	ω	α	β	γ
in set theory	finite set	set	classes	(aggregates)
in category theory	finite	small = α-small	large = β-small	very large = γ-small

practice of ST is here

practice of CT is here

now: fix in. card $\omega < \alpha < \beta < \gamma$

Recall κ is regular

if $Set_{<\kappa}$ has $\coprod$

indexed by κ-small sets

- κ is inaccessible if

$Set_{<\kappa}$ has $\coprod$ $\prod$

indexed by κ-small sets

Theorem

- if κ is regular $Set_{<\kappa}$

has all colimits indexed

by a κ-small category

- if κ is inaccessible $Set_{<\kappa}$

has all colimits and limits

indexed by κ-small categories

proof of the colimit statement

- $\mathrm{Set}_{<\kappa}$ has $\underset{\kappa\text{-small}}{\coprod}$ ✓

- $\mathrm{Set}_{<\kappa}$ has coequalizers ✓

$$\underset{\kappa\text{-small}}{A} \rightrightarrows \underset{\kappa\text{-small}}{B} \dashrightarrow C$$

the coequalizer is it κ-small? yes

C is a quotient of B

$$|C| \leq |B| < \kappa$$

to finish the proof we use that any colimit can be computed from coproducts and equalizers.

I κ-small cat.

$$X : I \longrightarrow Set_{<\kappa}$$

$$\operatorname{colim}_{i} X(i) = \,?$$

+ prove that it is κ-small.

$ob(I) = I_0 \qquad |I_0| \leq |I_1| < \kappa$

$Arr(I) = I_1$

$$\coprod_{i \in I_0} X(i) \qquad \coprod_{u \in I_1} X(s(u))$$

where $s : I_1 \longrightarrow I$ is the 'source' or 'domain' map

both exist in $Set_{<\kappa}$

$$P = \{ 1 \rightrightarrows 0 \}$$

$$\begin{tikzcd}
P \arrow[r] & \mathrm{Set}_{<\kappa} \\
1 \arrow[r, mapsto] \arrow[d, shift right] \arrow[d, shift left] & \coprod_{u \in I_1} X(s(u)) \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\beta"] \\
0 \arrow[r, mapsto] & \coprod_{i \in I_0} X(i)
\end{tikzcd}$$

$$\alpha : \coprod_{u \in I_1} X(s(u)) \longrightarrow \coprod_{i \in I_0} X(i)$$

$$\iff$$

$$\alpha_u : X(s(u)) \xrightarrow{inc_{s(u)}} \coprod_{i \in I_0} X(i)$$

for all $u \in I_1$ $\qquad s(u) \in I_0$

$$\beta : \coprod_{u \in I_1} X(s(u)) \longrightarrow \coprod_{i \in I_0} X(i)$$

$\iff$ for any $u \in I_1$

$$\beta_u : X(s(u)) \xrightarrow{X(u)} X(t(u)) \xrightarrow{inc_{t(u)}} \coprod_{i \in I_0} X(i)$$

<u>claim</u> the coequalizer of

$$\coprod_{u \in I_1} X(s(u)) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \coprod_{i \in I_0} X(i) \longrightarrow Z$$

is also the colimit of the diagram X.

strategy for the proof: construct bijection

$$\mathrm{Cocone}\left(\coprod \dots \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \coprod \dots , Y \right) \simeq \mathrm{Cocone}\left(X(i), Y \right)$$

for any object Y

cocone $(X(i), Y)$

maps $X(i) \xrightarrow{y_i} Y$

s.t. for any $u : i \to j \in I_1$

$$\begin{tikzcd} X(i) \arrow[r, "y_i"] \arrow[d, "X(u)"'] & Y \\ X(j) \arrow[ur, "y_j"'] & \end{tikzcd}$$

commutes.

cocone$\left(\sqcup \cdots \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \sqcup \cdots , Y\right) = ?$

recall cocone$\left(A \overset{\alpha}{\underset{\beta}{\rightrightarrows}} B, Y\right)$

$$\begin{tikzcd} A \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] \arrow[dr, bend right, "g"'] & B \arrow[d, bend left, "f"] \\ & Y \end{tikzcd}$$

s.t. $f\alpha = f\beta = g$

$\iff$ $f : B \to Y$ st.

$f\alpha = f\beta$

$$\mathrm{Cocone}\left(\coprod \ldots \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \coprod \ldots, Y\right)$$

$$\parallel$$

$$\coprod_{i \in I_0} X(i) \xrightarrow{y} Y$$

s.t. $y \circ \alpha = y \circ \beta$

y is equivalent to a family

$$y_i : X(i) \longrightarrow Y \quad \text{for } i \in I_0$$

<u>exercise</u>: the condition

$$y \circ \alpha = y \circ \beta$$

implies the expected condition on y_i

$X(u)$

$$\begin{tikzcd} X(i) \arrow[r, "y_i"] \arrow[d] & Y \\ X(j) \arrow[ur, "y_j"'] & \end{tikzcd}$$

commutes.

end of the proof:

the bijection

$$\mathrm{cocones}(U_{\cdot\cdot} \rightrightarrows U_{\cdot\cdot\cdot}, Y)$$
$$\wr|$$
$$\mathrm{cocone}(X(i), Y)$$

can be used to construct an isomorphism between the categories of cocones on the 2 diagrams.

Hence the colimit, i.e. the initial object of the cat. of cocones, are the same object. ☒

<u>Rem</u> for the limits, need to use the <u>equalizer</u> of

$$\prod_{i \in I_0} X(i) \rightrightarrows \prod_{i \in I_1} X(t(u))$$

Corollary $\boxed{\kappa \text{ inaccessible}}$

for any κ-small cat C

the category of C-diagrams

in $Set_{<\kappa}$: $Fun(C, Set_{<\kappa})$

has all κ-small colimits

and limits.

proof: Homework!

other examples of

categories with

small colimits & limits

- $Mon_{<\kappa}$
- $Gp_{<\kappa}$
- $Poset_{<\kappa}$
- $Abelian\ gp_{<\kappa}$
- $Cat_{<\kappa}$
- ...

(proof are more involved

than for $Set_{<\kappa}$)

I small cat

C a category with all colimits of shape I.

let $C^I = \mathrm{Fun}(I, C)$

= cat of I-diagram in C + natural transformations

there exist a functor

$$\Delta : C \longrightarrow C^I$$

$$X \longmapsto \Delta(X) = \text{constant diagram with value } X$$

<u>Lemma</u> (exercise) for $X : I \to C$, there is a bijection

$$\mathrm{Cocone}(X(i), Y) \simeq \mathrm{Hom}_{C^I}(X, \Delta(Y))$$

<u>sketch of the proof</u> $f : X \to \Delta Y$ nat. tr.

$$\begin{tikzcd}
X(i) \arrow[r, "f(i)"] \arrow[d, "X(u)"'] & \Delta(Y)(i) = Y \arrow[d, "\Delta(Y)(u)"', "id_Y"] \\
X(j) \arrow[r, "f(j)"'] & \Delta(Y)(j) = Y
\end{tikzcd}$$

define a cone $X(i) \xrightarrow{f(i)} Y$

<u>proposition</u>. The colimit of I-diagrams in C define a functor

$$C^I \longrightarrow C$$

$$X(i) \longmapsto \underset{i}{\mathrm{colim}}\, X(i)$$

(apex of the colimit cone)

proof: Homework.

Discussion about the "true" size of categories $\mathrm{Set}_{<\kappa}$

(these issues will be properly explained latter in the course)

(not necessary for Homework!)

there's a problem in the definition of $Set_{<\kappa}$

because we need a "collection" of objects
"
sets

need of a "set of set"
"universe of set" ...

in ZF + universe axiom : $\mathbb{U}$ $ob(Set_{<\kappa})$ = sets in $\mathbb{U}$

in CT what we care about is sets up to bijection

Variation : define $ob(Set_{<\kappa})$ as sets that are in bijection with a set in $\mathbb{U}$

strictly speaking the 2 def of $\mathrm{Set}_{<\kappa}$ do not produce isomorphic categories. But they should be the same categories! they are equivalent categories.

because they have the same classes of objects up to isomorphism (sets up to bijection)

introducing <u>equivalent categories</u> we can define a category C to be <u>essentially small</u> if it is equivalent to a small category.

ex $\alpha < \beta$

$Set_{<\alpha}$ is only essentially β-small.

FinSet is — ———— small.

0 is an ordinal

$\{0\}$ singletons.

$\Rightarrow$ FinSet has a non small collection of objects