Today : size issue

size of objects are a problem in Set Theory and a fonction in Cat. theory.

Qst is it possible for a category to have all colimits or all limit?
(without size restrictions)

Answer (Freyd's fact):

Yes but the category has to be a preorder!

<u>Consequence</u> : the category of Sets does not have arbitrary colimits or limits.

$\longrightarrow$ need a <u>restriction</u> on the type of colimits / limits that can exist in Set

$\rightsquigarrow$ notion of "small colimit" and "small limit" the core of size issues is the following result

Cantor's lemma

given a set E, then the cardinal of $P(E)$ is strictly bigger than the cardinal of E.

$$|E| < |P(E)|$$

in other words: there is no bijection $E \simeq P(E)$

proof

suppose a map
$$E \xrightarrow{A} P(E)$$
$$x \longmapsto A(x)$$

then A cannot be surjective

to see this: picture A by a table

Recall $P(E)$ = set of fct $E \to \{0,1\}$

$$A : E \longrightarrow P(E) \simeq \{0,1\}^E$$

$$x \longmapsto A(x) \leftrightarrow \chi_{A(x)} : E \to \{0,1\}$$

$A(x) \subset E$ characteristic fct.

x, y, z are all elt of E

	x	y	z	t	....	elt of E
$\chi_{A(x)}$	0	1	1	0	1	...
$\chi_{A(y)}$	0	0	1	0	1	...
$\chi_{A(z)}$	1	1	1	0	0	...
⋮				... 1 ...		

define the set

$B \subset E$

$x \in B$ if

$x \notin A(x)$

	x	y	z	t
χ_B	1	1	0	0 ...

"dual" value than the diagonal

claim : $B \neq A(x)$ for all x.

if $B = A(x)$

$$\chi_B = \chi_{A(x)}$$

$$\chi_B(x) = \chi_{A(x)}(x)$$

but by definition of B

$$\chi_B(x) \neq \chi_{A(x)}(x)$$

so $B = A(x)$ impossible.

B is not in the image of $A : E \to P(E)$

the map cannot be surjective nor a <u>bijection</u>.

One the other side, there exist an <u>injection</u>

$$\begin{aligned} E &\longrightarrow P(E) \\ x &\longmapsto \{x\} \end{aligned}$$

so $|E| \underset{\text{strict}}{<} |P(E)|$

proof of Freyd's fact

Suppose $\mathcal{C}$ is a cat with arbitrary sums

Then $\mathcal{C}$ is a preorder.

proof : $C_0 = ob(\mathcal{C})$

$C_1 = Arr(\mathcal{C})$

fix $x \in \mathcal{C}$

Consider $\coprod_{C_1} x$ = sum of C_1 copies of x.

for any other $y \in \mathcal{C}$

$$Hom(x, y) \subset C_1$$

$$|Hom(x, y)| \leq |C_1|$$

in part. $|Hom(\coprod_{C_1} x, y)| \leq |C_1|$

$$Hom(\coprod_{C_1} x, y) \simeq \prod_{C_1} Hom(x, y)$$

$$\simeq Hom(x, y)^{C_1}$$

Suppose $|Hom(x,y)| \geqslant 2$

then $\{0,1\} \subset Hom(x,y)$

$$P(C_1) \simeq \{0,1\}^{C_1} \subset Hom(x,y)^{C_1}$$

$$|P(C_1)| = |\{0,1\}^{C_1}| \leqslant |Hom(x,y)^{C_1}| = |Hom(\coprod_{C_1} x, y)|$$

$\vee$ by Cantor's lemma

$|C_1|$

$$|Hom(\coprod_{C_1} x, y)| \leqslant |C_1|$$

$$|C_1| < |Hom(\coprod_{C_1} x, y)|$$

contradiction!

Hence

$|Hom(x,y)| = 1$ or 0

$\Rightarrow C$ preorder!

$\Rightarrow$ Set does not have arbitrary sums.

$\rightarrow$ need to restrict the size of sums existing in Set.

Cantor's fact : if I is a set and E_i is a family of sets indexed by I. Then there exists a set F which is not in bijection with any of the E_i

proof consider $\coprod_{i \in I} E_i =: E$

we have $|E_i| \leq |E| < |P(E)|$

chose $F := P(E)$.

Cantor's paradox

the collection of all sets is not a set.

proof: suppose it is a set and apply Cantor's fact to find a set which is not in the collection of all sets:

absurd!

Solutions to the paradox?

first solution: introduce a new type of object classes such that any set is a class (but not the other way) and such that the collection of all sets is a class.

problems of this solution

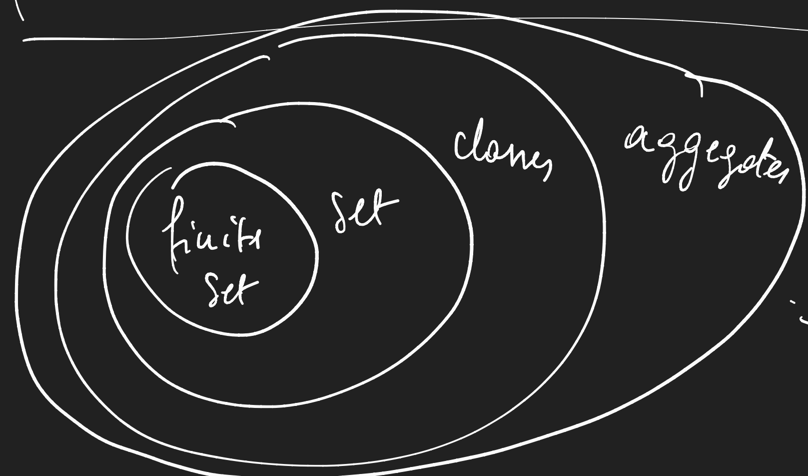
- not clear what the difference between sets and classes is (both are collections of things)
- the collection of all classes is still a problem.

(need to invent a new notion <u>aggregates</u> to collect classes)

<u>second solution</u>

keep a single notion of <u>set</u>

but introduce <u>size bounds</u>.




0
ω
countable cardinal
α inaccessible cardinal
β in. card
γ in. co
ℕ, ℝ
total order of cardinals
finite sets
sets ∐ Π
classes
aggregates

Cantor set theory

intuitively : set = collections.

formally : sets are things
with some "algebraic" operations
given by $\bigsqcup$ indexed by set, $\prod$ indexed by a set
+ extracting subset ...

any theory of sets has to be stable
by these operations.

Definition

- if κ is a cardinal
a set E is called κ-small
if $|E| < \kappa$

ex $\kappa = \omega$ countable cardinal

E is ω-small iff
it is finite

$\mathrm{Set}_{<\kappa}$ category of

κ-small sets.

ex $\mathrm{Set}_{<\omega} = \mathrm{FinSet}$

• κ is called <u>regular</u>

if $\mathrm{Set}_{<\kappa}$ has $\underset{\kappa\text{-small}}{\coprod}$

(indexed by κ-small sets)

ex $\mathrm{Set}_{<\omega} = \mathrm{FinSet}$ has

finite coproducts / sums

so ω is <u>regular</u>

• κ is (strongly) <u>inaccessible</u> if

$\mathrm{Set}_{<\kappa}$ has $\underset{\kappa\text{-small}}{\coprod}$ $\underset{\kappa\text{-small}}{\prod}$

ex ω is inaccessible.

Suppose we have a set theory

with $\amalg$ $\prod$

then the cardinal κ of

the collection of all sets

is <u>inaccessible</u>

<u>proof</u>: any set is κ-small

$\rightsquigarrow$ Cantor

Second solution is to

replace the hierarchy of

objects

set / classes / aggregates ...

by

$\mathrm{Set}_{<\alpha}$ / $\mathrm{Set}_{<\beta}$ / $\mathrm{Set}_{<\gamma}$ ...

for some inaccessible

cardinals

$\omega < \alpha < \beta < \gamma < \ldots$

there is no paradox if we assume that sets are bounded by an <u>inaccessible cardinal</u> κ

there is a <u>clean beyond</u>: sets larger than κ

implementation of this idea in formal <u>Set theory</u>

- ZFC inaccessible cardinal are equivalent to <u>Grothendieck universes</u>

instead of $\alpha < \beta < \gamma$

$$\mathbb{U} \in \mathbb{V} \in \mathbb{W}$$

hierarchy of universes.

- in <u>Dependent type theory</u>

inaccessible cardinal is equivalent to a universe (type of types)

stable by Σ and Π

(dependent sums) (dep. products)