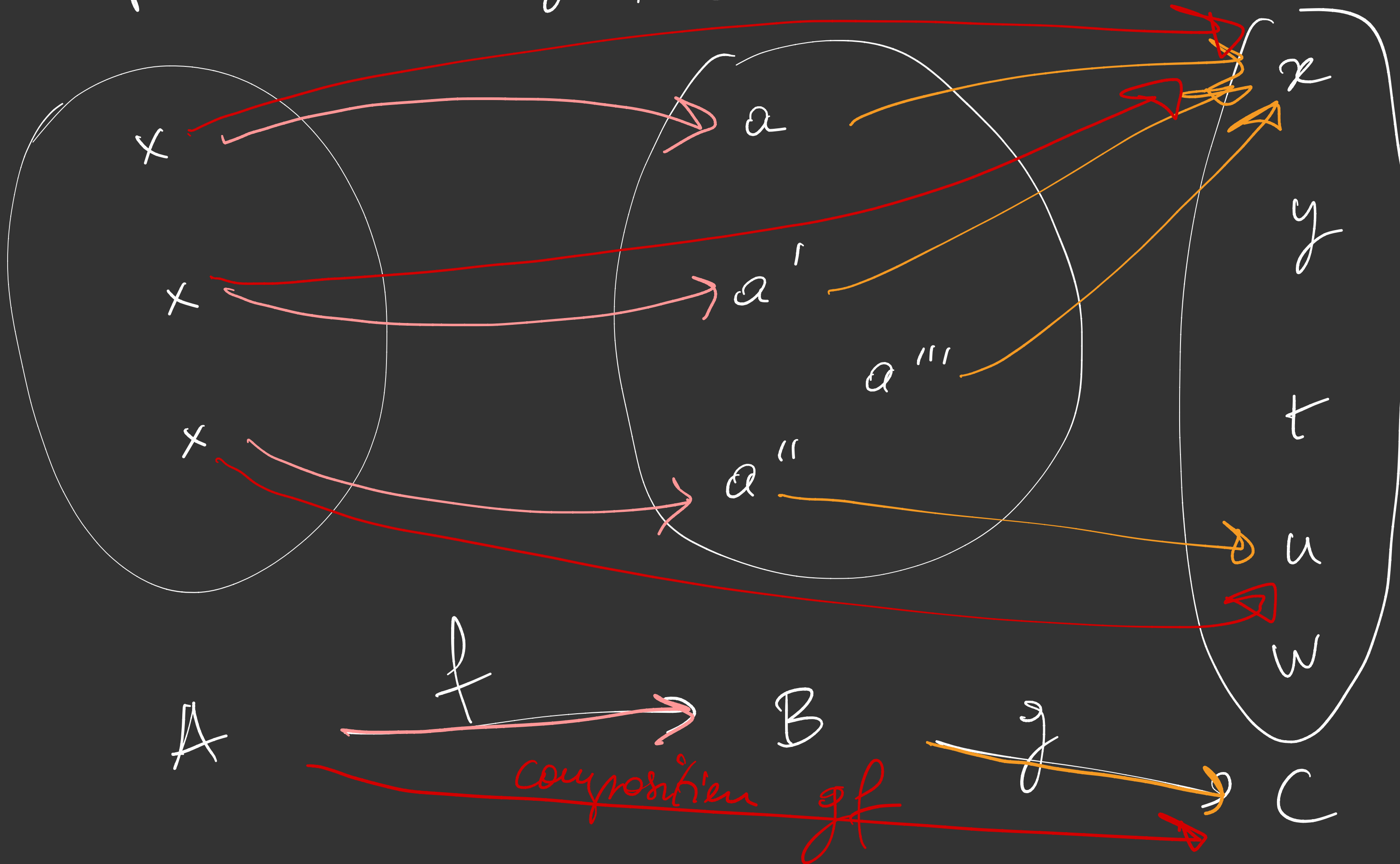
CATEGORY THEORY

WHAT IS A CATEGORY?

structure providing a synthesis for

- set
- preorder, poset, total order, well-order
- equivalence relation
- monoid, monoid action
- group, group action
- groupoid
- path algebra of a graph

main example : the category of sets



composition of functions is **associative**

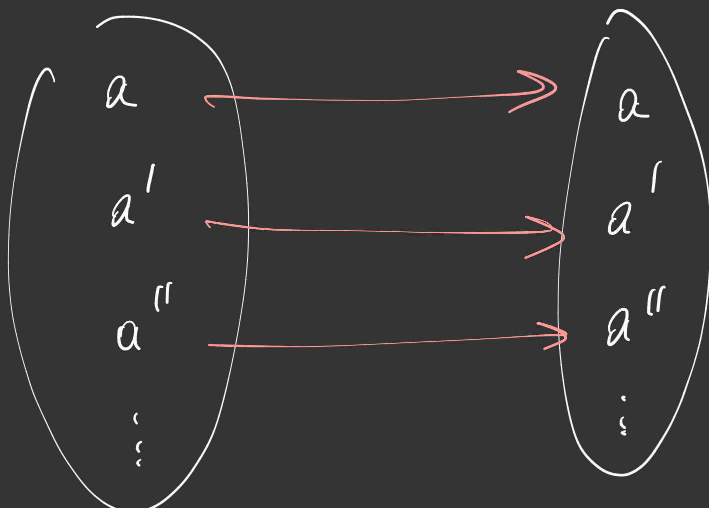
$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend right, "gf"'] \arrow[rrr, bend left=60, "(hg)f"] \arrow[rrr, bend right=60, "h(gf)"'] & B \arrow[r, "g"] \arrow[rr, bend left, "hg"] & C \arrow[r, "h"] & D
\end{tikzcd}$$

$$(hg)f = h(gf)$$

same function.

Special function: <u>identity function</u>

$$A \xrightarrow{id_A,\ 1_A} A$$
$$a \longmapsto a$$



important because of this property:

for any function

$$f: A \longrightarrow B$$

$$\begin{tikzcd} & A \arrow[r, "f"] & B \\ A \arrow[ur, "id_A"] \arrow[urr] & & \end{tikzcd}$$

$$f\, id_A = f$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right] & B \arrow[r, "id_B"] & B \end{tikzcd}$$

$$id_B f = f$$

Definition of a Category

a category C is the following data

- a collection of objects $ob(C)$
- a collection of morphisms / maps / arrows $arr(C)$

each arrow has a domain (source)

and a codomain (target)

which are objects

domain (source) $\longrightarrow$ codomain (target)

$$arr(C) \underset{\text{codomain}}{\overset{\text{domain}}{\rightrightarrows}} ob(C)$$

- composition structure

X, Y, Z are objects

$X \xrightarrow{f} Y = Y \xrightarrow{g} Z$ are "composable"

if $\operatorname{cod}(f) = \operatorname{dom}(g)$

for each pair of composable function $\underset{X \to Y}{f}$ and $\underset{Y \to Z}{g}$

we are given a composition $gf : X \longrightarrow Z$

- identity arrows

for each object X, we are given a distinguished arrow $id_X : X \longrightarrow X$

$\mathrm{dom}(id_X) = \mathrm{cod}(id_X) = X$

these data are assumed to satisfy the conditions

- associativity of composition.

$$\begin{tikzcd} & Y \arrow[r, "g"] \arrow[rrd, "hg"] & Z \arrow[rd, "h"] & \\ X \arrow[ru, "f"] \arrow[rru, "gf"] \arrow[rrr, "h(gf)"] \arrow[rrr, bend right, "(hg)f"'] & & & T \end{tikzcd}$$

$$h(gf) = (hg)f$$

$$(hg)f = h(gf)$$

means that can forget parenthesis in hgf

- identity relations

for any arrow $f : X \to Y$

$$\begin{tikzcd} X \arrow[r, "id_X"] \arrow[rr, bend right] & X \arrow[r, "f"] & Y \end{tikzcd}$$

$$f\, id_X = f$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right] & Y \arrow[r, "id_Y"] & Y \end{tikzcd}$$

$$id_Y\, f = f$$

Some examples

- the category of sets $\mathbf{Set}$

coll of objects = coll of all sets

coll of arrows = coll. of all fct between sets.

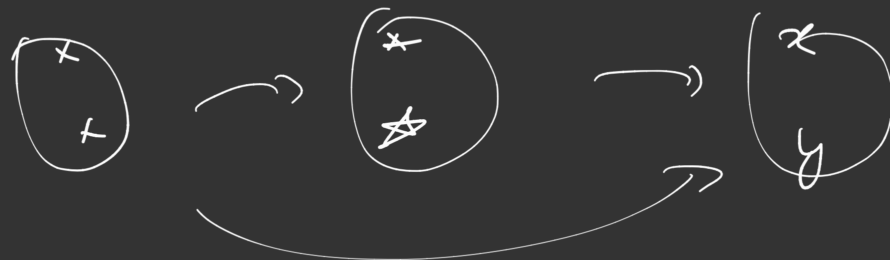
arrow with domain A and codomain B is a function from A to B

$$f : A \longrightarrow B$$

identity arrows = identity functions

composition of arrows = composition of functions.

- cat. of <u>finite sets</u> Fin
- ______ sets of cardinality 2



- cat. of <u>pairs of sets</u>. Set^2

objects : pairs of sets (A, B)

arrows $(A, B) \longrightarrow (A', B')$

pair of functions $(f : A \to A', \ g : B \to B')$

- fix a set I (index set of the family)

Cat of I- families of sets. "Set^I"

objects : fam. of set $=$ A_i a set for each $i \in I$

index

arrows $(A_i)_{i\in I} \xrightarrow{f} (B_i)_{i \in I}$

family of functions $(f_i : A_i \to B_i)_{i \in I}$

More examples

- 2 trivial categories:
  - empty category : $ob(C) = \emptyset$

    $ar(C) = \emptyset$

  - "one object" category $ob(C) = \{*\}$

    "punctual" ———

    $ar(C) = \{id_*\}$

    (single object and a single arrow)

- Any set can be looked as a category

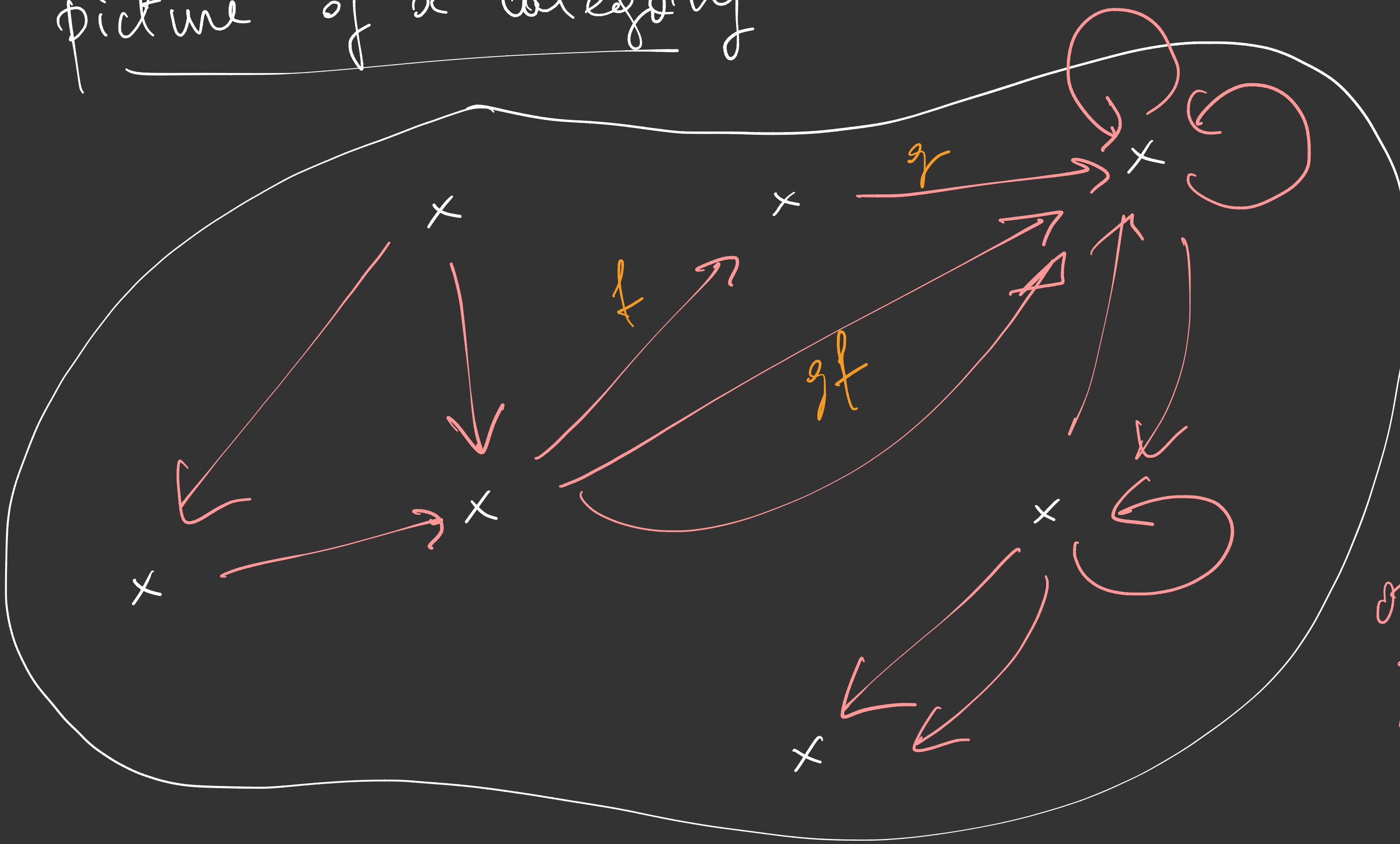
fix a set E

define a category $\underline{E}$

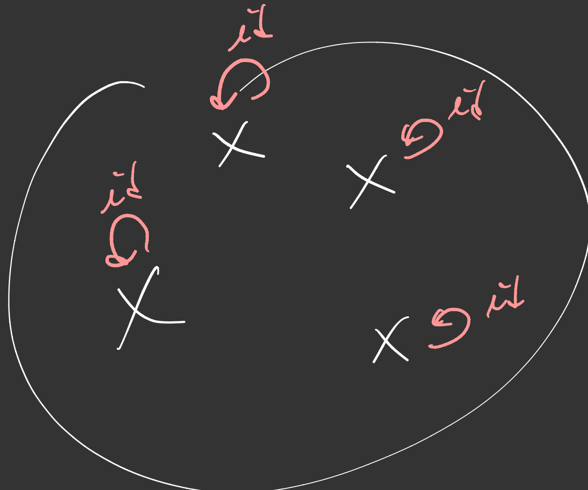
$ob(\underline{E}) = E$ (the <u>objects</u> are the <u>elements</u> of E)

$arr(\underline{E}) =$ identity arrows only.

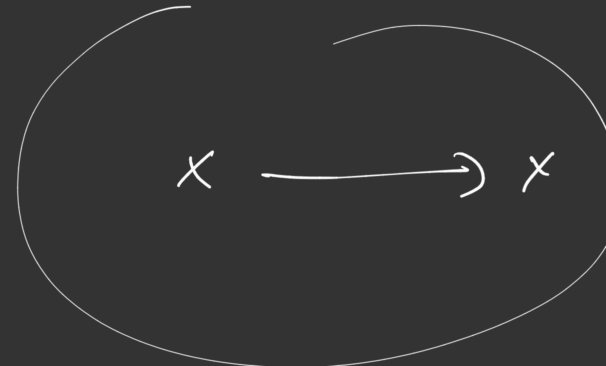
picture of a category



oriented graph.



a category with no non-identity arrows.



category with two objects and a single arrow between them.